Int. J. Nonlinear Anal. Appl. 11 (2020) No. 2, 305-316
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/IJNAA.2020.4480

Approximate Quartic Lie $*$-Derivations with perturbing terms

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The aim of this paper is to investigate stable approximation of almost quartic Lie $*$-derivations associated with approximate quartic homogeneous mappings by quartic Lie $*$-derivations on ρ-complete convex modular algebras by using Δ_2-condition via convex modular ρ.

Keywords: ρ-complete convex modular algebras; quartic Lie $*$-derivations; Δ_2-condition.
2010 MSC: 39B72, 46H30, 16W25

1. Introduction

Ulam [20] raised the question concerning the stability of group homomorphisms: *Let G be a group and let G' be a metric group with the metric $d(\cdot,\ \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \to G'$ satisfies the inequality*

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \to G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$? Hyers [6] had answered affirmatively the question of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1] and for approximately linear mappings was presented by Rassias [17]. Since then, many interesting results of the stability problems to a number of functional equations have

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Received: February 2019 *Revised:* June 2020

been investigated. The reader is referred to the references [2, 3, 4, 15, 19] for many information of stability problem with a large variety of applications.

Now, we recall some basic definitions and remarks of modular spaces with modular functionals, which are primitive notions corresponding to norms and metrics, as in the followings [10, 21]. The concept of modular spaces was first introduced by Nakano [13], and then by Musielak and Orlicz [12]. As for the stability theory in modular spaces, Sadeghi [18] has established generalized stability via the fixed point method of a generalized Jensen functional equation in convex modular spaces. The authors [21] have presented the generalized stability of quadratic functional equations via the extensive studies of fixed point theory in the framework of modular spaces whose modular is convex, lower semicontinuous but does not satisfy any relatives of Δ_2-condition (refer to [9, 25]). Lately, the stability problems of various functional equations in modular spaces have been intensively investigated (see for example, [10, 21, 22]).

Definition 1.1. *Let χ be a linear space.*

(1) *A function $\rho : \chi \to [0, \infty]$ is called a convex modular if for arbitrary $x, y \in \chi$,*

(m1) *$\rho(x) = 0$ if and only if $x = 0$,*

(m2) *$\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,*

(m3) *$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for every scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,*

acting on real linear space χ. In this case, we say that ρ is a convex modular on real linear space χ.

(2) *Alternatively, if* (m3) *is replaced by*

(m3)' *$\rho(\alpha x + \beta y) \leq |\alpha|\rho(x) + |\beta|\rho(y)$ for every scalars $\alpha, \beta \in \mathbb{C}$, where $|\alpha| + |\beta| = 1$,*

acting on complex linear space χ, then it is said that ρ is a convex modular on complex linear space χ. As a matter of fact, it is well known that a modular ρ defines a corresponding modular space, i.e., the linear space χ_ρ given by

$$\chi_\rho = \{x \in \chi : \rho(\lambda x) \to 0 \quad \textit{as} \quad \lambda \to 0\}.$$

Now let ρ be a modular on χ_ρ. Then we observe that $\rho(tx)$ is an increasing function in $t \geq 0$ for each fixed $x \in \chi$, that is, $\rho(ax) \leq \rho(bx)$ whenever $0 \leq a < b$. In particular, if ρ is a convex modular on χ, then $\rho(\alpha x) \leq \alpha\rho(x)$ for all $x \in \chi$ and for all α with $0 \leq \alpha \leq 1$. Moreover, we see that $\rho(\alpha x) \leq |\alpha|\rho(x)$ for all $x \in \chi$ and all α with $|\alpha| \leq 1$.

Remark 1.2. (1) *In general, we note that $\rho\big(\sum_{i=1}^n \alpha_i x_i\big) \leq \sum_{i=1}^n \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \geq 0$ $(i = 1, \cdots, n)$ whenever $0 < \alpha := \sum_{i=1}^n \alpha_i \leq 1$ (Cf. [10]).* (2) *Consequently, we lead to $\rho\big(\sum_{i=1}^n \alpha_i x_i\big) \leq \sum_{i=1}^n |\alpha_i|\rho(x_i)$ for all $x_i \in \chi$ and all $\alpha_i \in \mathbb{C}$ whenever $0 < \alpha := \sum_{i=1}^n |\alpha_i| \leq 1$.*

Definition 1.3. *Let χ_ρ be a modular space and let $\{x_n\}$ be a sequence in χ_ρ. Then,*

(1) *$\{x_n\}$ is ρ-convergent to $x \in \chi_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$;*

(2) *$\{x_n\}$ is called ρ-Cauchy in χ_ρ if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$;*

(3) *A subset K of χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence in K is ρ-convergent to an element in K.*

It is said that χ_ρ is called a *convex modular* $*$*-algebra* if the fundamental space χ is a $*$-algebra with convex modular ρ subject to $\rho(ab) \le \rho(a)\rho(b)$ and $\rho(c^*) = \rho(c)$ for all $a, b, c \in \chi$. We say that a linear mapping d is a *Lie* $*$*-derivation* if $d([x, y]) = [d(x), y] + [x, d(y)]$ and $d(z^*) = d(z)^*$ for all x, y, z, where $[a, b] = ab - ba$. Similarly, a quartic mapping d is said to be *quartic homogeneous* if $d(\lambda x) = \lambda^4 d(x)$ for all x and scalars λ, and a quartic homogeneous mapping d is called a *quartic Lie* $*$*-derivation* if $d([x, y]) = [d(x), y^4] + [x^4, d(y)]$ and $d(z^*) = d(z)^*$ for all x, y, z.

