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Korovkin type approximation of Abel transforms of q-Meyer-König and Zeller operators

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Abstract

In this paper we investigate some Korovkin type approximation properties of the q-Meyer-König and Zeller operators and Durrmeyer variant of the q-Meyer-König and Zeller operators via Abel summability method which is a sequence-to-function transformation and which extends the ordinary convergence. We show that the approximation results obtained in this paper are more general than some previous results. We also obtain the rate of Abel convergence for the corresponding operators. Finally, we conclude our results with some graphical analysis.

Keywords: Meyer-König and Zeller Operators, Abel convergence, Rate of convergence.
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1. Preliminaries

Korovkin type approximation theory aims to provide some simple criteria for the convergence of a sequence of positive linear operators in some senses [24]. There is a number of main motivations in this theory. One of them is obtaining some suitable conditions for the convergence of arbitrary sequence of positive linear operators acting from one certain space to another one. Next motivation is studying some particular conditions for convergence for certain sequence of positive linear operators by using known criteria (see e.g., [4]). It is also possible to introduce the summability theory whose main idea is to make a non-convergent sequence or series to converge in some general senses whenever the sequence of positive linear operators does not converge in the ordinary sense. The leading study

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with this motivation gives criteria for the statistical convergence of a sequence of positive linear operators over $C[a,b]$, the space of all real continuous functions defined on the interval $[a,b]$ [17]. Following that study many authors have given several approximation results via summability theory [7, 13, 38]. Related results with approximation theory may be found in [23, 30, 31, 41, 42, 43, 40].

In 1987, Lupaş [25] introduced the first q-analogue of Bernstein operators and investigated its approximation and shape preserving properties. In 1997, Phillips [35] defined another q-generalization of Bernstein operators. Afterwards, many generalizations of positive linear operators based on q-integers were introduced and studied by several authors, for example, we refer the readers to [1, 2, 3, 5, 6, 9, 15, 26, 28, 29, 32, 33, 50, 52].

Now, let us recall some notations from q- analysis [16, 36]:
For any fixed real number $q>0$, the q-integer $[n]$ is defined by

$$[n]:=[n]_q=\sum_{k=1}^{n}q^{k-1}=\begin{cases}\frac{1-q^n}{1-q} & , \quad q\neq 1\\ n & , \quad q=1\end{cases},$$

where n is a positive integer and $[0]=0$, the q- factorial $[n]!$ of $[n]$ is given with

$$[n]!:=\begin{cases}\prod\limits_{k=1}^{n}[k] & , \quad n=1,2,...\\ 1 & , \quad n=0.\end{cases}$$

For integers $n\geq r\geq 0$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n\\ r\end{bmatrix}_q=\frac{[n]_q!}{[r]_q!\,[n-r]_q!}$$

and q-shifted factorial is defined by

$$(t;q)_n:=\begin{cases}1 & , \quad n=0\\ \prod\limits_{j=0}^{n-1}\left(1-tq^j\right) & , \quad n=1,2,...\end{cases}.$$

Thomae [46] introduced the q-integral of function f defined on the interval $[0,a]$ as follows:

$$\int_0^a f(t)\,d_qt:=a(1-q)\sum_{n=0}^{\infty}f(aq^n)\,q^n, \qquad 0<q<1.$$

Finally, the q-beta function [46] is defined by

$$B_q(m,n)=\int_0^1 t^{m-1}(qt;q)_{n-1}\,d_qt.$$

The original Meyer-König and Zeller operators were introduced for $f \in C[0,1]$ in 1960 (see [27]). Later, Cheney and Sharma [12] rearranged these operators as follows:

$$M_n(f;x) = \begin{cases} (1-x)^{n+1} \sum\limits_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \begin{pmatrix} n+k \\ k \end{pmatrix} x^k & , \quad x \in [0,1) \\ \\ f(1) & , \quad x = 1. \end{cases} \tag{1.1}$$

Trif [47] defined the q- generalization of the Meyer-König and Zeller operators as

$$M_n^q(f;x) = \begin{cases} \prod\limits_{j=0}^{n} (1-q^j x) \sum\limits_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right) \begin{bmatrix} n+k \\ k \end{bmatrix} x^k & , \quad x \in [0,1) \\ \\ f(1), & , \quad x = 1. \end{cases} \tag{1.2}$$

