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Approximating Fixed Points for Nonexpansive Mappings and Generalized Mixed Equilibrium Problems in Banach Spaces

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Abstract

We introduce a new iterative scheme for finding a common element of the solutions set of a generalized mixed equilibrium problem and the fixed points set of an infinitely countable family of nonexpansive mappings in a Banach space setting. Strong convergence theorems of the proposed iterative scheme are also established by the generalized projection method. Our results generalize the corresponding results in the literature.

Keywords: Generalized Mixed Equilibrium Problem, Nonexpansive Mappings, Common Fixed Point, Strong Convergence, Generalized Projection.
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1. Introduction

Let C be a closed convex subset of a Banach space E. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of a fixed point of T.

Let $f : C \times C \to \mathbb{R}$ be a bifunction, $A : C \to E^*$ a mapping, and $\varphi : C \to \mathbb{R}$ a real-valued function. The generalized mixed equilibrium problem is to find $x \in C$ such that

$$f(x, y) + \langle Ax, y - x\rangle + \varphi(y) \geq \varphi(x), \ \forall y \in C. \tag{1.1}$$

The solutions set of (1.1) is denoted by $GMEP(f, A, \varphi)$.

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If $A \equiv 0$, then the generalized mixed equilibrium problem (1.1) reduces to the following mixed equilibrium problem: finding $x \in C$ such that

$$f(x, y) + \varphi(y) \geq \varphi(x), \ \forall y \in C. \tag{1.2}$$

Problem (1.2) was introduced by Ceng and Yao [7]. The solutions set of (1.2) is denoted by $MEP(f, \varphi)$.

If $f \equiv 0$, then the generalized mixed equilibrium problem (1.1) reduces to the following mixed variational inequality problem: finding $x \in C$ such that

$$\langle Ax, y - x \rangle + \varphi(y) \geq \varphi(x), \ \forall y \in C. \tag{1.3}$$

The solutions set of (1.3) is denoted by $VI(C, A, \varphi)$.

If $\varphi \equiv 0$, then the mixed equilibrium problem (1.2) reduces to the following equilibrium problem: finding $x \in C$ such that

$$f(x, y) \geq 0, \ \forall y \in C. \tag{1.4}$$

The solutions set of (1.4) is denoted by $EP(f)$.

If $f \equiv 0$, then the mixed equilibrium problem (1.2) reduces to the following convex minimization problem: finding $x \in C$ such that

$$\varphi(y) \geq \varphi(x), \ \forall y \in C. \tag{1.5}$$

The solutions set of (1.5) is denoted by $CMP(\varphi)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others; see for instance, [5, 11, 13, 20].

For solving the equilibrium problem, let us assume that:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
(A4) for all $x \in C$, $f(x, .)$ is convex and lower semicontinuous.

In 1953, Mann [19] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.6}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in (0,1).

However, we note that Mann's iteration process (1.6) has only weak convergence, in general; for instance, see [4, 14, 27].

Let C be a nonempty, closed and convex subset of a Banach space E and let $\{T_n\}$ be sequence of mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy the NST-condition if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n \to \infty} \|z_n - T_n z_n\| = 0$$

implies $\omega_w(z_n) \subset \bigcap_{n=1}^{\infty} F(T_n)$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$; see [3, 21, 22].

In 2008, Takahashi et al. [33] has adapted Nakajo and Takahashi [23]'s idea to modify the process (1.6) so that strong convergence is guaranteed. They proposed the following modification for nonexpansive mappings in a Hilbert space: $x_0 \in H,\ C_1 = C,\ u_1 = P_{C_1}x_0$ and

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$ and P_K is a metric projection from a Hilbert space H onto a nonempty, closed and convex subset K of H. They proved that if $\{T_n\}$ satisfies the NST-condition, then $\{u_n\}$ generated by (1.7) converges strongly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Xu [36] introduced the following iterative scheme for finding a fixed point of a nonexpansive mapping in a Banach space: $x_0 = x \in C$ and

$$\begin{cases} C_n = \overline{co}\{z \in C : \|z - Tz\| \le t_n\|x_n - Tx_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n\cap D_n}x, \quad n \ge 0, \end{cases} \tag{1.8}$$

where $\overline{co}D$ denotes the convex closure of the set D, $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$, and $\Pi_{C_n\cap D_n}$ is a generalized projection from a Banach space E onto $C_n \cap D_n$. Then, he proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to a fixed point of T.

