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A series of refinements on the Young and reverse Young inequalities through a recursive algorithm

Somayeh Saiedinezhad[a]

[a]*School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran*

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Abstract

In this article, we present a recursive algorithm to obtain a series of refinements of the classical Young inequality. These inequalities conduce to equalities whenever the number of the iteration in the recursive algorithm tends to infinity. Also these refinements applied to establish some improved reverse Young and matrix Young inequalities with Hilbert- Schmidt norm.

Keywords: Young inequality; Reverse; Young inequality; Semidefinite positive matrix; Hilbert-Schmidt norm.
2010 MSC: 15A45; 15A60.

1. Introduction And Preliminarily

The known classical Young inequality for two scalars assert that if $a, b \geq 0$ and $v \in [0,1]$ then $va + (1-v)b \geq a^v b^{1-v}$. This inequality may be written in the form $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ which $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In the literature, there are a lot of researches specially in the last decade which improved the Young inequality to a sharper version in the scalar or matrix form. F. Kittaneh and Y. Manasrah in [6] presented a refinement of the Young inequality by the following theorem.

Theorem 1.1. *[6]. If $a, b \geq 0$ and $v \in [0,1]$ then*

$$va + (1-v)b \geq a^v b^{1-v} + \min\{v, 1-v\}(\sqrt{a} - \sqrt{b})^2. \tag{1.1}$$

M. Sababheh and D. Choi in [8] discussed a refinement of the Young and the reversed Young inequalities as follows.

Email address: ssaiedinezhad@iust.ac.ir (Somayeh Saiedinezhad)

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Theorem 1.2. *[8]. Let $a, b > 0$, $v \in [0,1]$ and $N \in \mathbb{N}$ then*

$$va + (1-v)b \geq$$
$$a^v b^{1-v} + \sum_{j=1}^{N}((-1)^{r_j}2^{j-1}v + (-1)^{r_j+1}[\frac{r_j+1}{2}])(a^{\frac{k_j}{2^j}} b^{\frac{2^{j-1}-k_j}{2^j}} - a^{\frac{k_j+1}{2^j}} b^{\frac{2^{j-1}-k_j-1}{2^j}})^2$$

Where $r_j = [2^j v]$ and $k_j = [2^{j-1}v]$. Moreover for $a, b > 0$, $v \geq 0$ and $N \in \mathbb{N}$

$$(1+v)a - vb \leq a^{1+v}b^{-v} - v\sum_{j=1}^{N} 2^{j-1}(\sqrt{a} - a^{\frac{2^{j-1}-1}{2^j}} b^{\frac{1}{2^j}})^2.$$

Obviously, from (1.1) we insert that,

$$(va + (1-v)b)^2 \geq a^{2v}b^{2(1-v)}; \tag{1.2}$$

for $a, b \geq 0$ and $v \in [0,1]$. Hirzallah and Kittaneh in [4], obtained a sharper version of (1.2) as follows,

$$(va + (1-v)b)^2 \geq a^{2v}b^{2(1-v)} + (\min(v, 1-v))^2(a-b)^2. \tag{1.3}$$

These types of inequalities, which involve the square of the terms are noteworthy for the sake of their application in the 2- norms matrix inequalities. Let us to denote the space of $n \times n$ complex matrixes by $M_n(\mathbb{C})$ with Hilbert-Schmidt norm $\|.\|_2$ which for every $A = [a_{i,j}] \in M_N(\mathbb{C})$, $\|A\|_2 = (\Sigma_{i,j=1}^2 |a_{ij}|^2)^{\frac{1}{2}}$.

A matrix version of the young inequality, that is

$$\varphi(A^\nu XB^{1-\nu}) \leq \varphi(vAX + (1-v)XB)$$

where $A, B, X \in M_n(\mathbb{C})$ and A, B are semidefinite positive, for some specified functions φ is studied; for example in [1], φ is defined by $\varphi(A) := \lambda_j(A)$ (j -th eigenvalue of A when the set of eigenvalues are arranged by $\lambda_1(A) \geq ... \geq \lambda_N(A)$); in [5] φ is defined by $\varphi(A) := det(A)$ and in [3] φ is defined by $\varphi(A) := \|A\|_2$.

In this paper we describe a recursive algorithm which develops a series of refinements on the Young and the reversed Young inequalities and leads to its application in the matrix form. These inequalities are notable since, if the number of the iteration of the algorithm tends to infinity the difference of the both sides of the inequalities tend to zero and so we get equalities.

2. Main results

Theorem 2.1. *Suppose that $a, b \geq 0$ and $v \in [0,1]$ such that for some $n \in \mathbb{N}$ and integer $k \in \{1, 2, ..., 2^{f(n)} := 2^{2^{n-1}}\}$, $\frac{k-1}{2^{f(n)}} \leq v < \frac{k}{2^{f(n)}}$. Then*

$$va + (1-v)b \geq a^v b^{1-v} + G_{k,1}^n(v,a,b) + ... + G_{k,f(n)}^n(v,a,b); \tag{2.1}$$

in which by letting $k = m \times 2^{f(n-1)} + r$ with $m \in \{0, ..., 2^{f(n-1)} - 1\}$ and $r \in \{1, ..., 2^{f(n-1)}\}$ the following recursive formula describes $G_{k,i}^n$.

$$G_{k,i}^n(v,a,b) = \begin{cases} G_{m+1,i}^{n-1}(v,a,b); \quad 1 \leq i \leq f(n-1); \\ G_{r,i-f(n-1)}^{n-1}(2^{f(n-1)}v - m, a^{\frac{m+1}{2^{f(n-1)}}} b^{\frac{2^{f(n-1)}-(m+1)}{2^{f(n-1)}}}, a^{\frac{m}{2^{f(n-1)}}} b^{\frac{2^{f(n-1)}-m}{2^{f(n-1)}}}); \quad i > f(n-1). \end{cases} \tag{2.2}$$

where $G_{1,1}^1(v,a,b) := v(\sqrt{a} - \sqrt{b})^2$ and $G_{2,1}^1(v,a,b) := (1-v)(\sqrt{a} - \sqrt{b})^2$.

Example 2.2.

$$va+(1-v)b \geq \begin{cases} a^v b^{1-v} + v(\sqrt{a}-\sqrt{b})^2 + 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2; & 0 \leq v \leq \frac{1}{4}; \\ a^v b^{1-v} + v(\sqrt{a}-\sqrt{b})^2 + (1-2v)(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2; & \frac{1}{4} \leq v \leq \frac{1}{2}; \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2v-1)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2; & \frac{1}{2} \leq v \leq \frac{3}{4}; \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2-2v)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2; & \frac{3}{4} \leq v \leq 1. \end{cases}$$

Remark 2.3. *Theorem 2.1 is the recursive expression of the first inequality of the Theorem 1.2, indeed our approach to get the recursive formulation is more comprehensible although is more complex.*

Remark 2.4. *For every n, (2.2) generate $2^{f(n)}$ inequalities where each of them holds for v belongs to a sub interval of $[0,1]$. By increasing n to $n+1$ each of them refines to $2^{f(n)}$ other inequalities and so by $n+1$ we obtain $(2^{f(n)})^2$ inequalities. Note that $(2^{f(n)})^2 = 2^{f(n+1)}$.*

For simplicity, let me to name the inequality (2.1), which is dependent on v,a,b,n,k, by $Inq(v,a,b,n,k)$.
Proof .(Proof of the Theorem 2.1). If $v \in [0,\frac{1}{2}]$ then

$$va+(1-v)b - v(\sqrt{a}-\sqrt{b})^2 = 2v\sqrt{ab} + (1-2v)b,$$

so by applying inequality (1.1) for $2v, \sqrt{ab}$ instead of v, a respectively, we obtain

$$2v\sqrt{ab} + (1-2v)b \geq a^v b^{1-v} + \min(2v, 1-2v)(\sqrt{\sqrt{ab}} - \sqrt{b})^2.$$

