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An extension and refinement of Hermite-Hadamard inequality and related results

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Abstract

In this paper we establish an extension of Hermite-Hadamard inequality and as a result we obtain the Hermite-Hadamard inequality for fractional integral and logarithmical integral. Also we get a new refinement of it. Some examples are given.

Keywords: Hermite-Hadamard inequality, Riemann-Liouville, Fractional integrals, Integral inequality.
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1. Introduction

Let f be a convex function on $[a, b]$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as the Hermite-Hadamard inequality (H-H inequality). It is well known that the H-H inequality plays an important role in nonlinear analysis. In recent years there have been many extension, generalization and refinement of the inequality (1.1), see [1, 2, 5, 6, 7] and the references therein.
M. Z. Sarikaia et al. [8] proved the following inequalities of H-H type involving RiemannLiouville fractional integrals.

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Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L^1[a,b]$. If f is convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\leqslant\frac{f(a)+f(b)}{2}\tag{1.2}$$

where $\alpha>0$, and

$$J_{a^+}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha)}\int_a^b(b-t)^{\alpha-1}f(t)dt,J_{b^-}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha)}\int_a^b(t-a)^{\alpha-1}f(t)dt$$

In this paper we extend the H-H inequality by a positive function $g\in L^1([0,1])$ and we prove the following inequality

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{2I(b-a)}\int_a^b\left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right]f(x)dx\leqslant\frac{f(a)+f(b)}{2}\tag{1.3}$$

where f is a positive convex function on $[a,b]$ and $I=\int_0^1 g(t)dt$.
As a result of (1.3) we obtain inequality (1.2) and also the following inequality,

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leqslant\frac{1}{2(b-a)\Gamma(\alpha)}\int_a^b\left[\ln^{\alpha-1}\frac{b-a}{x-a}+\ln^{\alpha-1}\frac{b-a}{b-x}\right]f(x)dx\\&\leqslant\frac{f(a)+f(b)}{2}\end{aligned}\tag{1.4}$$

In Theorem 2.4 and Corollaries 2.5, 2.6 we get a new refinement of the inequalities (1.2), (1.3) and (1.4).
Finally we will give some examples via the convexity of $f(x)=x^n$.

2. Main results

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a positive convex function on $[a,b]$ and $g:[0,1]\to\mathbb{R}$ be a positive function such that $g\in L^1([0,1])$. Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{2I(b-a)}\int_a^b\left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right]f(x)dx\leqslant\frac{f(a)+f(b)}{2}\tag{2.1}$$

where $I=\int_0^1 g(x)dx$.

Proof . Since f is convex on $[a,b]$, by change of variable $x=tb+(1-t)a$ we have

$$\begin{aligned}&\frac{1}{2I(b-a)}\int_a^b\left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right]f(x)dx=\frac{1}{2I(b-a)}\int_0^1[g(t)+g(1-t)]\\&f(tb+(1-t)a)(b-a)dt\leqslant\frac{1}{2I}\int_0^1[g(t)+g(1-t)]\left(tf(b)+(1-t)f(a)\right)dt\\&=\frac{f(b)}{2I}\int_0^1 t\left[g(t)+g(1-t)\right]dt+\frac{f(a)}{2I}\int_0^1(1-t)\left[g(t)+g(1-t)\right]dt\end{aligned}$$

By easy calculation we see that

$$\int_0^1 t\,[g(t)+g(1-t)]\,dt = \int_0^1 (1-t)\,[g(t)+g(1-t)]\,dt = \int_0^1 g(t)dt = I$$

So

$$\frac{1}{2I(b-a)}\int_a^b \left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right] f(x)dx \leqslant \frac{f(a)+f(b)}{2}$$

For proving the second part of the inequality, considering the convexity of f, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b+(1-t)a+tb}{2}\right) \\ &\leqslant \frac{1}{2}f(ta+(1-t)b)+\frac{1}{2}f((1-t)a+tb) \end{aligned}$$