Very recently, Kimura and Nakajo [16], by using the Mosco convergence technique, obtained strong convergence theorems in a Banach space. They also proposed the following algorithm: $x_1 = x \in C$ and

$$\begin{cases} C_n = \overline{co}\{z \in C : \|z - T_n z\| \le t_n\|x_n - T_n x_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n\cap D_n}x, \quad n \ge 1, \end{cases} \tag{1.9}$$

where $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$ as $n \to \infty$. They proved that if $\{T_n\}$ satisfies the NST-condition, then the sequence $\{x_n\}$ generated by (1.9) converges strongly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

The problem of finding a common element of the fixed points set and the solutions set of an equilibrium problem in the framework of Hilbert spaces and Banach spaces has been studied by many authors; for instance, see [8, 9, 24, 25, 26, 29, 30, 32, 35, 37] and the references therein.

Motivated and inspired by Xu [36], Kimura and Nakajo [16], we introduce a new hybrid projection algorithm for finding a common element of the solutions set of a generalized mixed equilibrium problem and the fixed points set of an infinitely countable family of nonexpansive mappings in the framework of Banach spaces.

2. Preliminaries and lemmas

Let E be a real Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if for any $x, y \in U$,

$$x \ne y \ \text{ implies } \left\|\frac{x+y}{2}\right\| < 1.$$

It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \ge \varepsilon \ \text{ implies } \left\|\frac{x+y}{2}\right\| < 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function $\delta : [0,2] \to [0,1]$ called the *modulus of convexity* of E as follows:

$$\delta(\varepsilon) = \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \ x, y \in E, \ \|x\| = \|y\| = 1, \ \|x - y\| \geq \varepsilon\Big\}.$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{\ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E; see [31] for more details.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. In a Hilbert space H, we have $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

Lemma 2.1 (*Kamimura and Takahashi* **[15]**)**.** *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty, closed and convex subset of E. The *generalized projection mapping*, introduced by Alber [1], is a mapping $\Pi_C : E \to C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}.$$

In fact, we have the following result.

Lemma 2.2 (*Alber* **[1]**)**.** *Let C be a nonempty, closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then, there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.*

The existence and uniqueness of the operator Π_C follows from the properties of the functional ϕ and strict monotonicity of the duality mapping J; for instance, see [1, 2, 10, 15, 31]. In a Hilbert space, Π_C is coincident with the metric projection.

Lemma 2.3 (*Alber* **[1]** *and Kamimura and Takahashi* **[15]**)**.** *Let C be a nonempty, closed and convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4 (*Alber* **[1]** *and Kamimura and Takahashi* **[15]**)**.** *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 2.5 (*Bruck* [**6**]). *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E. Then, there exists a strictly increasing convex continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\Big(\Big\|T\Big(\sum_{i=1}^{n}\lambda_i x_i\Big) - \sum_{i=1}^{n}\lambda_i T x_i\Big\|\Big) \le \max_{1\le j\le k\le n}(\|x_j - x_k\| - \|Tx_j - Tx_k\|)$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, ..., x_n\} \subset C$, $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset [0,1]$ with $\sum_{i=1}^{n}\lambda_i = 1$ and nonexpansive mapping T of C into E.

Lemma 2.6 (*Blum and Oettli* [**5**]). *Let C be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions* (A1)-(A4)*, and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$$

The following result can be found in [38].