In [47], the author studied Korovkin type approximation properties, calculated the rate of convergence and also gave a result for monotonicity properties of these operators. Heping [22] proved some approximation results for the operators $(M_n^q f)$ using q-hypergeometric series. Another q-generalization of the classical Meyer-König and Zeller operators can be found in [14]. Besides, Durrmeyer variant of the q-Meyer König and Zeller operators [21] was introduced for $f \in C[0,1]$, $x \in [0,1]$, $n \in \mathbb{N}$ and $\alpha > 0, q \in (0,1]$ as follows:

$$D_n^q(f;x) = \begin{cases} \sum\limits_{k=1}^{\infty} m_{n,k,q}(x) \int\limits_0^1 \frac{1}{B_q(n,k)} t^{k-1} (qt;q)_{n-1} f(t)\, d_q t & , \quad x \in [0,1) \\ + m_{n,0,q}(x) f(0) & , \\ & , \quad x = 1, \\ f(1) \end{cases} \tag{1.3}$$

where

$$m_{n,k,q}(x) = (x;q)_{n+1} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k.$$

The authors investigated some approximation properties with the help of well-known Korovkin's theorem and compute the rate of convergence for these operators in terms of the second-order modulus of continuity [21].

It is well known that if the classical conditions

$$\lim_{n\to\infty} q_n = 1 \text{ and } \lim_{n\to\infty} \frac{1}{[n]} = 0 \tag{1.4}$$

hold, then for each $f \in C[0,1]$ the sequences $(M_n^q f)$ and $(D_n^q f)$ converge uniformly to f over $[0,1]$ (see [21, 47]). Furthermore, we use the norm of the space $C[a,b]$ defined for any $f \in C[a,b]$ by

$$||f|| := \sup_{a \le x \le b} |f(x)|. \tag{1.5}$$

In the present paper, taking into account the Abel convergence we obtain some approximation results for the q-Meyer-König and Zeller operators and Durrmeyer variant of q-Meyer-König and Zeller operators. We also study the rate of the convergence of these operators. We also show that the results obtained in this paper are stronger than some previous ones.

Let $x = (x_j)$ be a real sequence. If the series

$$\sum_{j=0}^{\infty} x_j y^j \tag{1.6}$$

is convergent for any $y \in (0,1)$ and

$$\lim_{y \to 1^-} (1-y) \sum_{j=0}^{\infty} x_j y^j = \alpha$$

then x is said to be Abel convergent to real number α [37]. Korovkin type approximation via Abel convergence and other power series methods may be found in [8, 10, 11, 34, 44, 45, 48, 49, 51].

The fact given in the following remark helps us throught the paper:

Remark 1.1. *Let (f_n) be a sequence in $C[0,1]$. If there exists a positive integer n_0 such that*

$$\lim_{y \to 1^-} (1-y) \left\| \sum_{j=n_0}^{\infty} f_j y^j \right\| = 0$$

then it is not difficult to see that

$$\lim_{y \to 1^-} (1-y) \left\| \sum_{j=0}^{\infty} f_j y^j \right\| = 0,$$

i.e., while studying the Abel convergence finitely many terms do not make sense as with the ordinary convergence.

Before studying the announced approximation properties of the operators, we recall some well-known lemmas:

Lemma 1.2. *[47] Let $n \geq 3$ be a positive integer. Then the followings hold for the operators* (1.2) :

$$M_n^q (e_0; x) = 1 \tag{1.7}$$

$$M_n^q (e_1; x) = x \tag{1.8}$$

$$x^2 \leq M_n^q (e_2; x) \leq \frac{x}{[n-1]} + x^2 \tag{1.9}$$

where $e_i(x) = x^i$ for $i = 0, 1, 2$.