Multiplying both sides by $g(t)$ and integrating on $[0,1]$ we obtion

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_0^1 g(t)dt &\leqslant \frac{1}{2}\int_0^1 g(t)f(ta+(1-t)b)dt+\frac{1}{2}\int_0^1 g(t)f(tb+(1-t)a)dt \\ &= \frac{1}{2(b-a)}\int_a^b \left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right] f(x)dx \end{aligned}$$

Hence

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2I(b-a)}\int_a^b \left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right] f(x)dx$$

The proof is complete. □

Corollary 2.2. *Let f be a positive convex function on $[a,b]$. Then the following inequalities hold:*

(*i*)

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right] \leqslant \frac{f(a)+f(b)}{2} \quad (\alpha>0) \tag{2.2}$$

(*ii*)

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2(b-a)\Gamma(\alpha)}\int_a^b f(x)\left[\ln^{\alpha-1}\frac{b-a}{x-a}+\ln^{\alpha-1}\frac{b-a}{b-x}\right]dx \leqslant \frac{f(a)+f(b)}{2} \tag{2.3}$$

Proof . (*i*) Let $g(x)=x^{\alpha-1}(\alpha>0)$ on $[0,1]$. Then $I=\int_0^1 g(x)dx=\dfrac{1}{\alpha}$. By Theorem 2.1 we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\alpha}{2(b-a)}\int_a^b \left[\left(\frac{x-a}{b-a}\right)^{\alpha-1}+\left(\frac{b-x}{b-a}\right)^{\alpha-1}\right] f(x)dx \leqslant \frac{f(a)+f(b)}{2}$$

So

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\alpha}{2(b-a)^\alpha}\int_a^b \left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right] f(x)dx \leqslant \frac{f(a)+f(b)}{2}$$

Finally by putting

$$J^{\alpha}_{a^+}f(b) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-x)^{\alpha-1}f(x)dx, J^{\alpha}_{b^-}f(a) = \frac{1}{\Gamma(\alpha)}\int_a^b (x-a)^{\alpha-1}f(x)dx$$

we deduce the Hermite-Hadamard inequality for fractional integral by Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ (see [8])
(ii) Let $g(x) = (-\ln x)^{\alpha-1}(\alpha > 0)$ on $[0, 1]$. Then

$$I = \int_0^1 (-\ln x)^{\alpha-1}dx = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt = \Gamma(\alpha)$$

Using Theorem 2.1 we obtion

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2(b-a)\Gamma(\alpha)}\int_a^b f(x)\left[\ln^{\alpha-1}\frac{b-a}{x-a} + \ln^{\alpha-1}\frac{b-a}{b-x}\right]dx \leqslant \frac{f(a)+f(b)}{2}$$

□ Now we want to refine the right side of the inequalities (2.1), (2.2) and (2.3). First we need the following Lemma.

Lemma 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a positive convex function and $g : [0, 1] \to \mathbb{R}$ be a positive function such that $g \in L^1([0, 1])$. Then the following inequalities hold:*

(i) $\int_a^{\frac{a+b}{2}} g\left(\frac{x-a}{b-a}\right) f(x)dx \leqslant \frac{1}{2}\int_a^b f(x)\left(\int_x^b g\left(\frac{t-a}{2(b-a)}\right)\frac{dt}{t-a}\right)dx$
$= \frac{1}{2}\int_a^b f(x)G\left(\frac{x-a}{b-a}\right)dx$

(ii) $\int_{\frac{a+b}{2}}^b g\left(\frac{b-x}{b-a}\right) f(x)dx \leqslant \frac{1}{2}\int_a^b f(x)\left(\int_a^x g\left(\frac{b-t}{2(b-a)}\right)\frac{dt}{t-a}\right)dx$
$= \frac{1}{2}\int_a^b f(x)G\left(\frac{b-x}{b-a}\right)dx$