From now on, χ_ρ will denote a ρ-complete convex modular $*$-algebra. It is said that the modular ρ has the *Fatou property* if and only if $\rho(x) \le \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x. A modular function ρ is said to satisfy the Δ_2*-condition* if there exists $\kappa > 0$ such that $\rho(2x) \le \kappa\rho(x)$ for all $x \in \chi_\rho$.

In [16], Rassias has studied the stability problem of the quartic functional equation

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y) + 24f(y), \tag{1.1}$$

of which the general solution is called a quartic mapping. The stability theorems of $*$-derivations on Banach $*$-algebras and C^*-algebras, respectively, can be found in the references [7, 11, 14, 23, 24]. Concerning the stability theory of approximate quartic Lie $*$-derivations in ρ-complete convex modular algebras, we first investigate stable approximation of almost quartic Lie $*$-derivations associated with the following functional equation

$$f(3x - y) + f(3x - 2y) + f(y) = 9f(2x - y) + 9f(x - y) + 9f(x) \tag{1.2}$$

in ρ-complete convex modular algebras without using both Fatou property and Δ_2-condition, and then alternatively present generalized stability result of the equation (1.2) associated with almost quartic Lie $*$-derivations using necessarily Δ_2-condition but not using the Fatou property in ρ-complete convex modular algebras.

2. Approximate quartic Lie ∗-derivations

We first note that the equation (1.2) is equivalent to the original quartic functional equation (1.1). In this case, every solution of equation (1.2) is a quartic mapping.

For convenience, we denote the difference operators for quartic equation (1.2) and quartic derivation, respectively, as follows:

$$\begin{aligned} QE_f^\lambda(x, y) &:= f(3\lambda x - \lambda y) + f(3\lambda x - 2\lambda y) + f(\lambda y) \\ &\qquad -9\lambda^4 f(2x - y) - 9\lambda^4 f(x - y) - 9\lambda^4 f(x), \\ QD_f(x, y) &:= f([x, y]) - [f(x), y^4] - [x^4, f(y)] \end{aligned}$$

for all x, y and $\lambda \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, which act as perturbing terms of quartic Lie $*$-derivations.

Now we present a generalized stability of the equation (1.2) via direct method associated with approximate quartic Lie $*$-derivations in ρ-complete modular algebras without using both Fatou property and Δ_2-condition.

Theorem 2.1. *Suppose that a mapping* $f : \chi_\rho \to \chi_\rho$ *with* $f(0) = 0$ *satisfies*

$$\rho(QE_f^\lambda(x, y) + f(z^*) - f(z)^*) \le \phi_1(x, y, z), \tag{2.1}$$

$$\rho(QD_f(x, y)) \le \phi_2(x, y) \tag{2.2}$$

and $\phi_1 : \chi_\rho^3 \to [0,\infty)$ *and* $\phi_2 : \chi_\rho^2 \to [0,\infty)$ *are mappings such that*

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} \frac{\phi_1(2^j x, 2^j y, , 2^j z)}{2^{4j}} < \infty, \ \lim_{n\to\infty} \frac{\phi_2(2^n x, 2^n y)}{4^{4n}} = 0 \tag{2.3}$$

for all $x,y,z \in \chi_\rho$ *and* $\lambda \in \mathbb{T}$. *Furthermore, if for each* $x \in \chi_\rho$ *the mapping* $r \to f(rx)$ *from* $\mathbb{R}$ *to* χ_ρ *is continuous, then we can find a unique quartic Lie* $*$*-derivation* $d_1 : \chi_\rho \to \chi_\rho$ *near* f *which satisfies the equation* $QE^\lambda_{d_1}(x,y) = 0$, $QD_{d_1}(x,y) = 0$ *and*

$$\rho(f(x) - d_1(x)) \le \frac{1}{16}\Phi(x,x,0) \tag{2.4}$$

for all $x,y \in \chi_\rho$ *and all* $\lambda \in \mathbb{C}$.

Proof . Putting $y := x$ and $z := 0$ in (2.1) with $\lambda = 1$, we obtain

$$\rho(QE^1_f(x,x)) = \rho(f(2x) - 16f(x)) \le \phi_1(x,x,0), \tag{2.5}$$

which yields

$$\rho\Big(f(x) - \frac{f(2x)}{16}\Big) \le \frac{1}{16}\rho(f(2x) - 16f(x)) \le \frac{1}{16}\phi_1(x,x,0)$$

for all $x \in \chi_\rho$. Since $\sum_{j=0}^{n-1} \frac{1}{2^{4(j+1)}} \le 1$, we prove the following functional inequality

$$\begin{aligned} \rho\Big(f(x) - \frac{f(2^n x)}{2^{4n}}\Big) &= \rho\Big[\sum_{j=0}^{n-1} \big(\frac{f(2^j x)}{2^{4j}} - \frac{f(2^{j+1}x)}{2^{4(j+1)}}\big)\Big] \\ &= \rho\Big[\sum_{j=0}^{n-1} \frac{1}{2^{4(j+1)}}\big(2^4 f(2^j x) - f(2^{j+1}x)\big)\Big] \\ &\le \sum_{j=0}^{n-1} \frac{1}{2^{4(j+1)}}\rho\Big(2^4 f(2^j x) - f(2^{j+1}x)\Big) \\ &\le \frac{1}{2^4}\sum_{j=0}^{n-1} \frac{\phi_1(2^j x, 2^j x, 0)}{2^{4j}} \end{aligned} \tag{2.6}$$

for all $x \in \chi_\rho$ by using the property of convex modular ρ.