Therefore,

$$va+(1-v)b \geq a^v b^{1-v} + v(\sqrt{a}-\sqrt{b})^2 + \min(2v,1-2v)(\sqrt{\sqrt{ab}}-\sqrt{b})^2. \tag{2.3}$$

Similarly for $v \in [\frac{1}{2},1]$ we have $va+(1-v)b-(1-v)(\sqrt{a}-\sqrt{b})^2 = (2v-1)a + (2-2v)\sqrt{ab}$, and by applying inequality (1.1) for $2v-1, \sqrt{ab}$ instead of v, b respectively, we obtain

$$(2v-1)a + (2-2v)\sqrt{ab} \geq a^v b^{1-v} + \min(2v-1, 2-2v)(\sqrt{a} - \sqrt{\sqrt{ab}})^2.$$

Thus

$$va+(1-v)b \geq a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + \min(2v-1,2-2v)(\sqrt{a}-\sqrt{\sqrt{ab}})^2. \tag{2.4}$$

We see that $G^2_{k,i}(v,a,b)$ has a regularity which one can find it out in the following table. For abbreviation denote $G^n_{k,i}(v,a,b)$ by $G^n_{k,i}$.

$G^2_{1,1} = G^1_{1,1} = v(\sqrt{a}-\sqrt{b})^2$	$G^2_{1,2} = G^1_{1,1}(2v, a^{\frac{1}{2}}b^{\frac{1}{2}}, b) = 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2$
$G^2_{2,1} = G^1_{1,1} = v(\sqrt{a}-\sqrt{b})^2$	$G^2_{2,2} = G^1_{2,1}(2v, a^{\frac{1}{2}}b^{\frac{1}{2}}, b) = (1-2v)(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2$
$G^2_{3,1} = G^1_{2,1} = (1-v)(\sqrt{a}-\sqrt{b})^2$	$G^2_{3,2} = G^1_{1,1}(2v-1, a, a^{\frac{1}{2}}b^{\frac{1}{2}}) = (2v-1)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2$
$G^2_{4,1} = G^1_{2,1} = (1-v)(\sqrt{a}-\sqrt{b})^2$	$G^2_{4,2} = G^1_{2,1}(2v-1, a, a^{\frac{1}{2}}b^{\frac{1}{2}}) = (2-2v)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2$

The table of the functions $G^2_{k,i}$

Let us to denote

$$R(v,a,b,2,1) := va + (1-v)b - G^2_{1,1}(v,a,b) - G^2_{1,2}(v,a,b). \tag{2.5}$$

We readily check that $R(v,a,b,2,1) = 4va^{\frac{1}{4}}b^{\frac{3}{4}} + (1-4v)b$. Hence by applying the inequalities (2.3) and (2.4) with $4v, a^{\frac{1}{4}}b^{\frac{3}{4}}$ instead of v, a respectively we deduce four below inequalities.

$$4va^{\frac{1}{4}}b^{\frac{3}{4}} + (1-4v)b \geq$$
$$\begin{cases} a^v b^{1-v} + 4v(a^{\frac{1}{2^3}}b^{\frac{3}{2^3}} - b^{\frac{1}{2}})^2 + 8v(a^{\frac{1}{2^4}}b^{\frac{7}{2^4}} - b^{\frac{1}{2}})^2; & 0 \leq v \leq \frac{1}{16}; \\ a^v b^{1-v} + 4v(a^{\frac{1}{2^3}}b^{\frac{3}{2^3}} - b^{\frac{1}{2}})^2 + (1-8v)(a^{\frac{1}{2^4}}b^{\frac{7}{2^4}} - b^{\frac{1}{2}})^2; & \frac{1}{16} \leq v \leq \frac{1}{8}; \\ a^v b^{1-v} + (1-4v)(a^{\frac{1}{2^3}}b^{\frac{3}{2^3}} - b^{\frac{1}{2}})^2 + (8v-1)(a^{\frac{1}{8}}b^{\frac{3}{8}} - a^{\frac{1}{2^4}}b^{\frac{7}{2^4}})^2; & \frac{1}{8} \leq v \leq \frac{3}{16}; \\ a^v b^{1-v} + (1-4v)(a^{\frac{1}{2^3}}b^{\frac{3}{2^3}} - b^{\frac{1}{2}})^2 + (2-8v)(a^{\frac{1}{8}}b^{\frac{3}{8}} - a^{\frac{1}{2^4}}b^{\frac{7}{2^4}})^2; & \frac{3}{16} \leq v \leq \frac{1}{4}. \end{cases} \tag{2.6}$$

By applying these results in the inequality (2.5), a group of four inequalities produced which is a refinement of the the first line of the inequalities in the example 2.2 as follows.

$$va + (1-v)b \geq$$
$$\begin{cases} a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2 + 4v(a^{\frac{1}{2^3}}b^{\frac{3}{2^3}} - b^{\frac{1}{2}})^2 + 8v(a^{\frac{1}{2^4}}b^{\frac{7}{2^4}} - b^{\frac{1}{2}})^2; \\ \qquad 0 \leq v \leq \frac{1}{16}; \\ a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2 + 4v(a^{\frac{1}{2^3}}b^{\frac{3}{2^3}} - b^{\frac{1}{2}})^2 + (1-8v)(a^{\frac{1}{2^4}}b^{\frac{7}{2^4}} - b^{\frac{1}{2}})^2; \\ \qquad \frac{1}{16} \leq v \leq \frac{1}{8}; \\ a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2 + (1-4v)(a^{\frac{1}{2^3}}b^{\frac{3}{2^3}} - b^{\frac{1}{2}})^2 + (8v-1)(a^{\frac{1}{8}}b^{\frac{3}{8}} - a^{\frac{1}{2^4}}b^{\frac{7}{2^4}})^2; \\ \qquad \frac{1}{8} \leq v \leq \frac{3}{16}; \\ a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2 + (1-4v)(a^{\frac{1}{2^3}}b^{\frac{3}{2^3}} - b^{\frac{1}{2}})^2 + (2-8v)(a^{\frac{1}{8}}b^{\frac{3}{8}} - a^{\frac{1}{2^4}}b^{\frac{7}{2^4}})^2; \\ \qquad \frac{3}{16} \leq v \leq \frac{1}{4}. \end{cases}$$

Similarly one can compute

$$R(v,a,b,2,2) := va + (1-v)b - G^2_{2,1}(v,a,b) - G^2_{2,2}(v,a,b) = (4v-1)a^{\frac{1}{2}}b^{\frac{1}{2}} + (2-4v)a^{\frac{1}{4}}b^{\frac{3}{4}}. \tag{2.7}$$

So, by applying the inequalities (2.3) and (2.4) with $4v-1, a^{\frac{1}{2}}b^{\frac{1}{2}}, a^{\frac{1}{4}}b^{\frac{3}{4}}$ instead of v, a, b respectively we deduce a group of four inequalities which if we apply them in (2.7) a group of four inequalities will be produced which is a refinement of the second inequality in the example 2.2 as follows.