Lemma 1.3. *[21] Let $n \geq 3$ be a positive integer. Then the followings hold for the operators* (1.3) :

$$D_n^q (e_0; x) = 1 \tag{1.10}$$

$$D_n^q (e_1; x) = x \tag{1.11}$$

$$D_n^q (e_2; x) = x^2 + \frac{[2]\, x (1-x)(1-q^n x)}{[n-1]} - E_{n,q}(x), \tag{1.12}$$

where

$$0 \leq E_{n,q}(x) \leq \frac{x\,[2]\,[3]\, q^{n-1}}{[n-1]\,[n-2]} (1-x)(1-qx)(1-q^n x).$$

The following lemma can be proved easily:

Lemma 1.4. *Let $n \geq 3$ be a positive integer. Then we have*

$$\begin{aligned} D_n^q(e_2;x) - x^2 &\geq \frac{[2]\,x\,(1-x)\,(1-q^n x)}{[n-1]} \\ &- \frac{x\,[2]\,[3]\,q^{n-1}}{[n-1]\,[n-2]}(1-x)\,(1-qx)\,(1-q^n x) \\ &\geq 0. \end{aligned}$$

2. Main Results

2.1. Abel Transform of the sequence (M_n^q)

In this subsection, we study Korovkin type approximation of the operators (M_n^q) defined with (1.2) by considering the Abel method. Throughout this subsection, we study with the sequence (q_n) such that $0 < q_n \leq 1$ and $q_0 = 0$ and we define $M_0^q f = 0$ for any $f \in C[0,1]$.

Unver (see [48], Theorem 1) proved the following Korovkin-type theorem via Abel summability method.

Theorem 2.1. *Let (L_n) be a sequence of positive linear operators from $C[a,b] \to B[a,b]$ such that $\sum_{n=0}^{\infty} \|L_n(e_0)\|\, y^n < \infty$ for any $y \in (0,1)$. Then for any $f \in C[a,b]$ we have*

$$\lim_{y\to 1^-}(1-y)\left\|\sum_{n=0}^{\infty}(L_n f - f)\,y^n\right\| = 0$$

if and only if

$$\lim_{y\to 1^-}(1-y)\left\|\sum_{n=0}^{\infty}(L_n e_i - e_i)\,y^n\right\| = 0,\ \ i = 0,1,2. \tag{2.1}$$

We are now ready to prove the following theorem:

Theorem 2.2. *If the sequence $\left(\frac{1}{[n-1]}\right)_{n=3}^{\infty}$ is Abel null then for each $f \in C[0,1]$ we have*

$$\lim_{y\to 1^-}(1-y)\left\|\sum_{n=0}^{\infty}(M_n^q f - f)\,y^n\right\| = 0.$$

Proof . From Lemma 1.2 we see that $\sum_{n=0}^{\infty} \|M_n^q(e_0)\|\, y^n < \infty$ for any $y \in (0,1)$. If we consider Theorem 2.1, it suffices to show that (2.1) holds for (M_n^q). Now, considering Lemma 1.2 , one can get for $i = 0,1$ that

$$\lim_{y\to 1^-}(1-y)\left\|\sum_{n=0}^{\infty}(M_n^q e_i - e_i)\,y^n\right\| = 0.$$

Moreover, using (1.9), we have for $n \geq 3$ that

$$0 \leq M_n^q(e_2;x) - x^2 \leq \frac{x}{[n-1]}$$

which gives

$$
\begin{aligned}
0 &\leq (1-y)\left\|\sum_{n=3}^{\infty}\left(M_n^q e_2 - e_2\right)y^n\right\| \\
&\leq (1-y)\sup_{0\leq x\leq 1}\sum_{n=3}^{\infty}\frac{x}{[n-1]}y^n \\
&\leq (1-y)\sum_{n=3}^{\infty}\frac{y^n}{[n-1]}.
\end{aligned}
$$

Finally, we have

$$
\lim_{y\to 1^-}(1-y)\left\|\sum_{n=3}^{\infty}\left(M_n^q e_2 - e_2\right)y^n\right\| = 0.
$$

Hence, considering Remark 1.1 we get

$$
\lim_{y\to 1^-}(1-y)\left\|\sum_{n=0}^{\infty}\left(M_n^q e_2 - e_2\right)y^n\right\| = 0
$$

which conclude the proof. □

The following example shows that the conditions of Theorem 2.2 are weaker than the classical conditions:

Example 2.3. *It is not difficult to see that the classical conditions* (1.4) *entail that the sequence* $\left(\frac{1}{[n-1]}\right)_{n=3}^{\infty}$ *is Abel null. Conversely, if we define the sequence* (q_n) *with*

$$
q_n := \begin{cases} 0 & , \ n \text{ is a perfect cube} \\ 1 & , \ \text{otherwise} \end{cases}
$$

then (q_n) *does not satisfy the classical conditions. Besides, we have for any* $n\geq 3$ *that*

$$
\frac{1}{[n-1]} = \begin{cases} 1 & , \ n \text{ is a perfect cube} \\ \frac{1}{n-1} & , \ \text{otherwise.} \end{cases}
$$

Now, since the sequence $\left(\frac{1}{[n-1]}\right)_{n=3}^{\infty}$ *is bounded and statistically convergent to zero it is Abel null [37, 39].*

2.2. Abel Transform of the operators (D_n^q)

In this subsection, we study Korovkin type approximation of the operators (D_n^q) defined with (1.3) by considering the Abel method as well.