(iii) $\int_a^{\frac{a+b}{2}} g\left(\frac{b-x}{b-a}\right) f(x)dx + \int_{\frac{a+b}{2}}^b g\left(\frac{x-a}{b-a}\right) f(x)dx$
$\leqslant (b-a)(f(a)+f(b))\int_{\frac{1}{2}}^1 g(t)dt$

where $G\left(\frac{x-a}{b-a}\right) = \int_x^b g\left(\frac{t-a}{2(b-a)}\right)\frac{dt}{t-a}$ *and* $G\left(\frac{b-x}{b-a}\right) = \int_a^x g\left(\frac{b-t}{2(b-a)}\right)\frac{dt}{t-a}$.

Proof . (i) By change of variable $x = \dfrac{a+t}{2}$, using the left side of Hermite-Hadamard inequality and Fubini's Theorem we obtain

$$\begin{aligned}\int_a^{\frac{a+b}{2}} g\left(\frac{x-a}{b-a}\right) f(x)dx &= \frac{1}{2}\int_a^b g\left(\frac{t-a}{2(b-a)}\right) f\left(\frac{a+t}{2}\right)dt \\ &\leqslant \frac{1}{2}\int_a^b g\left(\frac{t-a}{2(b-a)}\right)\left(\frac{1}{t-a}\int_a^t f(x)dx\right)dt \\ &= \frac{1}{2}\int_a^b f(x)\left(\int_x^b g\left(\frac{t-a}{2(b-a)}\right)\frac{dt}{t-a}\right)dx\end{aligned}$$

Put $\int_x^b g\left(\frac{t-a}{2(b-a)}\right)\frac{dt}{t-a} = G\left(\frac{x-a}{b-a}\right)$, then

$$G\left(\frac{b-x}{b-a}\right) = G\left(\frac{(b+a-x)-a)}{b-a}\right) = \int_{b+\alpha-x}^{b} g\left(\frac{t-a}{2(b-a)}\right)\frac{dt}{t-a}$$
$$= \int_a^x g\left(\frac{b-t}{2(b-a)}\right)\frac{dt}{b-t}$$

The proof of (i) is complete. Using a similar method the part (ii) is clear.
(iii) By change of variable $x = tb + (1-t)a$ and convexity of f we have

$$\int_a^{\frac{a+b}{2}} g\left(\frac{b-x}{b-a}\right) f(x)dx + \int_{\frac{a+b}{2}}^{b} g\left(\frac{x-a}{b-a}\right) f(x)dx$$
$$= (b-a)\int_0^{\frac{1}{2}} g(1-t)f(tb+(1-t)a)dt + (b-a)\int_{\frac{1}{2}}^{1} g(t)f(tb+(1-t)a)dt$$
$$\leqslant (b-a)\int_0^{\frac{1}{2}} g(1-t)(tf(b)+(1-t)f(a))dt + (b-a)\int_{\frac{1}{2}}^{1} g(t)(tf(b)+(1-t)f(a))dt$$

$$= (b-a)f(b)\left[\int_0^{\frac{1}{2}} tg(1-t)dt + \int_{\frac{1}{2}}^{1} tg(t)dt\right]$$
$$+ (b-a)f(a)\left[\int_0^{\frac{1}{2}} (1-t)g(1-t)dt + \int_{\frac{1}{2}}^{1} (1-t)g(t)\right] dt$$
$$= (b-a)(f(a)+f(b))\int_{\frac{1}{2}}^{1} g(t)dt$$

Because by easy calculation we see that

$$\int_0^{\frac{1}{2}} tg(1-t)dt + \int_{\frac{1}{2}}^{1} tg(t)dt = \int_0^{\frac{1}{2}} (1-t)g(1-t)dt + \int_{\frac{1}{2}}^{1} (1-t)g(t)dt = \int_{\frac{1}{2}}^{1} g(t)dt$$