$$va + (1-v)b \geq$$
$$\begin{cases} a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + (1-2v)(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2 + (4v-1)(a^{\frac{1}{2^2}}b^{\frac{1}{2^2}} - a^{\frac{1}{2^3}}b^{\frac{3}{2^3}})^2 & \\ \qquad + (8v-2)(a^{\frac{3}{2^4}}b^{\frac{5}{2^4}} - a^{\frac{1}{2^3}}b^{\frac{3}{2^3}})^2; & \frac{1}{4} \leq v \leq \frac{5}{16}; \\ a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + (1-2v)(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2 + (4v-1)(a^{\frac{1}{2^2}}b^{\frac{1}{2^2}} - a^{\frac{1}{2^3}}b^{\frac{3}{2^3}})^2 & \\ \qquad + (3-8v)(a^{\frac{3}{2^4}}b^{\frac{5}{2^4}} - a^{\frac{1}{2^3}}b^{\frac{3}{2^3}})^2; & \frac{5}{16} \leq v \leq \frac{6}{16}; \\ a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + (1-2v)(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2 + (2-4v)(a^{\frac{1}{2^2}}b^{\frac{1}{2^2}} - a^{\frac{1}{2^3}}b^{\frac{3}{2^3}})^2 & \\ \qquad + (8v-3)(a^{\frac{1}{2^2}}b^{\frac{1}{2^2}} - a^{\frac{3}{2^4}}b^{\frac{5}{2^4}})^2; & \frac{6}{16} \leq v \leq \frac{7}{16}; \\ a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 + (1-2v)(a^{\frac{1}{4}}b^{\frac{1}{4}} - b^{\frac{1}{2}})^2 + (2-4v)(a^{\frac{1}{2^2}}b^{\frac{1}{2^2}} - a^{\frac{1}{2^3}}b^{\frac{3}{2^3}})^2 & \\ \qquad + (4-8v)(a^{\frac{1}{2^2}}b^{\frac{1}{2^2}} - a^{\frac{3}{2^4}}b^{\frac{7}{2^4}})^2; & \frac{5}{16} \leq v \leq \frac{1}{2}. \end{cases}$$

If we follow, in the same way on

$$R(v,a,b,2,3) := va + (1-v)b - G^2_{3,1}(v,a,b) - G^2_{3,2}(v,a,b) = (4v-2)a^{\frac{3}{4}}b^{\frac{1}{4}} + (3-4v)a^{\frac{1}{2}}b^{\frac{1}{2}}$$

and $R(v,a,b,2,4) = va + (1-v)b - G^2_{4,1}(v,a,b) - G^2_{4,2}(v,a,b) = (4v-3)a + (4-4v)a^{\frac{3}{4}}b^{\frac{1}{4}}$,
then two groups of four inequalities will be produced which the first one is a refinement of the third inequality in the example 2.2 and the second one is a refinement of the fourth inequality in the example 2.2. Indeed,

$$va + (1-v)b \geq$$

$$\begin{cases} a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2v-1)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2 + (4v-2)(a^{\frac{3}{2^3}}b^{\frac{1}{2^3}} - a^{\frac{1}{2^2}}b^{\frac{1}{2^2}})^2 & \\ \qquad +(8v-4)(a^{\frac{5}{2^4}}b^{\frac{3}{2^4}} - a^{\frac{1}{2^2}}b^{\frac{1}{2^2}})^2; & \frac{1}{2} \leq v \leq \frac{9}{16}; \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2v-1)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2 + (4v-2)(a^{\frac{3}{2^3}}b^{\frac{1}{2^3}} - a^{\frac{1}{2^2}}b^{\frac{1}{2^2}})^2 & \\ \qquad +(5-8v)(a^{\frac{5}{2^4}}b^{\frac{3}{2^4}} - a^{\frac{1}{2^2}}b^{\frac{1}{2^2}})^2; & \frac{9}{16} \leq v \leq \frac{10}{16}; \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2v-1)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2 + (3-4v)(a^{\frac{3}{2^3}}b^{\frac{1}{2^3}} - a^{\frac{1}{2^2}}b^{\frac{1}{2^2}})^2 & \\ \qquad +(8v-5)(a^{\frac{3}{2^3}}b^{\frac{1}{2^3}} - a^{\frac{5}{2^4}}b^{\frac{3}{2^4}})^2; & \frac{10}{16} \leq v \leq \frac{11}{16}; \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2v-1)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2 + (3-4v)(a^{\frac{3}{2^3}}b^{\frac{1}{2^3}} - a^{\frac{1}{2^2}}b^{\frac{1}{2^2}})^2 & \\ \qquad +(6-8v)(a^{\frac{3}{2^3}}b^{\frac{1}{2^3}} - a^{\frac{5}{2^4}}b^{\frac{3}{2^4}})^2; & \frac{11}{16} \leq v \leq \frac{3}{4}; \end{cases}$$

and
$$va + (1-v)b \geq$$

$$\begin{cases} a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2-2v)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2 + (4v-3)(a^{\frac{1}{2}} - a^{\frac{3}{2^3}}b^{\frac{1}{2^3}})^2 & \\ \qquad +(8v-6)(a^{\frac{7}{2^4}}b^{\frac{1}{2^4}} - a^{\frac{3}{2^3}}b^{\frac{1}{2^3}})^2; & \frac{3}{4} \leq v \leq \frac{13}{16}; \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2-2v)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2 + (4v-3)(a^{\frac{1}{2}} - a^{\frac{3}{2^3}}b^{\frac{1}{2^3}})^2 & \\ \qquad +(7-8v)(a^{\frac{7}{2^4}}b^{\frac{1}{2^4}} - a^{\frac{3}{2^3}}b^{\frac{1}{2^3}})^2; & \frac{13}{16} \leq v \leq \frac{14}{16}; \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2-2v)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2 + (4-4v)(a^{\frac{1}{2}} - a^{\frac{3}{2^3}}b^{\frac{1}{2^3}})^2 & \\ \qquad +(8v-7)(a^{\frac{1}{2}} - a^{\frac{7}{2^4}}b^{\frac{1}{2^4}})^2; & \frac{14}{16} \leq v \leq \frac{15}{16}; \\ a^v b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + (2-2v)(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2 + (4-4v)(a^{\frac{1}{2}} - a^{\frac{3}{2^3}}b^{\frac{1}{2^3}})^2 & \\ \qquad +(8-8v)(a^{\frac{1}{2}} - a^{\frac{7}{2^4}}b^{\frac{1}{2^4}})^2; & \frac{15}{16} \leq v \leq 1. \end{cases}$$

In the table of the functions $G^3_{k,i}$, as follows, the regularity that produce $G^3_{k,i}$ for $k \in \{1,...,16\}$ and $i \in \{1,...,4\}$ is comprehensible. In this recursive approach if for $n \in \mathbb{N}$, $k \in \{1,2,...,2^{f(n)}\}$ and $v \in [\frac{k-1}{2^{f(n)}}, \frac{k}{2^{f(n)}})$, we consider the corresponding inequality $Inq(v,a,b,n,k)$ which is

$$va + (1-v)b \geq a^v b^{1-v} + G^n_{k,1}(v,a,b) + ... + G^n_{k,f(n)}(v,a,b);$$

one can readily check that
$R(v,a,b,n,k) := va + (1-v)b - G^n_{k,1} - ... - G^n_{k,f(n)} =$

$$(2^{f(n)}v - (k-1))a^{\frac{k}{2^{f(n)}}}b^{\frac{2^{f(n)}-k}{2^{f(n)}}} + (1 - (2^{f(n)}v - (k-1)))a^{\frac{k-1}{2^{f(n)}}}b^{\frac{2^{f(n)}-(k-1)}{2^{f(n)}}}.$$

By applying each of the $2^{f(n)}$ inequalities $\{Inq(v,a,b,n,j)\}_{j\in\{1,...,2^{f(n)}\}}$ on R(v,a,b,n,k) with $2^{f(n)}v - (k-1)$, $a^{\frac{k}{2^{f(n)}}}b^{\frac{2^{f(n)}-k}{2^{f(n)}}}$, $a^{\frac{k-1}{2^{f(n)}}}b^{\frac{2^{f(n)}-(k-1)}{2^{f(n)}}}$ respectively instead of v,a,b and so apply their results on