Throughout this subsection, we study with the sequence (q_n) such that $0 < q_n < 1$ and $q_0 = 0$ and we define $D_0^q f = 0$ for any $f \in C[0,1]$.

Theorem 2.4. *If the sequence* $\left(\frac{[2]}{[n-1]}\right)_{n=3}^{\infty}$ *is Abel null then for each* $f\in C[0,1]$ *we have*

$$
\lim_{y\to 1^-}(1-y)\left\|\sum_{n=0}^{\infty}\left(D_n^q f - f\right)y^n\right\| = 0.
$$

Proof . Lemma 1.3 implies that $\sum_{n=0}^{\infty}\left\|D_n^q(e_0)\right\|y^n<\infty$. From Theorem 2.1, it sufficies to show that (2.1) holds for (D_n^q). Using (1.10) and (1.11) , we obtain for $i=0,1$ that

$$\lim_{y\to 1^-}(1-y)\left\|\sum_{n=0}^{\infty}\left(D_n^q e_i-e_i\right)y^n\right\|=0.$$

On the other hand from (1.12) and Lemma 1.4 we have for any $n\geq 3$ that

$$\begin{aligned}&\frac{[2]\,x\,(1-x)\,(1-q^n x)}{[n-1]}-\frac{x\,[2]\,[3]\,q^{n-1}}{[n-1]\,[n-2]}\,(1-x)\,(1-qx)\,(1-q^n x)\\&\leq D_n^q\left(e_2;x\right)-x^2\\&\leq\frac{[2]\,x\,(1-x)\,(1-q^n x)}{[n-1]}\end{aligned}$$

which implies

$$\begin{aligned}0&\leq(1-y)\left\|\sum_{n=3}^{\infty}\left(D_n^q e_2-e_2\right)y^n\right\|\\&\leq(1-y)\sup_{0\leq x\leq 1}\sum_{n=3}^{\infty}\left(\frac{[2]\,x\,(1-x)\,(1-q^n x)}{[n-1]}\right)y^n\\&\leq(1-y)\sum_{n=3}^{\infty}\frac{[2]}{[n-1]}y^n.\end{aligned}$$

Now from the hypothesis we obtain

$$\lim_{y\to 1^-}(1-y)\left\|\sum_{n=3}^{\infty}\left(D_n^q e_2-e_2\right)y^n\right\|=0.$$

Therefore, from Remark 1.1 we can write

$$\lim_{y\to 1^-}(1-y)\left\|\sum_{n=0}^{\infty}\left(D_n^q e_2-e_2\right)y^n\right\|=0.$$

which ends the proof. □

Following example shows that the condition of Theorem 2.4 is weaker than the classical conditions (1.4):

Example 2.5. *Note that if the classical conditions* (1.4) *hold then condition of Theorem 2.4 holds. In fact, if* $\lim_{n\to\infty}q_n=1$ *and* $\lim_{n\to\infty}\frac{1}{[n]_n}=0$ *then we have*

$$\lim_{n\to\infty}\frac{[2]}{[n-1]}=\lim_{n\to\infty}\frac{1+q_n}{[n]-q_n^{n-1}}=0.$$

Therefore, it is Abel null. Conversely, consider the sequence (q_n) *given by*

$$q_n:=\begin{cases}\frac{1}{5} & ,\ n\ \textit{is a prime}\\ 1-\frac{1}{n} & ,\ \textit{otherwise.}\end{cases}$$

We see that (q_n) does not satisfy the classical conditions. On the other hand, one can have for any $n \geq 2$ that

$$\frac{[2]}{[n-1]} = \begin{cases} \frac{24}{25}\left(1-\frac{1}{5^{n-1}}\right)^{-1} & , \quad n \text{ is a prime} \\ \frac{\frac{1}{n}\left(2-\frac{1}{n}\right)}{1-\left(1-\frac{1}{n}\right)^{n-1}} & , \quad \text{otherwise.} \end{cases}$$

Thus the sequence $\left(\frac{[2]}{[n-1]}\right)_{n=2}^{\infty}$ is Abel convergent to zero (since it is bounded and statistically null) but not convergent.