□

Theorem 2.4. *With the assumptions of Theorem 2.1 the following inequalities hold:*

$$\frac{1}{2I(b-a)}\int_a^b \left[g\left(\frac{x-a}{b-a}\right) + g\left(\frac{b-x}{b-a}\right)\right] f(x)dx$$
$$\leqslant \frac{1}{4I(b-a)}\int_a^b \left[G\left(\frac{x-a}{b-a}\right) + G\left(\frac{b-x}{b-a}\right)\right] dx + \frac{\int_{\frac{1}{2}}^{1} g(t)dt}{2I}(f(a)+f(b))$$
$$\leqslant \frac{f(a)+f(b)}{2}, \quad \textit{where } I = \int_0^1 g(t)dt$$

Proof . Using Lemma 2.3 we have

$$\frac{1}{2I(b-a)}\int_a^b\left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right]f(x)dx=$$

$$=\frac{1}{2I(b-a)}\left[\int_a^{\frac{a+b}{2}}\left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right]f(x)dx+\int_{\frac{a+b}{2}}^b\left[g\left(\frac{x-a}{b-a}\right)+g\left(\frac{b-x}{b-a}\right)\right]f(x)dx\right]$$

$$\leqslant\frac{1}{2I(b-a)}\left[\frac{1}{2}\int_a^b f(x)\int_x^b g\left(\frac{t-a}{2(b-a)}\right)\frac{dt}{t-a}dx+\frac{1}{2}\int_a^b f(x)\int_a^x g\left(\frac{b-t}{2(b-a)}\right)\frac{dt}{b-t}dx\right]$$

$$+\left(\frac{f(b)+f(a)}{2I}\right)\int_{\frac{1}{2}}^1 g(t)dt=\frac{1}{4I(b-a)}\left[\int_a^b f(x)\left[G\left(\frac{x-a}{b-a}\right)+G\left(\frac{b-x}{b-a}\right)\right]dx\right]$$

$$+\frac{f(a)+f(b)}{2I}\int_{\frac{1}{2}}^1 g(t)dt \tag{2.4}$$

By change of variable $x=tb+(1-t)a$ and convexity of f we get

$$\int_a^b f(x)\left[G\left(\frac{x-a}{b-a}\right)+G\left(\frac{b-x}{b-a}\right)\right]dx=(b-a)\int_0^1 f(tb+(1-t)a)\left[G(t)+G(1-t)\right]dt$$

$$\leqslant(b-a)\int_0^1(tf(b)+(1-t)f(a))\left[G(t)+G(1-t)\right]dt$$

$$=(b-a)f(b)\left[\int_0^1 tG(t)dt+\int_0^1 tG(1-t)dt\right]$$

$$+(b-a)f(a)\left[\int_0^1(1-t)G(t)dt+\int_0^1(1-t)G(1-t)dt\right]$$

$$=(b-a)(f(a)+f(b))\int_0^1 G(t)dt \tag{2.5}$$

Via integration by parts we have

$$\int_0^1 G(t)dt=tG(t)\big|_0^1-\int_0^1 tG'(t)dt$$

$$=G(1)-\int_0^1 t\left(-\frac{1}{t}g(\frac{t}{2})\right)dt=\int_0^1 g(\frac{t}{2})dt=\int_0^{\frac{1}{2}}2g(t)dt \tag{2.6}$$

$$\left(G(1)=0,G(y)=\int_{a+y(b-a)}^b g\left(\frac{t-a}{2(b-a)}\right)\frac{dt}{t-a},G'(y)=-\frac{1}{y}g(\frac{y}{2})\right)$$