Now, replacing x by $2^m x$ in (2.6), we have

$$\rho\Big(\frac{f(2^m x)}{2^{4m}} - \frac{f(2^{m+n}x)}{2^{4(m+n)}}\Big) \le \frac{1}{16}\sum_{j=m}^{m+n-1} \frac{\phi_1(2^j x, 2^j x, 0)}{2^{4j}} \tag{2.7}$$

which converges to zero as $m \to \infty$ by the assumption (2.3). Thus the above inequality implies that the sequence $\{\frac{f(2^n x)}{2^{4n}}\}$ is a ρ-Cauchy for all $x \in \chi_\rho$ and so it is convergent in χ_ρ since the space χ_ρ is ρ-complete. Thus, we may define a mapping $d_1 : \chi_\rho \to \chi_\rho$ as

$$d_1(x) := \rho - \lim_{n\to\infty} \frac{f(2^n x)}{2^{4n}} \Longleftrightarrow \lim_{n\to\infty} \rho\Big(\frac{f(2^n x)}{2^{4n}} - d_1(x)\Big) = 0,$$

for all $x \in \chi_\rho$.

Now, we proclaim d_1 is a quartic mapping satisfying the equation (1.2) and the approximation (2.4). In fact, if we put $(x, y, z) := (2^n x, 2^n y, 0)$ in (2.1), and then divide the resulting inequality by 2^{4n}, one obtains

$$\rho\Big(\frac{QE_f^\lambda(2^n x, 2^n y)}{2^{4n}}\Big) \leq \frac{\rho(QE_f^\lambda(2^n x, 2^n y))}{2^{4n}} \leq \frac{\phi_1(2^n x, 2^n y, 0)}{2^{4n}},$$

which implies

$$\begin{aligned}\rho\Big(\frac{QE_f^\lambda(2^n x, 2^n y)}{2^{4n}}\Big) &\leq \frac{\rho(QE_f^\lambda(2^n x, 2^n y))}{2^{4n}}\\ &\leq \frac{\phi_1(2^n x, 2^n y, 0)}{2^{4n}}\\ &\to 0\end{aligned}$$

for all $x, y \in \chi_\rho$ and all $\lambda \in \mathbb{T}$. Thus, noting $\frac{27|\lambda^4|+4}{31} \leq 1$, we figure out by use of Remark 1.2

$$\begin{aligned}&\rho(\frac{1}{31}QE_{d_1}^\lambda(x, y))\\ &= \rho\Big(\frac{1}{31}QE_{d_1}^\lambda(x, y) - \frac{QE_f^\lambda(2^n x, 2^n y)}{31\cdot 2^{4n}} + \frac{QE_f^\lambda(2^n x, 2^n y)}{31\cdot 2^{4n}}\Big)\\ &\leq \frac{1}{31}\rho\Big(d_1\big(3\lambda x - \lambda y\big) - \frac{f\big(2^n(3\lambda x - \lambda y)\big)}{2^{4n}}\Big) + \frac{1}{31}\rho\Big(d_1(3\lambda x - 2\lambda y) - \frac{f\big(2^n(3\lambda x - 2\lambda y)\big)}{2^{4n}}\Big)\\ &\quad + \frac{9\lambda^4}{31}\rho\Big(\frac{f\big(2^n(2x - y)\big)}{2^{2n}} - d_1\big(2x - y\big)\Big) + \frac{9\lambda^4}{31}\rho\Big(\frac{f(2^n(x - y))}{2^{4n}} - d_1(x - y)\Big)\\ &\quad + \frac{1}{31}\rho\Big(d_1(\lambda y) - \frac{f(2^n\lambda y)}{2^{4n}}\Big) + \frac{9\lambda^4}{31}\rho\Big(\frac{f(2^n x)}{2^{4n}} - d_1(x)\Big) + \frac{1}{31}\rho\Big(\frac{QE_f^\lambda(2^n x, 2^n y)}{2^{4n}}\Big)\end{aligned}$$

for all $x, y \in \chi_\rho$ and all positive integers n. Taking the limit as $n \to \infty$, one obtains $\rho(\frac{1}{9}QE_{d_1}^\lambda(x, y)) = 0$, and so

$$QE_{d_1}^\lambda(x, y) = 0 \tag{2.8}$$

for all $x, y \in \chi_\rho$ and all $\lambda \in \mathbb{T}$. Hence d_1 satisfies the equation (1.2) for the case of $\lambda = 1$, and so it is quartic. Next, since $\sum_{i=0}^n \frac{1}{2^{4(i+1)}} + \frac{1}{2^4} \leq 1$ for all $n \in \mathbb{N}$, it follows from (2.5) and Remark 1.2 that