2.3. Rate of Abel convergence

In this section, we compute the rate of the Abel convergence by means of the modulus of continuity. The modulus of continuity of $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) = \sup_{\substack{|x-t|\leq\delta \\ x,t\in[0,1]}} |f(x) - f(t)|$$

It is well known that, for any $f \in C[a,b]$,

$$\lim_{\delta\to 0^+} \omega(f,\delta) = 0 \tag{2.2}$$

and for any $\delta > 0$

$$|f(x) - f(t)| \leq \omega(f,\delta)\left(\frac{|x-t|}{\delta} + 1\right) \tag{2.3}$$

and for all $c > 0$

$$\omega(f, c\delta) \leq (1 + \lfloor c \rfloor)\omega(f,\delta)$$

where $\lfloor c \rfloor$ is the greatest integer less than or equal to c. Now we are ready to give the following lemma:

Lemma 2.6. *For any $f \in C[0,1]$ we have*

$$(1-y)\left\|\sum_{n=3}^{\infty}(D_n^q f - f)\, y^n\right\| \leq 2\omega(f, \varphi(y)),$$

where

$$\varphi(y) := \left\{(1-y)\sup_{0\leq x\leq 1}\sum_{n=3}^{\infty} D_n^q (t-x)^2 y^n\right\}^{\frac{1}{2}} \tag{2.4}$$

and the series in (2.4) is convergent for each $y \in (0,1)$.

Proof . By using (1.10), for any $f \in C[0,1]$ and any $\delta > 0$ we can write

$$\begin{aligned}\left|\sum_{n=3}^{\infty}\left(D_n^q(f;x)-f(x)\right)y^n\right| &\le \sum_{n=3}^{\infty}\left(D_n^q\left|f(t)-f(x)\right|;x\right)y^n \\ &\le \sum_{n=3}^{\infty}D_n^q\left(\omega\left(f,\frac{|t-x|}{\delta}.\delta\right);x\right)y^n \\ &\le \sum_{n=3}^{\infty}D_n^q\left(\left(1+\left\lfloor\frac{|t-x|}{\delta}\right\rfloor\right)\omega(f,\delta);x\right)y^n \\ &\le \omega(f,\delta)\sum_{n=3}^{\infty}D_n^q\left(1+\frac{(t-x)^2}{\delta^2};x\right)y^n \\ &\le \omega(f,\delta)\sum_{n=3}^{\infty}D_n^q\left(e_0(t);x\right)y^n \\ &+\frac{1}{\delta^2}\omega(f,\delta)\sum_{n=3}^{\infty}D_n^q\left((t-x)^2;x\right)y^n \\ &\le \omega(f,\delta)\left(\frac{1}{1-y}-y-y^2\right)+ \\ &+\frac{1}{\delta^2}\omega(f,\delta)\sum_{n=3}^{\infty}D_n^q\left((t-x)^2;x\right)y^n.\end{aligned}$$

Thus, we reach to

$$\begin{aligned}(1-y)\left|\sum_{n=3}^{\infty}\left(D_n^q(f;x)-f(x)\right)y^n\right| &\le \omega(f;\delta) \\ &+\frac{1}{\delta^2}\omega(f;\delta)\sum_{n=3}^{\infty}D_n^q\left((t-x)^2;x\right)y^n\end{aligned}$$

Now if we take $\delta = \left\{(1-y)\sup_{0\le x\le 1}\sum\limits_{n=3}^{\infty}D_n^q\left((t-x)^2;x\right)y^n\right\}^{\frac{1}{2}}$, we get

$$0 \le (1-y)\left\|\sum_{n=3}^{\infty}\left(D_n^q f-f\right)y^n\right\| \le 2\omega\left(f,\varphi(y)\right)$$

which completes the proof. □

Remark 2.7. *If the sequence* $\left(\dfrac{[2]}{[n-1]}\right)_{n=2}^{\infty}$ *is Abel summable (need not to be zero) then the series in (2.4) is convergent for each* $y \in (0,1)$.