Now by (2.4), (2.5), and (2.6) we obtaion

$$\frac{1}{4I(b-a)}\int_a^b f(x)\left[G\left(\frac{x-a}{b-a}\right)+G\left(\frac{b-x}{b-a}\right)\right]dx+\frac{f(a)+f(b)}{2I}\int_{\frac{1}{2}}^1 g(t)dt$$

$$\leqslant\frac{1}{2I}\left(\int_0^{\frac{1}{2}}g(t)dt\right)(f(a)+f(b))+\frac{f(a)+f(b)}{2I}\int_{\frac{1}{2}}^1 g(t)dt$$

$$=\frac{f(a)+f(b)}{2I}\left(\int_0^{\frac{1}{2}}g(t)dt+\int_{\frac{1}{2}}^1 g(t)dt\right)=\frac{f(a)+f(b)}{2I}\int_0^1 g(t)dt$$

$$=\frac{f(a)+f(b)}{2}$$

□

Corollary 2.5. *Let f be a positive convex function on $[a,b]$ and $\alpha > 0$. Then the following inequalities hold:*

(i) $$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right] \leqslant \frac{\alpha}{1+2^{\alpha}(\alpha-1)(b-a)}\int_a^b f(x)dx+\frac{(\alpha-1)(2^{\alpha}-1)}{1+2^{\alpha}(\alpha-1)}\frac{f(a)+f(b)}{2}\leqslant\frac{f(a)+f(b)}{2}$$

(ii) $$\frac{1}{2(b-a)\Gamma(\alpha)}\int_a^b\left[\ln^{\alpha-1}\frac{b-x}{x-a}+\ln^{\alpha-1}\frac{b-a}{b-x}\right]f(x)dx \leqslant \frac{1}{4(b-a)\Gamma(\alpha+1)}\int_a^b\left[\ln^{\alpha}\frac{2(b-a)}{b-x}+\ln^{\alpha}\frac{2(b-a)}{x-a}-2\ln^{\alpha}2\right]f(x)dx + \frac{f(a)+f(b)}{2\Gamma(\alpha)}\int_0^{\ln 2}t^{\alpha-1}e^{-t}dt\leqslant\frac{f(a)+f(b)}{2}.$$

Proof . (i) In Theorem 2.4 put $g(x)=x^{\alpha-1}(\alpha>0)$ on $[0,1]$.
Then

$$I=\int_0^1 x^{\alpha-1}dx=\frac{1}{\alpha}, G\left(\frac{x-a}{b-a}\right)=\int_x^b\left(\frac{t-a}{2(b-a)}\right)^{\alpha-1}\frac{dt}{t-a}$$
$$=\frac{1}{2^{\alpha-1}(b-a)^{\alpha-1}}\int_x^b(t-a)^{\alpha-2}dt=\frac{1}{2^{\alpha-1}(b-a)^{\alpha-1}(\alpha-1)}(t-a)^{\alpha-1}\Big|_x^b$$
$$=\frac{1}{2^{\alpha-1}(b-a)^{\alpha-1}(\alpha-1)}\left[(b-a)^{\alpha-1}-(x-a)^{\alpha-1}\right]$$

By similar way

$$G\left(\frac{b-x}{b-a}\right)=\frac{1}{2^{\alpha-1}(b-a)^{\alpha-1}(\alpha-1)}\left[(b-a)^{\alpha-1}-(b-x)^{\alpha-1}\right]$$

So

$$\frac{\alpha}{2(b-a)}\int_a^b\left[\left(\frac{x-a}{b-a}\right)^{\alpha-1}+\left(\frac{b-x}{b-a}\right)^{\alpha-1}\right]f(x)dx$$
$$\leqslant\frac{\alpha}{4(b-a)}\int_a^b\frac{1}{2^{\alpha-1}(b-a)^{\alpha-1}(\alpha-1)}\left[2(b-a)^{\alpha-1}-(x-a)^{\alpha-1}-(b-x)^{\alpha-1}\right]f(x)dx$$
$$+\frac{f(a)+f(b)}{2}\alpha\int_{\frac{1}{2}}^1 t^{\alpha-1}dt\leqslant\frac{f(a)+f(b)}{2}\qquad\Rightarrow$$
$$\frac{\alpha}{2(b-a)^{\alpha}}\int_a^b\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]f(x)dx$$
$$\leqslant\frac{\alpha}{2^{\alpha+1}(b-a)^{\alpha}(\alpha-1)}\left[\int_a^b 2(b-a)^{\alpha-1}f(x)dx-\int_a^b\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]f(x)dx\right]$$
$$+\frac{f(a)+f(b)}{2}\left(1-\frac{1}{2^{\alpha}}\right)\leqslant\frac{f(a)+f(b)}{2}\qquad\Rightarrow$$