$Inq(v,a,b,n,k)$, consequently $Inq(v,a,b,n,k)$ refines to a group of $2^{f(n)}$ inequalities that each of them are sharper than $Inq(v,a,b,n,k)$ on a sub interval of $[\frac{k-1}{2^{f(n)}}, \frac{k}{2^{f(n)}})$. Moreover the first $f(n)+1$ terms of these new inequalities are similar to $Inq(v,a,b,n,k)$. Indeed, by applying $Inq(v,a,b,n,j)$ with $2^{f(n)}v-(k-1)$ instead of v, we have $\frac{j-1}{2^{f(n)}} \leq 2^{f(n)}v-(k-1) < \frac{j}{2^{f(n)}}$ or equivalently

$$\frac{j+(k-1)2^{f(n)}-1}{2^{f(n+1)}} \leq v < \frac{j+(k-1)2^{f(n)}}{2^{f(n+1)}}.$$

Hence by applying $Inq(v,a,b,n,j)$ on $R(v,a,b,n,k)$ with above manner, $Inq(v,a,b,n+1,(k-1)2^{f(n)}+j)$ will be obtained. By this explanation the formula (2.2) would be clear and so the proof is completed. □

$G^3_{1,1}=G^2_{1,1}$	$G^3_{1,2}=G^2_{1,2}$	$G^3_{1,3}=G^2_{1,1}(4v,a^{\frac{1}{4}}b^{\frac{3}{4}},b)$	$G^3_{1,4}=G^2_{1,2}(4v,a^{\frac{1}{4}}b^{\frac{3}{4}},b)$
$G^3_{2,1}=G^2_{1,1}$	$G^3_{2,2}=G^2_{1,2}$	$G^3_{2,3}=G^2_{2,1}(4v,a^{\frac{1}{4}}b^{\frac{3}{4}},b)$	$G^3_{2,4}=G^2_{2,2}(4v,a^{\frac{1}{4}}b^{\frac{3}{4}},b)$
$G^3_{3,1}=G^2_{1,1}$	$G^3_{3,2}=G^2_{1,2}$	$G^3_{3,3}=G^2_{3,1}(4v,a^{\frac{1}{4}}b^{\frac{3}{4}},b)$	$G^3_{3,4}=G^2_{3,2}(4v,a^{\frac{1}{4}}b^{\frac{3}{4}},b)$
$G^3_{4,1}=G^2_{1,1}$	$G^3_{4,2}=G^2_{1,2}$	$G^3_{4,3}=G^2_{4,1}(4v,a^{\frac{1}{4}}b^{\frac{3}{4}},b)$	$G^3_{4,4}=G^2_{4,2}(4v,a^{\frac{1}{4}}b^{\frac{3}{4}},b)$
$G^3_{5,1}=G^2_{2,1}$	$G^3_{5,2}=G^2_{2,2}$	$G^3_{5,3}=G^2_{1,1}(4v-1,a^{\frac{2}{4}}b^{\frac{2}{4}},a^{\frac{1}{4}}b^{\frac{3}{4}})$	$G^3_{5,4}=G^2_{1,2}(4v-1,a^{\frac{2}{4}}b^{\frac{2}{4}},b^{\frac{3}{4}})$
$G^3_{6,1}=G^2_{2,1}$	$G^3_{6,2}=G^2_{2,2}$	$G^3_{6,3}=G^2_{2,1}(4v-1,a^{\frac{2}{4}}b^{\frac{2}{4}},a^{\frac{1}{4}}b^{\frac{3}{4}})$	$G^3_{6,4}=G^2_{2,2}(4v-1,a^{\frac{2}{4}}b^{\frac{2}{4}},a^{\frac{1}{4}}b^{\frac{3}{4}})$
$G^3_{7,1}=G^2_{2,1}$	$G^3_{7,2}=G^2_{2,2}$	$G^3_{7,3}=G^2_{3,1}(4v-1,a^{\frac{2}{4}}b^{\frac{2}{4}},a^{\frac{1}{4}}b^{\frac{3}{4}})$	$G^3_{7,4}=G^2_{3,2}(4v-1,a^{\frac{2}{4}}b^{\frac{2}{4}},a^{\frac{1}{4}}b^{\frac{3}{4}})$
$G^3_{8,1}=G^2_{2,1}$	$G^3_{8,2}=G^2_{2,2}$	$G^3_{8,3}=G^2_{4,1}(4v-1,a^{\frac{2}{4}}b^{\frac{2}{4}},a^{\frac{1}{4}}b^{\frac{3}{4}})$	$G^3_{8,4}=G^2_{4,2}(4v-1,a^{\frac{2}{4}}b^{\frac{2}{4}},a^{\frac{1}{4}}b^{\frac{3}{4}})$
$G^3_{9,1}=G^2_{3,1}$	$G^3_{9,2}=G^2_{3,2}$	$G^3_{9,3}=G^2_{1,1}(4v-2,a^{\frac{3}{4}}b^{\frac{1}{4}},a^{\frac{1}{2}}b^{\frac{1}{2}})$	$G^3_{9,4}=G^2_{1,2}(4v-2,a^{\frac{3}{4}}b^{\frac{1}{4}},a^{\frac{1}{2}}b^{\frac{1}{2}})$
$G^3_{10,1}=G^2_{3,1}$	$G^3_{10,2}=G^2_{3,2}$	$G^3_{10,3}=G^2_{2,1}(4v-2,a^{\frac{3}{4}}b^{\frac{1}{4}},a^{\frac{1}{2}}b^{\frac{1}{2}})$	$G^3_{10,4}=G^2_{2,2}(4v-2,a^{\frac{3}{4}}b^{\frac{1}{4}},a^{\frac{1}{2}}b^{\frac{1}{2}})$
$G^3_{11,1}=G^2_{3,1}$	$G^3_{11,2}=G^2_{3,2}$	$G^3_{11,3}=G^2_{3,1}(4v-2,a^{\frac{3}{4}}b^{\frac{1}{4}},a^{\frac{1}{2}}b^{\frac{1}{2}})$	$G^3_{11,4}=G^2_{3,2}(4v-2,a^{\frac{3}{4}}b^{\frac{1}{4}},a^{\frac{1}{2}}b^{\frac{1}{2}})$
$G^3_{12,1}=G^2_{3,1}$	$G^3_{12,2}=G^2_{3,2}$	$G^3_{12,3}=G^2_{4,1}(4v-2,a^{\frac{3}{4}}b^{\frac{1}{4}},a^{\frac{1}{2}}b^{\frac{1}{2}})$	$G^3_{12,4}=G^2_{4,2}(4v-2,a^{\frac{3}{4}}b^{\frac{1}{4}},a^{\frac{1}{2}}b^{\frac{1}{2}})$
$G^3_{13,1}=G^2_{4,1}$	$G^3_{13,2}=G^2_{4,2}$	$G^3_{13,3}=G^2_{1,1}(4v-3,a,a^{\frac{3}{4}}b^{\frac{1}{4}})$	$G^3_{13,4}=G^2_{1,2}(4v-3,a,a^{\frac{3}{4}}b^{\frac{1}{4}})$
$G^3_{14,1}=G^2_{4,1}$	$G^3_{14,2}=G^2_{4,2}$	$G^3_{14,3}=G^2_{2,1}(4v-3,a,a^{\frac{3}{4}}b^{\frac{1}{4}})$	$G^3_{14,4}=G^2_{2,2}(4v-3,a,a^{\frac{3}{4}}b^{\frac{1}{4}})$
$G^3_{15,1}=G^2_{4,1}$	$G^3_{15,2}=G^2_{4,2}$	$G^3_{15,3}=G^2_{3,1}(4v-3,a,a^{\frac{3}{4}}b^{\frac{1}{4}})$	$G^3_{15,4}=G^2_{3,2}(4v-3,a,a^{\frac{3}{4}}b^{\frac{1}{4}})$
$G^3_{16,1}=G^2_{4,1}$	$G^3_{16,2}=G^2_{4,2}$	$G^3_{16,3}=G^2_{4,1}(4v-3,a,a^{\frac{3}{4}}b^{\frac{1}{4}})$	$G^3_{16,4}=G^2_{4,2}(4v-3,a,a^{\frac{3}{4}}b^{\frac{1}{4}})$

The table of the functions $G^3_{k,i}$

Remark 2.5. *Since* $\frac{k-1}{2^{f(n)}} \leq v < k$*, hence* $k-1=[2^{f(n)}v]$*, where* $[x]$ *is the biggest integer number which is less than or equal to* x*; and in view of that* $\lim_{x\to\infty}\frac{x}{[x]}=1$*, we deduce that* $\lim_{n\to\infty}R(v,a,b,n,k)=a^vb^{1-v}$*. Thus for big enough* n*, we can reduce the difference between the right and left hand sides of (2.1) and so when* n *tends to infinity (2.1) conduces to equality.*

Kittaneh and Manasrah in [7] established the following inequality which is of the reversed Young inequality's form.