Using Lemma 2.6 and Remark 1.1 the following theorem which gives the rate of the Abel convergence for (D_n^q) can be proved:

Theorem 2.8. *Let φ be defined as Lemma 2.6 If $\omega(f, \varphi(y)) = o(\mu(y))$ as $y \to 1^-$ then we have*

$$(1-y)\left\|\sum_{n=1}^{\infty}\left(D_n^q f - f\right) y^n\right\| = o(\mu(y)) \text{ as } y \to 1.$$

The rate of Abel convergence for (M_n^q) defined with (1.2) can be proved by using the similar idea for (D_n^q):

Theorem 2.9. *If $\omega(f, \phi(y)) = o(\mu(y))$ as $y \to 1^-$. then we have*

$$(1-y)\left\|\sum_{n=1}^{\infty}\left(M_n^q f - f\right) y^n\right\| = o(\mu(y)) \text{ as } y \to 1^-$$

where

$$\phi(y) := \left\{(1-y)\sup_{0\le x\le 1}\sum_{n=3}^{\infty} M_n^q (t-x)^2 y^n\right\}^{\frac{1}{2}}.$$

3. Conclusion

In this paper we have studied the Abel convergence of the q-Meyer-König and Zeller operators and Durrmeyer variant of the q-Meyer-König and Zeller operators. In both cases we have had stronger results than the classical Korovkin type approximation theorems of the corresponding operators. Thus, it may be useful to consider Theorem 2.2 and Theorem 2.4 when classical conditions fail.

Example 3.1. *Consider the $f \in C[0,1]$ defined by $f(x) = \sin 2\pi x$. Let $q = (q_n)$ be the sequence defined by $q_n = 1 - \frac{1}{n}$ for any $n \in \mathbb{N}$ which satisfies the condition of Theorem 2.2.* ***Figure 1*** *illustrates the Abel convergence of the sequence $(M_n^{q_n} f)$. Note that we have taken the series finite for the sake of convenience of calculation in the computer. The sum of the q-Meyer-König Zeller operators has calculated upto 90 and the sum of the Abel transform has calculated upto 100. With larger integers more consistent curves can be sketched.*

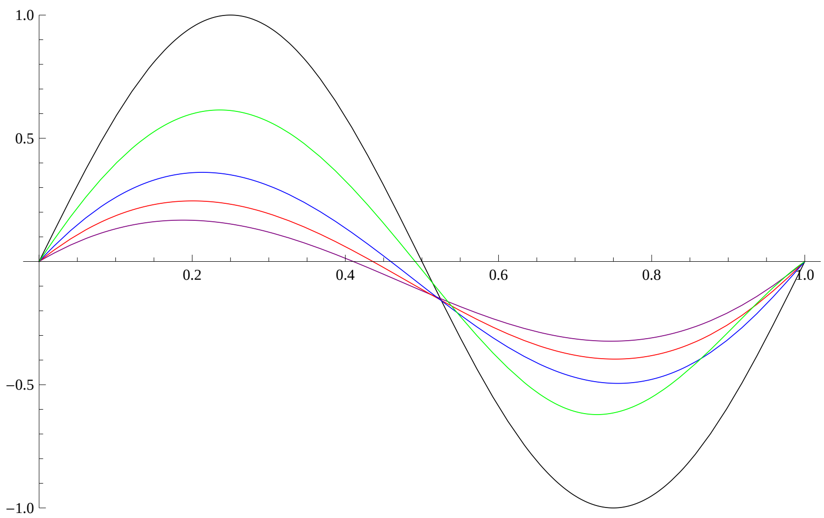

Figure 1: Function itself black, y=0.80 purple, y=0.85 red, y=0.90 blue, y=0.98 green

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References

[1] Acar, T. Quantitative q-Voronovskaya, q-Grüss-Voronovskaya-type results for q-Szász operators, Georgian Math.J., 23 (4) (2016), 459-468.

[2] Acu, A. M. Barbosu, D. and Sofonea, D. F. A Note on A q-Analogue of Stancu-Kantorovich Operators, Miskolc Mathematical Notes, 16(1)(2015), 3-15.

[3] Agratini O. and Doğru, O. Weighted approximation by q-Szász-King type operators, Taiwanese J. of Math., 14(4) (2010), 1283-1296.