Lemma 2.7 (*Zhang* [**38**]). *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. For all $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(z,y) + \langle Az, y - z\rangle + \varphi(y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge \varphi(z), \quad \forall y \in C.$$

Define a mapping $S_r : E \to 2^C$ as follows:

$$S_r(x) = \{z \in C : f(z,y) + \langle Az, y - z\rangle + \varphi(y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge \varphi(z), \quad \forall y \in C\}.$$

Then, the followings hold:
(1) S_r is single-valued;
(2) S_r is firmly nonexpansive-type mapping; [18], i.e., for all $x, y \in E$,

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \le \langle S_r x - S_r y, Jx - Jy\rangle;$$

(3) $F(S_r) = GMEP(f, A, \varphi)$;
(4) $GMEP(f, A, \varphi)$ is closed and convex.

3. Main Results

In this section, we prove the strong convergence theorem for finding a common element of the fixed points set for nonexpansive mappings and the solutions set of a generalized mixed equilibrium problem in Banach spaces.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4)*, $A : C \to E^*$ a continuous and monotone mapping, and φ a lower semicontinuous and convex function* *from C to $\mathbb{R}$. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad D_0 = C, \\ C_n = \bigcap_{i=1}^{\infty} \overline{co}\{z \in C : \|z - T_i z\| \leq t_n \|x_n - T_i x_n\|\}, \quad n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle S_{r_n} x_n - z, Jx_n - JS_{r_n} x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = \Pi_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences satisfying the conditions:
(C1) $\{t_n\} \subset (0, 1)$ *and* $\lim_{n\to\infty} t_n = 0$*;*
(C2) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} r_n > 0$*.*

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof . First, we show that the sequence $\{x_n\}$ is well-defined. It is easy to verify that $C_n \cap D_n$ is closed and convex and $F \subset C_n$ for all $n \geq 0$. Since $D_0 = C$, we also have $F \subset C_0 \cap D_0$. Suppose that $F \subset C_{k-1} \cap D_{k-1}$ for $k \geq 2$. It follows from Lemma 2.7 (2) that

$$\langle S_{r_k} x_k - S_{r_k} u, Jx_k - JS_{r_k} x_k - (Ju - JS_{r_k} u) \rangle \geq 0,$$

for all $u \in F$. This implies that

$$\langle S_{r_k} x_k - u, Jx_k - JS_{r_k} x_k \rangle \geq 0,$$

for all $u \in F$. Hence $F \subset D_k$. By the mathematical induction, we get that $F \subset C_n \cap D_n$ for each $n \geq 0$. By Lemma 2.7 (4), we know that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi)$ is nonempty, closed and convex. Then there exists a unique element $w \in F$ such that $w = \Pi_F x_0$. Since $F \subset C_{n-1} \cap D_{n-1}$ and $x_n = \Pi_{C_{n-1} \cap D_{n-1}} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(w, x_0), \quad n \geq 1. \tag{3.1}$$

Since $x_n = \Pi_{C_{n-1} \cap D_{n-1}} x_0$ and $x_{n+1} \in D_n \subset D_{n-1}$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad n \geq 1. \tag{3.2}$$

From (3.1) and (3.2) we can conclude that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists.

Next, we show that $\lim_{m,n\to\infty} \phi(x_m, x_n) = 0$. From $x_n = \Pi_{C_{n-1} \cap D_{n-1}} x_0$ and $x_m \in D_{m-1} \subset D_{n-1}$ for $m > n \geq 1$, we have by Lemma 2.4

$$\phi(x_m, x_n) + \phi(x_n, x_0) \leq \phi(x_m, x_0).$$

This implies that

$$\phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0).$$

Hence $\lim_{m,n\to\infty} \phi(x_m, x_n) = 0$. By Lemma 2.1, we obtain

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0.$$

In particular, we also have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

Thus $\{x_n\}$ is a Cauchy sequence in C. By the completeness of E and the closedness of C, we have $x_n \to v \in C$.