$$\begin{aligned}&\rho(f(x) - d_1(x))\\ &= \rho\left(\sum_{i=0}^n \frac{1}{2^{4(i+1)}}\Big(2^4 f(2^i x) - f(2^{i+1}x)\Big) + \frac{f(2^{n+1}x)}{2^{4(n+1)}} - \frac{d_1(2x)}{2^4}\right)\\ &\leq \sum_{i=0}^n \frac{1}{2^{4(i+1)}}\rho\Big(QE_f^\lambda(2^i x, 2^i x)\Big) + \frac{1}{2^4}\rho\Big(\frac{f(2^{n+1}x)}{2^{4n}} - d_1(2x)\Big)\\ &\leq \sum_{i=0}^n \frac{1}{2^{4(i+1)}}\phi_1(2^i x, 2^i x, 0) + \frac{1}{2^4}\rho\Big(\frac{f(2^n\cdot 2x)}{2^{4n}} - d_1(2x)\Big),\end{aligned}$$

without applying Fatou property of the modular ρ for all $x \in \chi_\rho$ and all $n \in \mathbb{N}$, from which we obtain the approximation (2.4) of f by the quartic mapping d_1 by taking $n \to \infty$ in the last inequality.

On the other hand, we claim that d_1 is a quartic Lie $*$-derivation. By (2.8), we have $QE_{d_1}^\lambda(x, x) = 0$ which yields $d_1(\lambda x) = \lambda^4 d_1(x)$ for all $x \in \chi_\rho$ and $\lambda \in \mathbb{T}$. By the assumption that the mapping $r \to$

$f(rx)$ is continuous, it follows from the same argument as in the paper [8, 17] that $d_1(rx) = r^4 d_1(x)$ for all $x \in \chi_\rho$ and $r \in \mathbb{R}$. Thus, for any nonzero $\lambda \in \mathbb{C}$,

$$\begin{aligned} d_1(\lambda x) &= d_1\left(2\frac{\lambda}{|\lambda|}\frac{|\lambda|}{2}x\right) = 2^4\left(\frac{\lambda}{|\lambda|}\right)^4 d_1\left(\frac{|\lambda|}{2}x\right) \\ &= 2^4\left(\frac{\lambda}{|\lambda|}\right)^4\left(\frac{|\lambda|}{2}\right)^4 d_1(x) = \lambda^4 d_1(x) \end{aligned}$$

for all $x \in \chi_\rho$, which concludes that d_1 is quartic homogeneous over $\mathbb{C}$. In addition, in view of the inequality in (2.2) and the second condition in (2.3), we arrive at

$$\begin{aligned} &\rho(\frac{1}{4}QD_{d_1}(x,y)) \\ &= \rho\Big(\frac{1}{4}QD_{d_1}(x,y) - \frac{QD_f(2^n x, 2^n y)}{4\cdot 4^{4n}} + \frac{QD_f(2^n x, 2^n y)}{4\cdot 4^{4n}}\Big) \\ &\le \frac{1}{4}\rho\Big(d_1([x,y]) - \frac{f(2^{2n}[x,y])}{4^{4n}}\Big) + \frac{1}{4}\rho\Big(\frac{[x^4, f(2^n y)]}{4^{2n}} - [x^4, d_1(y)]\Big) \\ &\quad + \frac{1}{4}\rho\Big(\frac{[f(2^n x), y^4]}{4^{2n}} - [d_1(x), y^4]\Big) + \frac{1}{4\cdot 4^{4n}}\rho\Big(QD_f(2^n x, 2^n y)\Big) \end{aligned}$$

for all $x, y \in \chi_\rho$, which tends to zero as n tends to ∞. Therefore, one obtains $\rho(\frac{1}{4}QD_{d_1}(x,y)) = 0$, and so d_1 is a quartic Lie derivation. In addition, we get the following inequality

$$\begin{aligned} \rho\Big(\frac{1}{3}\Big(d_1(z^*) - d_1(z)^*\Big)\Big) &\le \frac{1}{3}\rho\Big(d_1(z^*) - \frac{f(2^n z^*)}{2^{4n}}\Big) \\ &\quad + \frac{1}{3}\rho\Big(\frac{f(2^n z)^*}{2^{4n}} - d_1(z)^*\Big) + \frac{1}{3}\rho\Big(\frac{f(2^n z^*)}{2^{4n}} - \frac{f(2^n z)^*}{2^{4n}}\Big) \\ &\le \frac{1}{3}\rho\Big(d_1(z^*) - \frac{f(2^n z^*)}{2^{4n}}\Big) \\ &\quad + \frac{1}{3}\rho\Big(\frac{f(2^n z)^*}{2^{4n}} - d_1(z)^*\Big) + \frac{\phi_1(0,0,2^n z)}{3\cdot 2^{4n}} \end{aligned}$$

for all vector z. Taking $n \to \infty$, one concludes d_1 is a quartic Lie $*$-derivation.

Finally, applying the same argument as in the proof of Theorem [10], we prove the uniqueness of d_1 satisfying the approximation (2.4) near f.

Therefore, one concludes that the mapping d_1 is a unique quartic Lie $*$-derivation near f satisfying the approximation (2.4) in the modular algebra χ_ρ. □

As a result, we obtain a stability theorem of quartic Lie $*$-derivations by quartically contractive conditions of control functions ϕ_i for perturbing terms QE_f^λ and QD_f.