Theorem 2.6. *[7]. If* $a,b\geq 0$ *and* $v\in[0,1]$ *then*

$$va+(1-v)b \leq a^vb^{1-v}+\max\{v,1-v\}(\sqrt{a}-\sqrt{b})^2 \tag{2.8}$$

Now by applying the refinement of the Young inequality (2.1) we can refine (2.8) to a sharper version by the following approach. Firstly suppose $0\leq v\leq\frac{1}{2}$ then

$$(1-v)(\sqrt{a}-\sqrt{b})^2-va-(1-v)b=(1-2v)a+2v\sqrt{ab}-2\sqrt{ab}. \tag{2.9}$$

So, by applying the inequality $Inq(2v, \sqrt{ab}, a, 1, 1)$ on the two first terms in the right hand side of (2.9), we derive

$$va + (1-v)b \leq -a^{1-v}b^{v} + (1-v)(\sqrt{a}-\sqrt{b})^2 - 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - a^{\frac{1}{2}})^2 + 2\sqrt{ab},$$

for $v \in [0, \frac{1}{4}]$; and since

$$a^{1-v}b^{v} + a^{v}b^{1-v} \geq 2\sqrt{ab} \tag{2.10}$$

then

$$va + (1-v)b \leq a^{v}b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 - 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - a^{\frac{1}{2}})^2.$$

By applying the inequality $Inq(2v, \sqrt{ab}, a, 1, 2)$ on the second and third terms in the right hand side of (2.9), derive

$$va + (1-v)b \leq a^{v}b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 - (1-2v)(a^{\frac{1}{4}}b^{\frac{1}{4}} - a^{\frac{1}{2}})^2$$

which is satisfies $v \in [\frac{1}{4}, \frac{1}{2}]$. By applying the inequality $Inq(2v, \sqrt{ab}, a, 2, 1)$ on the second and third terms in the right hand side of (2.9), we derive

$$va + (1-v)b \leq a^{v}b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 - 2v(a^{\frac{1}{4}}b^{\frac{1}{4}} - a^{\frac{1}{2}})^2 - 4v(a^{\frac{3}{8}}b^{\frac{1}{8}} - a^{\frac{1}{2}})^2,$$

for $v \in [0, \frac{1}{8}]$. Indeed by applying inequality $Inq(2v, \sqrt{ab}, a, n, k)$ on the second and third terms in the right hand side of (2.9), we deduce

$$va + (1-v)b \leq a^{v}b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 + G^{n}_{k,1}(2v, a^{\frac{1}{2}}b^{\frac{1}{2}}, a) + ... + G^{n}_{k,f(n)}(2v, a^{\frac{1}{2}}b^{\frac{1}{2}}, a), \tag{2.11}$$

for $v \in [\frac{k-1}{2^{1+f(n)}}, \frac{k}{2^{1+f(n)}}]$.

Moreover if $\frac{1}{2} \leq v \leq 1$ then

$$v(\sqrt{a}-\sqrt{b})^2 - va - (1-v)b = (2v-1)b + (2-2v)\sqrt{ab} - 2\sqrt{ab}. \tag{2.12}$$

So by applying the inequality $Inq(2v-1, b, \sqrt{ab}, 1, 1)$ on the two first terms in the right hand side of (2.12) we deduce

$$va + (1-v)b \leq a^{v}b^{1-v} + v(\sqrt{a}-\sqrt{b})^2 - (2v-1)(b^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}})^2;$$

for $v \in [\frac{1}{2}, \frac{3}{4}]$. Further, by applying inequality $Inq(2v-1, b, a^{\frac{1}{2}}b^{\frac{1}{2}}, n, k)$ on the second and third terms in the right hand side of (2.12), we deduce

$$va + (1-v)b \leq a^{v}b^{1-v} + v(\sqrt{a}-\sqrt{b})^2 + G^{n}_{k,1}(2v-1, b, a^{\frac{1}{2}}b^{\frac{1}{2}}) + ... + G^{n}_{k,f(n)}(2v-1, b, a^{\frac{1}{2}}b^{\frac{1}{2}}); \tag{2.13}$$

for $v \in [\frac{k-1}{2^{1+f(n)}} + \frac{1}{2}, \frac{k}{2^{1+f(n)}} + \frac{1}{2}]$.

Let us to precise the above argument by the following theorem.

Theorem 2.7. *Suppose that $a, b \geq 0$, $v \in [0, 1]$, $n \in \mathbb{N}$ and $k \in \{1, 2, ..., 2^{f(n)}\}$. Then*

$$va+(1-v)b \leq \begin{cases} a^{v}b^{1-v} + (1-v)(\sqrt{a}-\sqrt{b})^2 - G^{n}_{k,1}(2v, a^{\frac{1}{2}}b^{\frac{1}{2}}, a) - ... - G^{n}_{k,f(n)}(2v, a^{\frac{1}{2}}b^{\frac{1}{2}}, a); \\ \qquad v \in [\frac{k-1}{2^{1+f(n)}}, \frac{k}{2^{1+f(n)}}]. \\ a^{v}b^{1-v} + v(\sqrt{a}-\sqrt{b})^2 - G^{n}_{k,1}(2v-1, b, a^{\frac{1}{2}}b^{\frac{1}{2}}) + ... - G^{n}_{k,f(n)}(2v-1, b, a^{\frac{1}{2}}b^{\frac{1}{2}}); \\ \qquad v \in [\frac{k-1}{2^{1+f(n)}} + \frac{1}{2}, \frac{k}{2^{1+f(n)}} + \frac{1}{2}]; \end{cases} \tag{2.14}$$

where $G^{n}_{k,i}$ is introduced in the Theorem 2.1.

Let us to recall the first line inequality of the Theorem 2.7 by $RInq^1(v,a,b,n,k)$ and the second one by $RInq^2(v,a,b,n,k)$.