[4] Altomare F. and Campiti, M. Korovkin-type approximaton theory and its applications, Walter de Gruyter, Berlin-New York, 1994.

[5] Aral, A. and Gupta, V. q-derivative and applications to the q-Szász Mirakyan operators, Calcolo, 43(2006), 151-170.

[6] Aral, A. and Gupta, V., R.P. Agarwal, Applications of q-Calculus in Operator Theory, Springer, New York, 2013

[7] Atlıhan Ö. G. and Orhan, C. Matrix summability and positive linear operators, Positivity, 11 (3), 387-398, 2007.

[8] Atlıhan Ö. G. and Ünver, M. Abel transforms of convolution operators. Georgian Math. J. 22 (3) (2015), 323-329.

[9] Aydın, D. Aral, A. and Başcanbaz-Tunca, G. A Generalization of Post-Widder Operators Based on q-Integers, Analele Ştiintifice ale Universităţii "Al.I. Cuza" din Iaşi Matematică, Tomul LXII, (2016), f.1.

[10] Braha N. L. Some properties of Baskakov-Schurer-Szász operators via power summability methods, Quaestiones Mathematicae, doi.org/10.2989/16073606.2018.1523248, (2018).

[11] Braha N. L. Some properties of new modified Szász-Mirakyan operators in polynomial weight spaces via power summability methods, Bull. Math. Anal. App. 10 (3) (2018), 53-65.

[12] Cheney E. W. and Sharma, A. Bernstein power series, Canad. J. Math., 16 (1964), 241-253.

[13] Doğru, O. Direct and Inverse Theorems on Statistical Aproximations by Positive Linear Operators, Rocky Mountain J. Math. 38, (2008), No: 6.

[14] Doğru O. and Duman, O. Statistical approximation of Meyer-König and Zeller operators based on q-integers. Publ. Math. Debrecen, 68 (2006)(1-2), 199-214

[15] Doğru O. and Gupta, V. Korovkin-type approximation properties of bivariate q- Meyer-König and Zeller operators. Calcolo, 43; (2006), 51-63.

[16] Ernst, T. The history of q-calculus and a new method, U.U.U.D.M Report 2000, 16, ISSN 1101-3591, (2000) Department of Mathematics, Upsala University.

[17] Gadjiev A. D. and Orhan, C. Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32, (2002), 129-138.

[18] Gairola, A. R., and Mishra, L. N. Rate of Approximation by Finite Iterates of q-Durrmeyer Operators. Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 86(2), 2016, 229-234.

[19] Gairola, A. R., and Mishra, L. N. On the q-Derivatives of a Certain Linear Positive Operators. Iranian Journal of Science and Technology, Transactions A: Science, 42(3), 2018, 1409-1417.

[20] Gandhi, R. B., Deepmala, and Mishra, V. N. Local and global results for modified Szász–Mirakjan operators. Mathematical Methods in the Applied Sciences, 40(7), 2017, 2491-2504.

[21] Govil, N. K. and Gupta, V. Convergence of q-Meyer-König-Zeller-Durrmeyer operators, Advanced Studies in Contemporary Mathematics, 19(1)(2009), 97-108.

[22] Heping, W. Properties of convergence for the q-Meyer-König-Zeller operators. J. Math. Anal. Appl. 335(2)(2007), 1360-1373.

[23] Homaeipour, S., and Razani, A. Convergence Theorems for Equilibrium Problems and Relatively Nonexpansive Multi-Valued Mappings in Banach Spaces. Numerical functional analysis and optimization, 33(12), 2012, 1386-1398.

[24] Korovkin, P. P. Convergence of linear positive operators in the spaces of continuous functions (Russian), Doklady Akad. Nauk. SSSR (N.S)

[25] Lupaş, A. A q-analogue of the Bernstein operators. Seminar on Numerical and Statistical Calculus 9, Universty of Cluj-Napoca, Cluj-Napoca, 1987.

[26] Mahmudov, N.I. Statistical approximation of Baskakov and Baskakov–Kantorovich operators based on the q-integers, Cent. Eur. J. Math. 8(4)(2010), 816-826.