Theorem 2.2. *Suppose there exist two functions $\phi_1 : \chi_\rho^3 \to [0,\infty)$ and $\phi_2 : \chi_\rho^2 \to [0,\infty)$ and two constants l_i with $0 < l_i < 1$ $(i = 1,2)$ for which a mapping $f : \chi_\rho \to \chi_\rho$ with $f(0) = 0$ satisfies*

$$\begin{aligned} \rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) &\le \phi_1(x,y,z),\ \phi_1(2x,2y,2z) \le 2^4 l_1 \phi_1(x,y,z), \\ \rho(QD_f(x,y)) &\le \phi_2(x,y), \quad \phi_2(2x,2y) \le 4^4 l_2 \phi_2(x,y) \end{aligned}$$

for all $x, y, z \in \chi_\rho$ and all $\lambda \in \mathbb{T}$. Moreover, if for each $x \in \chi_\rho$ the mapping $r \to f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then we can find a unique quartic Lie $*$-derivation $d_1 : \chi_\rho \to \chi_\rho$ near f which satisfies the equation $QE_{d_1}^\lambda(x,y) = 0$, $QD_{d_1}(x,y) = 0$ and

$$\rho(f(x) - d_1(x)) \le \frac{1}{2^4(1-l_1)}\phi_1(x,x,0) \tag{2.9}$$

for all $x, y \in \chi_\rho$ and all $\lambda \in \mathbb{C}$.

Proof . In view of quartically contractive conditions for control functions ϕ_1 and ϕ_2, one leads to $\phi_1(2^n x, 2^n y, 2^n z) \leq (2^4 l_1)^n \phi_1(x,y,z)$ and $\phi_2(2^n x, 2^n y) \leq (4^4 l_2)^n \phi_2(x,y)$ for all $x,y,z \in \chi_\rho$ and all $\lambda \in \mathbb{T}$. Hence, applying Theorem 2.1 to the theorem, we obtain the desired estimation. □

We recall that if the modular ρ satisfies the Δ_2-condition, then $\kappa \geq 1$ for nontrivial modular ρ, and $\kappa \geq 2$ for nontrivial convex modular ρ. See references [10, 18, 21].

Now in here, we are going to investigate alternatively generalized stability of the equation (1.2) associated with approximate quartic Lie $*$-derivations via direct method using necessarily Δ_2-condition but not using the Fatou property in ρ-complete convex modular algebras.

Theorem 2.3. *Let χ_ρ be a ρ-complete convex modular $*$-algebra with Δ_2-condition. Suppose there exist two functions $\varphi_1 : \chi_\rho^3 \to [0,\infty)$ and $\varphi_2 : \chi_\rho^2 \to [0,\infty)$ for which a mapping $f : \chi_\rho \to \chi_\rho$ satisfies*

$$\rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) \leq \varphi_1(x,y,z), \tag{2.10}$$

$$\sum_{j=1}^{\infty} \frac{\kappa^{5j}}{2^j}\varphi_1(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) := \Psi(x,y,z) < \infty, \tag{2.11}$$

$$\rho(QD_f(x,y)) \leq \varphi_2(x,y), \tag{2.12}$$

$$\lim_{n\to\infty} \kappa^{8n}\varphi_2(2^{-n}x, 2^{-n}y) = 0 \tag{2.13}$$

for all $x,y,z \in \chi_\rho$ and all $\lambda \in \mathbb{T}$. Then there exists a unique quartic Lie $*$-derivation $d_2 : \chi_\rho \to \chi_\rho$ which satisfies the equation $QE_{d_2}^\lambda(x,y) = 0$, $QD_{d_2}(x,y) = 0$ and

$$\rho(f(x) - d_2(x)) \leq \frac{1}{2\kappa^3}\Psi(x,x,0) \tag{2.14}$$

for all $x,y \in \chi_\rho$ and all $\lambda \in \mathbb{C}$.

Proof . First, we remark that since $\sum_{j=1}^{\infty} \frac{\kappa^{5j}}{2^j}\varphi_1(0,0,0) = \Psi(0,0,0) < \infty$ and $\rho(QE_f^1(0,0)) \leq \varphi_1(0,0,0)$, we lead to $\varphi_1(0,0,0) = 0$, $QE_f^1(0,0) = 0$ and so $f(0) = 0$. Thus, it follows from (2.5) that

$$\rho(f(x) - 16f(\frac{x}{2})) \leq \varphi_1(\frac{x}{2}, \frac{x}{2}, 0) \leq \frac{\kappa}{2}\varphi_1(\frac{x}{2}, \frac{x}{2}, 0)$$

for all $x \in \chi_\rho$. Thus, one obtains the following inequality by the convexity of the modular ρ and Δ_2-condition

$$\begin{aligned} \rho(f(x) - 16^2 f(\frac{x}{2^2})) &\leq \frac{1}{2}\rho\Big(2f(x) - 2\cdot 16 f(\frac{x}{2})\Big) + \frac{1}{2^2}\rho\Big(2^2\cdot 16 f(\frac{x}{2}) - 2^2\cdot 16^2 f(\frac{x}{2^2})\Big) \\ &\leq \frac{\kappa}{2}\varphi_1\Big(\frac{x}{2}, \frac{x}{2}, 0\Big) + \frac{\kappa^6}{2^2}\varphi_1\Big(\frac{x}{2^2}, \frac{x}{2^2}, 0\Big) \end{aligned}$$

for all $x \in \chi_\rho$. Then using the repeating process for any $n \geq 1$, we prove the following functional inequality