Remark 2.8. *In view of the Remark 2.5, we have*

$$\lim_{n\to\infty}(a^vb^{1-v}+(1-v)(\sqrt{a}-\sqrt{b})^2-\sum_{i=1}^{f(n)}G^n_{k,i}(2v,a^{\frac{1}{2}}b^{\frac{1}{2}},a)-va-(1-v)b)=a^vb^{1-v}+a^{1-v}b^v-2\sqrt{ab}.$$

Hence the right hand side of $RIneq^1(v,a,b,n,k)$ has a notable difference by $va+(1-v)b$ even for big n. With due attention to the process of our discussion, we derive that this error is occurred since we applied (2.10) which is not sufficiently sharp estimate. Hence by removing (2.10) from the process, we insert the improved version of the inequalities (2.14), as follows.

$$va+(1-v)b\leq\begin{cases}-a^{1-v}b^v+2\sqrt{ab}+(1-v)(\sqrt{a}-\sqrt{b})^2-\sum_{i=1}^{f(n)}G^n_{k,i}(2v,a^{\frac{1}{2}}b^{\frac{1}{2}},a); \\ \qquad\qquad v\in[\frac{k-1}{2^{1+f(n)}},\frac{k}{2^{1+f(n)}}]. \\ -a^{1-v}b^v+2\sqrt{ab}+v(\sqrt{a}-\sqrt{b})^2-\sum_{i=1}^{f(n)}G^n_{k,i}(2v-1,b,a^{\frac{1}{2}}b^{\frac{1}{2}}); \\ \qquad\qquad v\in[\frac{k-1}{2^{1+f(n)}}+\frac{1}{2},\frac{k}{2^{1+f(n)}}+\frac{1}{2}].\end{cases}$$

Remark 2.9. *It is remarkable that the above refinements are due to the specific form of the right hand sides of the equalities (2.9) and (2.12) which are caused by the difference of the terms involved in (2.8). It is surprising that we can derive the similar forms by computing the suitable difference in each produced refinement inequalities. For example consider inequality $RInq^1(v,a,b,n,k)$ for $n=1,k=1$, that is*

$$va+(1-v)b\leq a^vb^{1-v}+(1-v)(\sqrt{a}-\sqrt{b})^2-2v(a^{\frac{1}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}})^2; \tag{2.15}$$

for $v\in[0,\frac{1}{4}]$. Now, we readily check that

$$M^1(v,a,b,1,1):=(1-v)(\sqrt{a}-\sqrt{b})^2-2v(a^{\frac{1}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}})^2-va-(1-v)b=4va^{\frac{3}{4}}b^{\frac{1}{4}}+(1-4v)a-2\sqrt{ab}. \tag{2.16}$$

So by applying the inequalities $Inq(4v,a^{\frac{3}{4}}b^{\frac{1}{2}},a,1,1)$ and $Inq(4v,a^{\frac{3}{4}}b^{\frac{1}{2}},a,1,2)$ on the two first terms of the right hand side of (2.16), respectively we deduce

$$va+(1-v)b\leq a^vb^{1-v}+(1-v)(\sqrt{a}-\sqrt{b})^2-2v(a^{\frac{1}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}})^2-4v(a^{\frac{7}{8}}b^{\frac{1}{8}}-a^{\frac{1}{2}})^2;$$

for $v\in[0,\frac{1}{8}]$ and

$$va+(1-v)b\leq a^vb^{1-v}+(1-v)(\sqrt{a}-\sqrt{b})^2-2v(a^{\frac{1}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}})^2-(1-4v)(a^{\frac{7}{8}}b^{\frac{1}{8}}-a^{\frac{1}{2}})^2;$$

for $v\in[\frac{1}{8},\frac{1}{4}]$.

In this way, by applying the inequality $Inq(4v,a^{\frac{3}{4}}b^{\frac{1}{2}},a,n,k)$ with $k\in\{1,...,2^{f(n)}\}$, conclude the series of refined inequalities of (2.15) by the following formula.

$$\begin{aligned}va+(1-v)b\leq a^vb^{1-v}+(1-v)(\sqrt{a}-\sqrt{b})^2-2v(a^{\frac{1}{4}}b^{\frac{1}{4}}-a^{\frac{1}{2}})^2\\ -G^n_{k,1}(4v,a^{\frac{3}{4}}b^{\frac{1}{2}},a)-...-G^n_{k,f(n)}(4v,a^{\frac{3}{4}}b^{\frac{1}{2}},a);\end{aligned}$$

for $v\in[\frac{k-1}{2^{2+f(n)}},\frac{k}{2^{2+f(n)}}]$. We precise the above argument by the below theorem.

Theorem 2.10. *Suppose that* $a, b > 0$ *and* $v \in [0,1]$. *For* $n, r \in \mathbb{N}$ *let* $s \in \{1, ..., 2^{f(r)}\}$ *and* $k \in \{1, ..., 2^{f(n)}\}$. *Then for* $v \in [\frac{k-1}{2^{f(n)+f(r)+1}} + \frac{s-1}{2^{f(r)+1}}, \frac{k}{2^{f(n)+f(r)+1}} + \frac{s-1}{2^{f(r)+1}}]$ *we have*

$$
\begin{aligned}
va + (1-v)b \leq\, & a^v b^{1-v} + (1-v)(\sqrt{a} - \sqrt{b})^2 - G^r_{s,1}(2v, a^{\frac{1}{2}} b^{\frac{1}{2}}, a) - ... - G^r_{s,f(r)}(2v, a^{\frac{1}{2}} b^{\frac{1}{2}}, a) \\
& - G^n_{k,1}(2^{f(r)+1}v - (s-1), a^{\frac{2^{f(r)+1}-s}{2^{f(r)+1}}} b^{\frac{s}{2^{f(r)+1}}}, a^{\frac{2^{f(r)+1}-s+1}{2^{f(r)+1}}} b^{\frac{s-1}{2^{f(r)+1}}}) \\
& - ... \\
& - G^n_{k,f(n)}(2^{f(r)+1}v - (s-1), a^{\frac{2^{f(r)+1}-s}{2^{f(r)+1}}} b^{\frac{s}{2^{f(r)+1}}}, a^{\frac{2^{f(r)+1}-s+1}{2^{f(r)+1}}} b^{\frac{s-1}{2^{f(r)+1}}}).
\end{aligned}
$$

and for $v \in [\frac{k-1}{2^{f(n)+f(r)+1}} + \frac{s-1}{2^{f(r)+1}} + \frac{1}{2}, \frac{k}{2^{f(n)+f(r)+1}} + \frac{s-1}{2^{f(r)+1}} + \frac{1}{2}]$ *we have*

$$
\begin{aligned}
va + (1-v)b \leq\, & a^v b^{1-v} + v(\sqrt{a} - \sqrt{b})^2 - G^r_{s,1}(2v-1, b, a^{\frac{1}{2}} b^{\frac{1}{2}}) - ... - G^r_{s,f(r)}(2v-1, b, a^{\frac{1}{2}} b^{\frac{1}{2}}) \\
& - G^n_{k,1}(2^{f(r)+1}v - (s-1) - 2^{f(r)}, a^{\frac{2^{f(r)}-s}{2^{f(r)+1}}} b^{\frac{2^{f(r)}+s}{2^{f(r)+1}}}, a^{\frac{2^{f(r)}-(s-1)}{2^{f(r)+1}}} b^{\frac{2^{f(r)}+(s-1)}{2^{f(r)+1}}}) \\
& - ... \\
& - G^n_{k,f(n)}(2^{f(r)+1}v - (s-1) - 2^{f(r)}, a^{\frac{2^{f(r)}-s}{2^{f(r)+1}}} b^{\frac{2^{f(r)}+s}{2^{f(r)+1}}}, a^{\frac{2^{f(r)}-(s-1)}{2^{f(r)+1}}} b^{\frac{2^{f(r)}+(s-1)}{2^{f(r)+1}}});
\end{aligned}
$$

where $G^n_{k,i}$ is introduced in the Theorem 2.1.

Proof .