[27] Meyer-König, W. Zeller, K. Bernsteinsche Potenzreihen, Studia Math., 19(1960), 89-94.
[28] Mishra, V. N., Khatri, K. and Mishra, L. N. Statistical approximation by Kantorovich-type discrete q-Betaoperators. Advances in Difference Equations, 2013 (1), 345.
[29] Mishra, V. N., Khatri, K. and Mishra, L.N. Some approximation properties of-Baskakov-Beta-Stancu type operators. Journal of Calculus of Variations, 2013.
[30] Mishra, V. N., Khatri, K., and Mishra, L. N. Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators. Journal of Inequalities and Applications, 2013 (1), 586.
[31] Mishra, V. N., Khatri, K., and Mishra, L. N. On simultaneous approximation for Baskakov-Durrmeyer-Stancu type operators. Journal of Ultra Scientist of Physical Sciences, 24(3), 2012, 567-577.
[32] Mursaleen, M. and Ansari, K. J. Approximation of q-Stancu-Beta operators which preserve x^2, Bulletin of the Malaysian Mathematical Sciences Society, 2015.
[33] Ostrovska, S. q-Bernstein polynomials and their iterates. Journal of Approximation Theory, vol. 123(2003), no. 2, 232-255.
[34] Özgüç, İ. L_p-approximation via Abel convergence. Bull. Belg. Math. Soc. Simon Stevin 22(2)(2015), 271-279.
[35] Phillips, G. M. Bernstein polynomials based on the q-integers, Ann. Numer. Math. 4(1997) 511-518.
[36] Phillips, G. M. Interpolation and approximation by polynomials. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 14. Springer-Verlag, New York, 2003.
[37] Powell, R. E. and Shah, S. M. Summability Theory and Its Applications, Prentice-Hall of India, New Delhi, 1988.
[38] Sakaoğlu, İ and Ünver, M. Statistical approximation for multivariable integrable functions, Miskolc Math. Notes 13(2)(2012), 485-491.
[39] Schoenberg, I. J. The integrability of certain functions and related summability methods. The American Mathematical Monthly, 66 (5) (1959), 361-775.
[40] Sahebi, H. R., and Razani, A. An iterative algorithm for finding the solution of a general equilibrium problem system. Filomat, 28(7), 2014, 1393-1415.
[41] Razani, A., and Goodarzi, Z. Iteration by Cesáro means for quasi-contractive mappings. Filomat, 28(8), 2014, 1575-1584.
[42] Razani, A. Results in Fixed Point Throey, Andishe Zarin Publisher, Qazvin, 2010.
[43] Razani, A. and Moradi, M. Fixed Point Theory in Modular Spaces, Saieh Ghostar Publisher, Qazvin, 2006.
[44] Taş, E. and T. Yurdakadim and Ö. G. Atlıhan, Korovkin type approximation theorems in weighted spaces via power series method, Oper. Matr. 12 (2) (2018), 529-535.
[45] Taş, E. and T. Yurdakadim, Approximation to derivatives of functions by linear operators acting on weighted spaces by power series method. Computational analysis, (2016) , Springer Proc. Math. Stat., 155, Springer, 363-372.
[46] J. Thomae, Beitrage zur Theorie der durch die Heinesche Reihe, J. Reine. Angew. Math. 70 (1869), 258-281.
[47] Trif, T. Meyer-König and Zeller operators based on the q-integers, Rev. Anal. Numer. Theor. Approx., 29(2000), 221-229.
[48] Ünver, M. Abel transforms of positive linear operators. In: ICNAAM 2013. AIP Conference Proceedings, vol. 1558(2013), 1148-1151.
[49] Ünver, M. Abel transforms of positive linear operators on weighted spaces. Bulletin of the Belgian Mathematical Society-Simon Stevin, 21(5) (2014), 813-822.
[50] Söylemez D. On q-Bleimann, Butzer and Hahn-Type Operators, Abstract and Applied Analysis Volume 2015, Article ID 480925.
[51] Söylemez D. and Ünver M. Korovkin type theorems for Cheney–Sharma Operators via summability methods, Res. Math., 73(3)(2017), 1601-1612.
[52] Srivastava, H. M., Mursaleen, M., Alotaibi, A., Nasiruzzaman, Md. and Al-Abied, A. A. H., Some approximation results involving the q-Szász-Mirakjan-Kantrovich type operators via Dunkl's generalization, Math. Meth.App. Sci., 40(15)(2017), 5437-5452.