$$\rho(f(x) - 16^n f(\frac{x}{2^n})) \leq \frac{1}{\kappa^4}\sum_{j=1}^{n} \frac{\kappa^{5j}}{2^j}\varphi_1\Big(\frac{x}{2^j}, \frac{x}{2^j}, 0\Big) \tag{2.15}$$

for all $x \in \chi_\rho$. In fact, it is true for $j = 1$. Assume that the inequality (2.15) holds true for n. Then, using the convexity of modular ρ, we deduce

$$\begin{aligned}
&\rho(f(x) - 16^{n+1}f(\frac{x}{2^{n+1}}))\\
&= \rho\Big(\frac{1}{2}\{2f(x) - 2\cdot 16f(\frac{x}{2})\} + \frac{1}{2}\{2\cdot 16f(\frac{x}{2}) - 2\cdot 16^{n+1}f(\frac{x}{2^{n+1}})\}\Big)\\
&\le \frac{\kappa}{2}\rho\Big(f(x) - 16f(\frac{x}{2})\Big) + \frac{\kappa^5}{2}\rho\Big(f(\frac{x}{2}) - 16^n f(\frac{x}{2^{n+1}})\Big)\\
&\le \frac{\kappa}{2}\varphi_1\Big(\frac{x}{2},\frac{x}{2},0\Big) + \frac{\kappa^5}{2}\cdot\frac{1}{\kappa^4}\sum_{j=1}^{n}\frac{\kappa^{5j}}{2^j}\varphi_1\Big(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0\Big)\\
&= \frac{\kappa}{2}\varphi_1\Big(\frac{x}{2},\frac{x}{2},0\Big) + \frac{1}{\kappa^4}\sum_{j=1}^{n}\frac{\kappa^{5(j+1)}}{2^{j+1}}\varphi_1\Big(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0\Big)\\
&= \frac{1}{\kappa^4}\sum_{j=1}^{n+1}\frac{\kappa^{5j}}{2^j}\varphi_1\Big(\frac{x}{2^j},\frac{x}{2^j},0\Big),
\end{aligned}$$

which proves (2.15) for $n+1$. Now, replacing x by $2^{-m}x$ in (2.15), we have

$$\begin{aligned}
\rho\Big(16^m f(\frac{x}{2^m}) - 16^{m+n}f(\frac{x}{2^{m+n}})\Big) &\le \kappa^{4m}\rho\Big(f(\frac{x}{2^m}) - 16^n f(\frac{x}{2^{m+n}})\Big)\\
&\le \frac{\kappa^{4m}}{\kappa^4}\sum_{j=1}^{n}\frac{\kappa^{5j}}{2^j}\varphi_1\Big(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}},0\Big)\\
&\le \frac{\kappa^{4m}}{\kappa^4}\sum_{j=1}^{n}\frac{\kappa^{5j}}{2^j}\varphi_1\Big(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}},0\Big)\cdot\frac{\kappa^m}{2^m}\\
&= \frac{1}{\kappa^4}\sum_{j=m+1}^{m+n}\frac{\kappa^{5j}}{2^j}\varphi_1\Big(\frac{x}{2^j},\frac{x}{2^j},0\Big),
\end{aligned}$$

which converges to zero as $m \to \infty$ by the assumption (2.11). Thus, the sequence $\{16^n f(\frac{x}{2^n})\}$ is a ρ-Cauchy for all $x \in \chi_\rho$ and so it is ρ-convergent in χ_ρ since the space χ_ρ is ρ-complete. Hence, one may define a mapping $d_2 : \chi_\rho \to \chi_\rho$ as

$$d_2(x) := \rho - \lim_{n\to\infty} 16^n f(\frac{x}{2^n}) \iff \lim_{n\to\infty}\rho\Big(16^n f(\frac{x}{2^n}) - d_2(x)\Big) = 0,$$

for all $x \in \chi_\rho$.

Now, we prove the mapping d_2 satisfies the equation (1.2). Letting $z := 0$ and setting $(x, y) := (2^{-n}x, 2^{-n}y)$ in (2.10), and then multiplying the resulting inequality by 2^{4n}, we get

$$\begin{aligned}
\rho(2^{4n}QE_f^\lambda(2^{-n}x, 2^{-n}y)) &\le \kappa^{4n}\varphi_1(2^{-n}x, 2^{-n}y, 0)\\
&\le \kappa^{4n}\varphi_1(2^{-n}x, 2^{-n}y, 0)\cdot\frac{\kappa^n}{2^n}\\
&= \frac{\kappa^{5n}}{2^n}\varphi_1(2^{-n}x, 2^{-n}y, 0),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y \in \chi_\rho$. Thus, it follows from Remark 1.2 that