Next, we show that $v \in \bigcap_{i=1}^{\infty} F(T_i)$. Since $x_{n+1} \in C_n$ and $t_n > 0$, there exists $m \in \mathbb{N}$, $\{\lambda_0, \lambda_1, ..., \lambda_m\} \subset [0,1]$ and $\{y_0, y_1, ..., y_m\} \subset C$ such that

$$\sum_{j=0}^{m} \lambda_j = 1, \ \Big\|x_{n+1} - \sum_{j=0}^{m} \lambda_j y_j\Big\| < t_n, \ \text{and} \ \|y_j - T_i y_j\| \leq t_n \|x_n - T_i x_n\|$$

for each $j = 0, 1, ..., m$ and $i \in \mathbb{N}$. Put $M = \sup_{n\geq 0} \|x_n - w\|$. We note that $\|y_j - T_i y_j\| \leq t_n \|x_n - T_i x_n\| \leq 2t_n \|x_n - w\| \leq 2t_n M$ for each $j = 0, 1, ..., m$ and $i \in \mathbb{N}$. Since $\{x_n\}$ is bounded, by Lemma 2.5, we have

$$\begin{aligned}
\|x_n - T_i x_n\| \ \leq \ & \|x_n - x_{n+1}\| + \Big\|x_{n+1} - \sum_{j=0}^{m} \lambda_j y_j\Big\| + \Big\|\sum_{j=0}^{m} \lambda_j y_j - \sum_{j=0}^{m} \lambda_j T_i y_j\Big\| \\
& + \Big\|\sum_{j=0}^{m} \lambda_j T_i y_j - T_i(\sum_{j=0}^{m} \lambda_j y_j)\Big\| + \Big\|T_i(\sum_{j=0}^{m} \lambda_j y_j) - T_i x_n\Big\| \\
\leq \ & \|x_n - x_{n+1}\| + t_n + \sum_{j=0}^{m} \lambda_j \|y_j - T_i y_j\| \\
& + \gamma^{-1}\Big(\max_{0\leq j\leq k\leq m} (\|y_j - y_k\| - \|T_i y_j - T_i y_k\|)\Big) + \Big\|\sum_{j=0}^{m} \lambda_j y_j - x_n\Big\|
\end{aligned}$$

$$\begin{aligned}
\leq \ & \|x_n - x_{n+1}\| + t_n + (2t_n M) \sum_{j=0}^{m} \lambda_j \\
& + \gamma^{-1}\Big(\max_{0\leq j\leq k\leq m} (\|y_j - T_i y_j\| + \|y_k - T_i y_k\|)\Big) \\
& + \Big(\|\sum_{j=0}^{m} \lambda_j y_j - x_{n+1}\| + \|x_n - x_{n+1}\|\Big) \\
\leq \ & 2\|x_n - x_{n+1}\| + t_n + 2t_n M \\
& + \gamma^{-1}(4Mt_n) + t_n \\
= \ & 2\|x_n - x_{n+1}\| + (2 + 2M)t_n + \gamma^{-1}(4Mt_n).
\end{aligned}$$

It follows from (3.3) and (C1) that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0,$$

for all $i \in \mathbb{N}$. Thus $v \in \bigcap_{i=1}^{\infty} F(T_i)$.

Next, we show that $v \in GMEP(f, A, \varphi)$. By the construction of D_n, we see from Lemma 2.3 that $S_{r_n} x_n = \Pi_{D_{n-1}} x_n$. Since $x_{n+1} \in D_n \subset D_{n-1}$, we obtain

$$\phi(S_{r_n} x_n, x_n) \leq \phi(x_{n+1}, x_n) \to 0,$$

as $n \to \infty$. From Lemma 2.1, we have

$$\lim_{n\to\infty} \|S_{r_n} x_n - x_n\| = 0.$$

Since $x_n \to v$, we have $S_{r_n} x_n \to v$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous on the bounded set, we have

$$\lim_{n\to\infty} \|J S_{r_n} x_n - J x_n\| = 0.$$

By (C2) we also have

$$\lim_{n\to\infty} \frac{\|JS_{r_n}x_n - Jx_n\|}{r_n} = 0. \tag{3.4}$$

For each $y \in C$, we see that

$$f(S_{r_n}x_n, y) + \langle AS_{r_n}x_n, y - S_{r_n}x_n\rangle + \varphi(y) + \frac{1}{r_n}\langle y - S_{r_n}x_n, JS_{r_n}x_n - Jx_n\rangle \geq \varphi(S_{r_n}x_n).$$