By considering $RInq^1(v, a, b, r, s)$, let

$$M^1(v, a, b, r, s) := (1-v)(\sqrt{a} - \sqrt{b})^2 - G^r_{s,1}(2v, a^{\frac{1}{2}} b^{\frac{1}{2}}, a) - ... - G^r_{s,f(r)}(2v, a^{\frac{1}{2}} b^{\frac{1}{2}}, a) - va - (1-v)b.$$

Then, it is easy to see that

$$
\begin{aligned}
& M^1(v, a, b, r, s) = \\
& \qquad (2^{f(r)+1}v - (s-1)) a^{\frac{2^{f(r)+1}-s}{2^{f(r)+1}}} b^{\frac{s}{2^{f(r)+1}}} + (s - 2^{f(r)+1}v) a^{\frac{2^{f(r)+1}-s+1}{2^{f(r)+1}}} b^{\frac{s-1}{2^{f(r)+1}}} - 2\sqrt{ab}.
\end{aligned}
$$

Now, by applying $Inq(2^{f(r)+1}v - (s-1), a^{\frac{2^{f(r)+1}-s}{2^{f(r)+1}}} b^{\frac{s}{2^{f(r)+1}}}, a^{\frac{2^{f(r)+1}-s+1}{2^{f(r)+1}}} b^{\frac{s-1}{2^{f(r)+1}}}, n, k)$ on the two first terms of the above relation, we have

$$
\begin{aligned}
va + (1-v)b \leq\, & (1-v)(\sqrt{a} - \sqrt{b})^2 - G^r_{s,1}(2v, a^{\frac{1}{2}} b^{\frac{1}{2}}, a) - ... - G^r_{s,f(r)}(2v, a^{\frac{1}{2}} b^{\frac{1}{2}}, a) \\
& - G^n_{k,1}(2^{f(r)+1}v - (s-1), a^{\frac{2^{f(r)+1}-s}{2^{f(r)+1}}} b^{\frac{s}{2^{f(r)+1}}}, a^{\frac{2^{f(r)+1}-s+1}{2^{f(r)+1}}} b^{\frac{s-1}{2^{f(r)+1}}}) \\
& - ... \\
& - G^n_{k,f(n)}(2^{f(r)+1}v - (s-1), a^{\frac{2^{f(r)+1}-s}{2^{f(r)+1}}} b^{\frac{s}{2^{f(r)+1}}}, a^{\frac{2^{f(r)+1}-s+1}{2^{f(r)+1}}} b^{\frac{s-1}{2^{f(r)+1}}}).
\end{aligned}
$$

Similarly by considering $RInq^1(v, a, b, r, s)$ and letting

$$M^2(v, a, b, r, s) := v(\sqrt{a} - \sqrt{b})^2 - G^r_{s,1}(2v-1, b, a^{\frac{1}{2}} b^{\frac{1}{2}}) - ... - G^r_{s,f(r)}(2v-1, b, a^{\frac{1}{2}} b^{\frac{1}{2}}) - va - (1-v)b,$$

we have

$$
\begin{aligned}
M^2(v, a, b, r, s) =\, & (2^{f(r)+1}v - (s-1) - 2^{f(r)}) a^{\frac{2^{f(r)}-s}{2^{f(r)+1}}} b^{\frac{2^{f(r)}+s}{2^{f(r)+1}}} \\
& + (2^{f(r)} + s - 2^{f(r)+1}v) a^{\frac{2^{f(r)}-(s-1)}{2^{f(r)+1}}} b^{\frac{2^{f(r)}+(s-1)}{2^{f(r)+1}}} - 2\sqrt{ab}.
\end{aligned}
$$

Hence by applying $Inq(2^{f(r)+1}v - (s-1) - 2^{f(r)}, a^{\frac{2^{f(r)}-s}{2^{f(r)+1}}} b^{\frac{2^{f(r)}+s}{2^{f(r)+1}}}, a^{\frac{2^{f(r)}-(s-1)}{2^{f(r)+1}}} b^{\frac{2^{f(r)}+(s-1)}{2^{f(r)+1}}}, n, k)$ on the two first terms of $M^2(v, a, b, r, s)$, the desired result will be obtained. □

Remark 2.11. *By an argument similar to the the Remark 2.8, if we replace the first term of the right hand sides in the inequalities of Theorem 2.7, (that is $a^v b^{1-v}$), with $-a^{1-v}b^v + 2\sqrt{ab}$; we get a sharper one which leads to equality as n tends to infinity.*

Here, by applying Theorem 2.1, we refine the inequality (1.3) to a sharper version which also leads to equality while n tends to infinity, as follows:

Theorem 2.12. *Let $a, b \geq 0$, $n \in \mathbb{N}$ and $k \in \{1, 2, ..., 2^{f(n)} = 2^{2^{n-1}}\}$. If $\frac{k-1}{2^{f(n)+1}} \leq v < \frac{k}{2^{f(n)+1}}$ then*

$$(va + (1-v)b)^2 \geq a^{2v}b^{2(1-v)} + v^2(a-b)^2 + b(G^n_{k,1}(2v, a, b) + ... + G^n_{k,f(n)}(2v, a, b)); \tag{2.17}$$

and if $\frac{k-1}{2^{f(n)+1}} + \frac{1}{2} \leq v < \frac{k}{2^{f(n)+1}} + \frac{1}{2}$ then

$$(va+(1-v)b)^2 \geq a^{2v}b^{2(1-v)} + (1-v)^2(a-b)^2 + a(G^n_{k,1}(2v-1, a, b) + ... + G^n_{k,f(n)}(2v-1, a, b)). \tag{2.18}$$

Where $G^n_{k,i}$ is introduced in the Theorem 2.1.

Proof . One can readily check that

$$(va + (1-v)b)^2 - v^2(a-b)^2 = b(2va + (1-2v)b). \tag{2.19}$$

Now by applying $Inq(2v, a, b, 1, 1)$ on the right hand side of (2.19), we derive that when $0 \leq v \leq \frac{1}{4}$,

$$\begin{aligned}(va + (1-v)b)^2 - v^2(a-b)^2 \ &\geq a^{2v}b^{2(1-v)} + b(2v)(\sqrt{a} - \sqrt{b})^2 \\ &= a^{2v}b^{2(1-v)} + bG^1_{1,1}(2v, a, b).\end{aligned} \tag{2.20}$$

Also, by applying $Inq(2v, a, b, 1, 2)$ on the right hand side of (2.19), we derive that when $v \in [\frac{1}{4}, \frac{1}{2}]$,

$$\begin{aligned}(va + (1-v)b)^2 - v^2(a-b)^2 \ &\geq a^{2v}b^{2(1-v)} + b(1-2v)(\sqrt{a} - \sqrt{b})^2 \\ &= a^{2v}b^{2(1-v)} + bG^1_{2,1}(2v, a, b).\end{aligned}$$

So, we easily see that by applying $Inq(2v, a, b, n, k)$ on the right hand side of (2.19), the inequality (2.17) satisfies for $\frac{k-1}{2^{f(n)+1}} \leq v < \frac{k}{2^{f(n)+1}}$.

Moreover we have

$$(va + (1-v)b)^2 - (1-v)^2(a-b)^2 = a((2v-1)a + 2(1-v)b). \tag{2.21}$$

Thus by applying $Inq(2v-1, a, b, 1, 1)$ on the right hand side of (2.21), we have

$$\begin{aligned}(va + (1-v)b)^2 - (1-v)^2(a-b)^2 \ &\geq a^{2v}b^{2(1-v)} + a(2v-1)(\sqrt{a} - \sqrt{b})^2 \\ &= a^{2v}b^{2(1-v)} + aG^1_{1,1}(2v-1, a, b);\end{aligned}$$

for $v \in [\frac{1}{2}, \frac{3}{4}]$. By applying $Inq(2v-1, a, b, 1, 2)$ on the right hand side of (2.21), we have

$$\begin{aligned}(va + (1-v)b)^2 - (1-v)^2(a-b)^2 \ &\geq a^{2v}b^{2(1-v)} + a(2-2v)(\sqrt{a} - \sqrt{b})^2 \\ &= a^{2v}b^{2(1-v)} + aG^1_{2,1}(2v-1, a, b);\end{aligned}$$

for $v \in [\frac{3}{4}, 1]$. Indeed by applying $Inq(2v-1, a, b, n, k)$ on the right hand side of (2.21), the inequality (2.18) satisfies for $\frac{k-1}{2^{f(n)}} \leq 2v - 1 \leq \frac{k}{2^{f(n)}}$ or equivalently $\frac{k-1}{2^{f(n)+1}} + \frac{1}{2} \leq v < \frac{k}{2^{f(n)+1}} + \frac{1}{2}$.