$$\begin{aligned}
&\rho(\frac{1}{31}QE^{\lambda}_{d_2}(x,y)) \\
&= \rho\Big(\frac{1}{31}QE^{\lambda}_{d_2}(x,y) - \frac{1}{31}2^{4n}QE^{\lambda}_{f}(\frac{x}{2^n},\frac{y}{2^n}) + \frac{1}{31}2^{4n}QE^{\lambda}_{f}(\frac{x}{2^n},\frac{y}{2^n})\Big) \\
&\leq \frac{1}{31}\rho\Big(d_2(3\lambda x - \lambda y) - 2^{4n}f(\frac{3\lambda x - \lambda y}{2^n})\Big) + \frac{1}{31}\rho\Big(d_2(3\lambda x - 2\lambda y) - 2^{4n}f(\frac{3\lambda x - 2\lambda y}{2^n})\Big) \\
&\quad + \frac{9\lambda^4}{31}\rho\Big(2^{4n}f(\frac{2x-y}{2^n}) - d_2(2x-y)\Big) + \frac{9\lambda^4}{31}\rho\Big(2^{4n}f(\frac{x-y}{2^n}) - d_2(x-y)\Big) \\
&\quad + \frac{1}{31}\rho\Big(d_2(\lambda y) - 2^{4n}f(\frac{\lambda y}{2^n})\Big) + \frac{9\lambda^4}{31}\rho\Big(2^{4n}f(\frac{x}{2^n}) - d_2(x)\Big) + \frac{1}{31}\rho\Big(2^{4n}QE^{\lambda}_{f}(\frac{x}{2^n},\frac{y}{2^n})\Big)
\end{aligned}$$

for all $x, y \in \chi_\rho$ and all positive integers n. Taking the limit as $n \to \infty$, one obtains

$$QE^{\lambda}_{d_2}(x,y) = 0$$

for all $x, y \in \chi_\rho$ and all $\lambda \in \mathbb{T}$. Thus, for any nonzero $\lambda \in \mathbb{C}$ we deduce the identity $d_2(\lambda x) = \lambda^4 d_2(x)$ for all $x \in \chi_\rho$ and all $\lambda \in \mathbb{C}$, which concludes that d_2 is quartic homogeneous.

Now, we are going to prove that d_2 is a quartic Lie $*$-derivation. First, it is easy to see that the mapping d_2 is quartic homogeneous by the same reasoning as in Theorem 2.1. From the last inequality (2.12) and the last condition (2.13), it follows that

$$\begin{aligned}
&\rho\Big(\frac{1}{4}QD_{d_2}(x,y)\Big) \\
&= \rho\Big(\frac{1}{4}QD_{d_2}(x,y) - 16^{2n}\frac{QD_f(2^{-n}x, 2^{-n}y)}{4} + 16^{2n}\frac{QD_f(2^{-n}x, 2^{-n}y)}{4}\Big) \\
&\leq \frac{1}{4}\rho\Big(d_2([x,y]) - 16^{2n}f(2^{-2n}[x,y])\Big) + \frac{1}{4}\rho\Big(16^n[x^4, f(2^{-n}y)] - [x^4, d_2(y)]\Big) \\
&\quad + \frac{1}{4}\rho\Big(16^n[f(2^{-n}x), y^4] - [d_2(x), y^4]\Big) + \frac{1}{4}\rho\Big(16^{2n}QD_f(2^{-n}x, 2^{-n}y)\Big) \\
&\leq \frac{1}{4}\rho\Big(d_2([x,y]) - 16^{2n}f(2^{-2n}[x,y])\Big) + \frac{1}{4}\rho\Big([x^4, 16^n f(2^{-n}y) - d_2(y)]\Big) \\
&\quad + \frac{1}{4}\rho\Big([16^n f(2^{-n}x) - d_2(x), y^4]\Big) + \frac{\kappa^{8n}}{4}\varphi_2\Big(2^{-n}x, 2^{-n}y\Big)
\end{aligned}$$

for all $x, y \in \chi_\rho$, from which $QD_{d_2}(x,y) = 0$ by taking $n \to \infty$, and so d_2 is a quartic Lie derivation. In addition, it follows from the definition of d_2 that the following inequality

$$\begin{aligned}
\rho\Big(\frac{1}{3}\Big(d_2(z^*) - d_2(z)^*\Big)\Big) &\leq \frac{1}{3}\rho\Big(d_2(z^*) - 16^n f\Big(\frac{z^*}{2^n}\Big)\Big) \\
&\quad + \frac{1}{3}\rho\Big(16^n f\Big(\frac{z}{2^n}\Big)^* - d_2(z)^*\Big) + \frac{1}{3}\rho\Big(16^n f\Big(\frac{z^*}{2^n}\Big) - 16^n f\Big(\frac{z}{2^n}\Big)^*\Big) \\
&\leq \frac{1}{3}\rho\Big(d_2(z^*) - 16^n f\Big(\frac{z^*}{2^n}\Big)\Big) \\
&\quad + \frac{1}{3}\rho\Big(16^n f\Big(\frac{z}{2^n}\Big)^* - d_2(z)^*\Big) + \frac{\kappa^{4n}}{3}\varphi_1\Big(0, 0, \frac{z}{2^n}\Big) \cdot \frac{\kappa^n}{2^n}
\end{aligned}$$

holds for all vectors z, which goes to zero as $n \to \infty$. Hence, one concludes d_2 is a quartic Lie $*$-derivation.