By using the same argument as in the proof of [28], we can verify that

$$f(v, y) + \langle Av, y - v\rangle + \varphi(y) \geq \varphi(v), \quad \forall y \in C.$$

This shows that $v \in GMEP(f, A, \varphi)$ and hence $v \in F := \bigcap_{i=1}^{\infty} F(T_i) \cap GMEP(f, A, \varphi)$.

Finally, we show that $v = w = \Pi_F x_0$. Since $x_{n+1} = \Pi_{C_n\cap D_n}x_0$, we have

$$\langle Jx_0 - Jx_{n+1}, x_{n+1} - z\rangle \geq 0 \quad \forall z \in C_n \cap D_n.$$

Since $F \subset C_n \cap D_n$, we also have

$$\langle Jx_0 - Jx_{n+1}, x_{n+1} - z\rangle \geq 0 \quad \forall z \in F. \tag{3.5}$$

By taking limit in (3.5), we obtain that

$$\langle Jx_0 - Jv, v - z\rangle \geq 0 \quad \forall z \in F.$$

By Lemma 2.3, we can conclude that $v = \Pi_F x_0 = w$. This completes the proof. □

If we take $T_i = I$ for all $i \in \mathbb{N}$ in Theorem 3.1, then we obtain the following result.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *$A : C \to E^*$ a continuous and monotone mapping, and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that $GMEP(f, A, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad D_0 = C, \\ D_n = \{z \in D_{n-1} : \langle S_{r_n}x_n - z, Jx_n - JS_{r_n}x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = \Pi_{D_n}x_0, \quad n \geq 0. \end{cases}$$

If $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{GMEP(f,A,\varphi)}x_0$.

If we take $f \equiv 0, A \equiv 0$ and $\varphi \equiv 0$ in Theorem 3.1, we obtain the following result.

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty, closed and convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ C_n = \bigcap_{i=1}^{\infty} \overline{co}\{z \in C : \|z - T_i z\| \leq t_n\|x_n - T_i x_n\|\}, \\ x_{n+1} = \Pi_{C_n}x_0, \quad n \geq 0. \end{cases}$$

If $\{t_n\} \subset (0, 1)$ and $\lim_{n\to\infty} t_n = 0$, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 3.4. *Theorem 3.1 also can be applied to find solutions of mixed equilibrium problems, mixed variational inequality problems, convex minimization problems and so on.*

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References

[1] Ya. I. Alber, *Matric and generalized projection operators in Banach spaces: Properties and applications*, in: A.G.Kartsatos (Ed.), Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, Marcel Dekker, New York, (1996) 15–50.

[2] Ya.I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator equations in Banach spaces*, Panamer. Math. J., 4 (1994) 39–54.

[3] H. H. Bauschke and P. L. Combettes, *A weak-to-strong convergence principle for fejer-monotone methods in Hilbert spaces*, Math. Oper. Res., 26 (2001) 248–264.

[4] H. H. Bauschke, E. Matouskova and S. Reich, *Projection and proximal point methods: convergence results and counterexamples*, Nonlinear Anal., 56 (2004) 715–738.

[5] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, 63 (1994) 123–145.

[6] R. E. Bruck, *On the convex approximation property and the asymptotic behaviour of nonlinear contractions in Banach spaces*, Israel J. Math., 38 (1981) 304–314.

[7] L. -C. Ceng and J. -C. Yao, *hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math., 214 (2008) 186–201.

[8] P. Cholamjiak, *A hybrid iterative scheme for equilibrium problems*, variational inequality problems and fixed point problems in Banach spaces, Fixed Point Theory Appl., (2009) doi:10.1155/2009/719360.