□ Our refined scalar inequalities can be extended to the matrix version as follows. It is known that for all unitary matrixes $U, V \in M_n(\mathbb{C})$, we have $\|UAV\|_2 = \|A\|_2$. We denote by A^T the transpose of matrix A that $(a_{ij})_{A^T} = (a_{ji})$ and also by A^H the Hermition of matrix A that $(a_{ij})_{A^H} = (\overline{a_{ji}})$.

Definition 2.13. *A matrix $A \in M_N(\mathbb{C})$ is called semidefinite positive if and only if $A^T X A \geq 0$ for all nonzero $X \in \mathbb{R}^N$, which we denoted it by $A \geq 0$ and moreover A is called unitary if and only if $AA^H = A^H A = I$, where I is $N \times N$ identity matrix.*

It is known that, every semidefinite positive matrix $A \in M_n(\mathbb{C})$ has n nonnegative eigenvalues $\{\lambda_1, ..., \lambda_2\}$ and is unitary diagnosable in the sense that there is a unitary matrix U such that $A = U\Lambda U^H$ in which $\Lambda = diag(\lambda_1, ..., \lambda_2)$. It is easy to show that $A^k = U\Lambda^k U^H$ for every $k > 0$. For more details about the matrix analysis see [2].

Theorem 2.14. *Let $A, B \in M_N(\mathbb{C})$ where $A, B \geq 0$, $n \in \mathbb{N}$ and $k \in \{1, 2, ..., 2^{f(n)} = 2^{2^{n-1}}\}$. If $\frac{k-1}{2^{f(n)+1}} \leq v < \frac{k}{2^{f(n)+1}}$ then*

$$\|vA+(1-v)b\|_2^2 \geq \|A^v X B^{1-v}\|_2^2 + v^2\|AX-BX\|_2^2 + \|B^{\frac{1}{2}}\|_2^2(\|F_{k,1}^n(2v, A, B)\|_2^2 + ... + \|F_{k,f(n)}^n(2v, A, B)\|_2^2); \tag{2.22}$$

and if $\frac{k-1}{2^{f(n)+1}} + \frac{1}{2} \leq v < \frac{k}{2^{f(n)+1}} + \frac{1}{2}$ then

$$\|vAX + (1-v)XB\|_2^2 \geq \|A^v X B^{1-v}\|_2^2 + (1-v)^2\|AX - XB\|_2^2 + \|A^{\frac{1}{2}}\|_2^2(\|F_{k,1}^n(2v-1, A, B)\|_2^2 \tag{2.23}$$

$$+ ... + \|F_{k,f(n)}^n(2v-1, A, B)\|_2^2)$$

Where $F_{k,m}^n$ is defined corresponding to $G_{k,m}^n$ in the following way. If $G_{k,m}^n(v, a, b) = (\alpha v + \beta)(a^{\alpha_1} b^{\beta_1} - a^{\alpha_2} b^{\beta_2})^2$ we consider,

$$F_{k,m}^n(v, A, B) = (\alpha v + \beta)(A^{\alpha_1} X B^{\beta_1} - A^{\alpha_2} X B^{\beta_2}). \tag{2.24}$$

Proof . Suppose that for the unitary matrixes $U, V \in M_N(\mathbb{C})$, and diagonal matrixes $\Lambda = diag\{\lambda_1, ..., \lambda_N\}$ and $M = diag\{\mu_1, ..., \mu_N\}$ we have $A = U\Lambda U^H$ and $B = VMV^H$. Hence by letting $Y := U^H X V$, we deduce

$$\begin{aligned} vAX + (1-v)XB &= vU\Lambda U^H X + (1-v)XVMV^H \\ &= vU\Lambda Y V^H + (1-v)UYMV^H \\ &= U(v\Lambda Y + (1-v)YM)V^H \\ &= U[(v\lambda_i + (1-v)\mu_j)y_{ij}]V^H. \end{aligned}$$

So

$$\|vAX + (1-v)XB\|_2^2 = \sum_{i,j=1}^{N} (v\lambda_i + (1-v)\mu_j)^2 y_{ij}^2. \tag{2.25}$$

Now by applying Theorem 2.12 on the right hand side of (2.25) we deduce if $\frac{k-1}{2^{f(n)+1}} \leq v < \frac{k}{2^{f(n)+1}}$ then

$$\|vAX+(1-v)XB\|_2^2 \geq \sum_{i,j=1}^{N} (\lambda_i^{2v}\mu_j^{2(1-v)} + v^2(\lambda_i - \mu_j)^2 + \mu_j(G_{k,1}^n(2v, \lambda_i, \mu_j) + ... + G_{k,f(n)}^n(2v, \lambda_i, \mu_j)))y_{ij}^2. \tag{2.26}$$

Note that

$$\begin{aligned}\|A^v X B^{1-v}\|_2^2 &= \|U\Lambda^v U^H X V M^{1-v} V^H\|_2^2 \\ &= \|U\Lambda^v Y M^{1-v} V^H\|_2^2 = \|\Lambda^v Y M^{1-v}\|_2^2 \\ &= \|[\lambda_i^v \mu_j^{1-v} y_{ij}]\|_2^2 = \Sigma_{i,j=1}^N \lambda_i^{2v} \mu_j^{2(1-v)} y_{ij}^2,\end{aligned}$$

and

$$\begin{aligned}\|AX - XB\|_2^2 &= \|U(\Lambda Y - YM)V^H\|_2^2 \\ &= \|\Lambda Y - YM\|_2^2 = \|[(\lambda_i - \mu_j) y_{ij}]\|_2^2 \\ &= \Sigma_{i,j=1}^N (\lambda_i - \mu_j)^2 y_{ij}^2.\end{aligned}$$

Moreover, from the recursive formula (2.2) for $G_{k,m}^n$, we know that ultimately for some $\alpha, \beta, \alpha_1, \beta_1, \alpha_2, \beta_2$ where $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$, $G_{k,m}^n(v, a, b) = (\alpha v + \beta)(a^{\alpha_1} b^{\beta_1} - a^{\alpha_2} b^{\beta_2})^2$ for every k, m. Thus for every $1 \leq m \leq f(n)$

$$\begin{aligned}\Sigma_{i,j=1}^N \mu_j G_{k,m}^n(2v, \lambda_i, \mu_j) y_{ij}^2 &= (\alpha v + \beta) \Sigma_{i,j=1}^N \mu_j (\lambda_i^{\alpha_1} \mu_j^{\beta_1} - \lambda_i^{\alpha_2} \mu_j^{\beta_2})^2 y_{ij}^2 \\ &= (\alpha v + \beta) \|[\mu_j^{\frac{1}{2}} (\lambda_i^{\alpha_1} \mu_j^{\beta_1} - \lambda_i^{\alpha_2} \mu_j^{\beta_2}) y_{ij}]\|_2^2 \\ &= (\alpha v + \beta) \|M^{\frac{1}{2}} (\Lambda^{\alpha_1} Y M^{\beta_1} - \Lambda^{\alpha_2} Y M^{\beta_2})\|_2^2 \\ &= (\alpha v + \beta) \|B^{\frac{1}{2}} (A^{\alpha_1} X B^{\beta_1} - A^{\alpha_2} Y B^{\beta_2})\|_2^2;\end{aligned}$$

where by explanation of $F_{k,m}^n$ in (2.24), indeed the last expression in the above relations is

$$\|B^{\frac{1}{2}}\|_2^2 \|F_{k,m}^n(2v, A, B)\|_2^2.$$

Therefore the righthand side of (2.26) is equal to the right hand side of (2.22).

Similarly one can deduce the relation (2.23), where we omit its proof for briefness and so the proof is completed. □

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