On the other hand, by Δ_2-condition without using the Fatou property, one can see the following inequality

$$\begin{aligned}
\rho(f(x)-d_2(x)) &= \rho\Big(\frac{1}{2}\{2f(x)-2\cdot 16^n f(\frac{x}{2^n})\}+\frac{1}{2}\{2\cdot 16^n f(\frac{x}{2^n})-2d_2(x)\}\Big)\\
&\le \frac{\kappa}{2}\rho\Big(f(x)-16^n f(\frac{x}{2^n})\Big)+\frac{\kappa}{2}\rho\Big(16^n f(\frac{x}{2^n})-d_2(x)\Big)\\
&\le \frac{\kappa}{2}\cdot\frac{1}{\kappa^4}\sum_{j=1}^{n}\frac{\kappa^{5j}}{2^j}\varphi_1(\frac{x}{2^j},\frac{x}{2^j},0)+\frac{\kappa}{2}\rho\Big(16^n f(\frac{x}{2^n})-d_2(x)\Big)\\
&\le \frac{1}{2\kappa^3}\sum_{j=1}^{\infty}\frac{\kappa^{5j}}{2^j}\varphi_1(\frac{x}{2^j},\frac{x}{2^j},0)=\frac{1}{2\kappa^3}\Psi(x,x,0),
\end{aligned}$$

for all positive integers n, which yields the approximation (2.14) by taking $n\to\infty$.

Finally, applying the same argument as in the proof of Theorem [10], we prove the uniqueness of d_2 satisfying the approximation (2.14) near f.

Hence, one can find a unique quartic Lie $*$-derivation d_2 satisfying the estimation (2.14) near f. □

Remark 2.4. *In Theorem 2.3, if $\chi := \chi_\rho$ is a Banach $*$-algebra with norm $\|\cdot\| := \rho$, and so $\rho(2x)=2\rho(x)$, $\kappa := 2$, then it follows from (2.10),(2.11),(2.12) and (2.13) that there exists a unique quartic Lie $*$-derivation $d_2 : \chi\to\chi$, defined as $d_2(x)=\lim_{n\to\infty}16^n f(\frac{x}{2^n})$, $x\in\chi$, which satisfies the equation (1.2) and*

$$\rho(f(x)-d_2(x))\le\frac{1}{16}\sum_{j=1}^{\infty}2^{4j}\varphi_1(\frac{x}{2^j},\frac{x}{2^j},0)$$

for all $x\in\chi$.

As a corollary of Theorem 2.1 and Theorem 2.3, we obtain the following stability result of the equation (1.2) associated with quartic Lie $*$-derivations, which generalizes stability result in normed $*$-algebras.

Corollary 2.5. *Let $\chi=\chi_\rho$ be a complete normed $*$-algebra with norm $\|\cdot\|$. For given nonnegative real numbers θ_i,ϑ_i together with $4\ne r_i$ $(i=1,2,3)$ and a,b with $4\ne a+b$, suppose a mapping $f:\chi\to\chi$ satisfies*

$$\begin{aligned}
\|QE_f^\lambda(x,y)+f(z^*)-f(z)^*\| &\le \theta_1\|x\|^{r_1}+\theta_2\|y\|^{r_2}+\theta_3(\|x\|^a\|y\|^b+\|z\|^{r_3}),\\
\|QD_f(x,y)\| &\le \vartheta_1\|x\|^{2r_1}+\vartheta_2\|y\|^{2r_2}+\vartheta_3\|x\|^{2a}\|y\|^{2b}
\end{aligned}$$

for all $x,y,z\in\chi$ and all $\lambda\in\mathbb{T}$. If for each $x\in\chi$ the mapping $r\to f(rx)$ from $\mathbb{R}$ to χ is continuous, then there exists a unique quartic Lie $*$-derivation $d_2:\chi\to\chi$ such that

$$\rho(f(x)-d_2(x))\le\frac{\theta_1\|x\|^{r_1}}{|2^{r_1}-2^4|}+\frac{\theta_2\|x\|^{r_2}}{|2^{r_2}-2^4|}+\frac{\theta_3\|x\|^{a+b}}{|2^{a+b}-2^4|}$$

for all $x\in\chi$.

In the following, we obtain a stability theorem of quartic Lie $*$-derivations by quartically contractive conditions of control functions ϕ_i for perturbing terms QE_f^λ and QD_f.

Theorem 2.6. *Let χ_ρ be a ρ-complete convex modular $*$-algebra with Δ_2-condition. Suppose there exist two functions $\varphi_1 : \chi_\rho^3 \to [0,\infty)$ and $\varphi_2 : \chi_\rho^2 \to [0,\infty)$ and two positive constant l_i with $l_1 < \frac{2^5}{\kappa^5}$ and $l_2 < \frac{2^8}{\kappa^8}$ for which a mapping $f : \chi_\rho \to \chi_\rho$ satisfies*

$$\begin{aligned} \rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) &\leq \varphi_1(x,y,z),\ \varphi_1(\frac{x}{2},\frac{y}{2},\frac{z}{2}) \leq \frac{l_1}{16}\varphi_1(x,y,z), \\ \rho(QD_f(x,y)) &\leq \varphi_2(x,y), \quad\ \varphi_2(\frac{x}{2},\frac{y}{2}) \leq \frac{l_2}{16^2}\varphi_2(x,y) \end{aligned}$$

for all $x,y,z \in \chi_\rho$ and all $\lambda \in \mathbb{T}$. Then there exists a unique quartic Lie $*$-derivation $d_2 : \chi_\rho \to \chi_\rho$ which satisfies the equation $QE_{d_2}^\lambda(x,y) = 0$, $QD_{d_2}(x,y) = 0$ and

$$\rho(f(x) - d_2(x)) \leq \frac{\kappa^2 l_1}{2(2^5 - \kappa^5 l_1)}\varphi_1(x,x,0)$$

for all $x,y \in \chi_\rho$ and all $\lambda \in \mathbb{C}$.

Acknowledgements

This work was supported by research fund of Chungnam National University.

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