[9] P. Cholamjiak and S. Suantai, *A new hybrid algorithm for variational inclusions, generalized equilibrium problems and a finite family of quasi-nonexpansive mappings*, Fixed Point Theory Appl., (2009) doi:10.1155/2009/350979.

[10] I. Cioranescu, *Geometry of Banach spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Plublishers, Dordrecht, 1990.

[11] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., 6 (2005) 117–136.

[12] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Mathematische Annalen, 142 (1961) 305–310.

[13] S. D. Flam and A. S. Antipin, *Equilibrium programming using proximal-like algorithms*, Math. Program., 78 (1997) 29–41.

[14] A. Genal and J. Lindenstrass, *An example concerning fixed points*, Israel J. Math., 22 (1975) 81–86.

[15] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim., 13 (2002) 938–945.

[16] Y. Kimura and K. Nakajo, *Some characterizations for a family of nonexpansive mappings and convergence of a generated sequence to their common fixed point*, Fixed Point Theory Appl., (2010) doi:10.1155/2010/732872.

[17] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, The American mathematical monthly, 72 (1965) 1004–1006.

[18] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces*, SIAM J. Optim., 19 (2008) 824–835.

[19] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., 4 (1953) 506–510.

[20] A. Moudafi, *Weak convergence theorems for nonexpansive mappings and equilibrium problems*, J. Nonlinear Convex Anal. Appl., 9 (2008) 37–43.

[21] K. Nakajo, K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings in Banach spaces*, J. Nonlinear Convex Anal., 8 (2007) 11–34.

[22] K. Nakajo, K. Shimoji and W. Takahashi, *Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces*, Taiwanese J. Math., 10 (2006) 339–360.

[23] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., 279 (2003) 372–379.

[24] J. -W. Peng and J. -C. Yao, *A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems*, Taiwanese J. Math., 12 (2008) 1401–1432.

[25] J. -W. Peng and J. -C. Yao, *Two extragradient methods for generalized mixed equilibrium problems*, nonexpansive mappings and monotone mappings, Computer and Mathematics with Applications, (2009) doi:10.1016/j.camwa.2009.07.040.
[26] X. Qin, Y. J. Cho and S. M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces*, J. Comput. Appl. Math., 225 (2009) 20–30.
[27] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., 67 (1979) 274–276.
[28] S. Saewan and P. Kumam, *A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems*, Abstract and Applied Analysis, vol. 2010, Article ID 123027, 31 pages.
[29] S. Saewan and P. Kumam, *Modified hybrid block iterative algorithm for convex feasibility problems and generalized equilibrium problems for uniformly quasi-ϕ-asymptotically nonexpansive mappings*, Abstract and Applied Analysis, vol. 2010, Article ID 357120, 22 pages.
[30] A. Tada and W. Takahashi, *Weak and strong convergence theorems for nonexpansive mappings and an equilibrium problem*, J. Optim. Theory Appl., 133 (2007) 359–370.
[31] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, 2000.
[32] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mappings in a Hilbert space*, J. Nonlinear Anal., 69 (2008) 1025–1033.
[33] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., 341 (2008) 276–286.
[34] W. Takahashi and K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, J. Nonlinear Anal., 70 (2009) 45–57.
[35] W. Takahashi and K. Zembayashi, *Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings*, Fixed Point Theory Appl., (2008) doi:10.1155/2008/528476.
[36] H. -K. Xu, *Strong convergence of approximating fixed point sequences for nonexpansive mappings*, Bull. Austral. Math. Soc., 74 (2006) 143–151.
[37] Y. Yao, M. A. Noor and Y. C. Liou, *On iterative methods for equilibrium problems*, J. Nonlinear Anal., 70 (2009) 497–509.
[38] S. Zhang, *Generalized mixed equilibrium problem in Banach spaces*, Appl. Math. Mech., 30 (2009) 1105–1112.