$$\frac{\alpha}{2(b-a)^\alpha}\int_a^b \left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right] f(x)dx$$
$$\leqslant \frac{\alpha}{2^\alpha(b-a)(\alpha-1)}\int_a^b f(x)dx - \frac{\alpha}{2^{\alpha+1}(b-a)^\alpha(\alpha-1)}\int_a^b \left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right] f(x)dx$$
$$+\frac{f(a)+f(b)}{2}\left(1-\frac{1}{2^\alpha}\right)\leqslant \frac{f(a)+f(b)}{2} \qquad \Rightarrow$$
$$\frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\leqslant \frac{\alpha}{2^\alpha(\alpha-1)(b-a)}\int_a^b f(x)dx$$
$$-\frac{\alpha\Gamma(\alpha)}{2^{\alpha+1}(b-a)^\alpha(\alpha-1)}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]+\frac{f(a)+f(b)}{2}\left(1-\frac{1}{2^\alpha}\right)\leqslant \frac{f(a)+f(b)}{2}$$

Adding $\frac{\Gamma(\alpha+1)}{2^{\alpha+1}(b-a)^\alpha(\alpha-1)}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]$ to both sides we obtion

$$\frac{\Gamma(\alpha+1)\left[2^\alpha(\alpha-1)+1\right]}{2^{\alpha+1}(b-a)^\alpha(\alpha-1)}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]$$
$$\leqslant \frac{\alpha}{2^\alpha(\alpha-1)(b-a)}\int_a^b f(x)dx+\frac{f(a)+f(b)}{2}\left(1-\frac{1}{2^\alpha}\right)$$
$$\leqslant \frac{\Gamma(\alpha+1)}{2^{\alpha+1}(b-a)^\alpha(\alpha-1)}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]+\frac{f(a)+f(b)}{2}$$

Since $\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\leqslant \frac{f(a)+f(b)}{2}$, so

$$\leqslant \frac{1}{2^\alpha(\alpha-1)}\frac{f(a)+f(b)}{2}+\frac{f(a)+f(b)}{2}=\frac{1+2^\alpha(\alpha-1)}{2^\alpha(\alpha-1)}\frac{f(a)+f(b)}{2}$$

Multiplying both sides by $\frac{2^\alpha(\alpha-1)}{1+2^\alpha(\alpha-1)}$ we get

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\leqslant \frac{2^\alpha(\alpha-1)}{1+2^\alpha(\alpha-1)}\frac{\alpha}{2^\alpha(\alpha-1)(b-a)}\int_a^b f(x)dx$$
$$+\frac{2^\alpha(\alpha-1)}{1+2^\alpha(\alpha-1)}\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(a)+f(b)}{2}\leqslant \frac{f(a)+f(b)}{2} \qquad \Rightarrow$$
$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\leqslant \frac{\alpha}{(1+2^\alpha(\alpha-1))(b-a)}\int_a^b f(x)dx$$
$$+\frac{(\alpha-1)(2^\alpha-1)}{1+2^\alpha(\alpha-1)}\frac{f(a)+f(b)}{2}\leqslant \frac{f(a)+f(b)}{2}.$$

(ii) In Theorem 2.4 put $g(x)=(-\ln x)^{\alpha-1}(\alpha>0)$ on $[0,1]$. Then

$$I=\int_0^1(-\ln x)^{\alpha-1}dx=\int_0^\infty t^{\alpha-1}e^{-t}dt=\Gamma(\alpha),$$

$$G\left(\frac{x-a}{b-a}\right)=\int_x^b\left(\ln\frac{2(b-a)}{t-a}\right)^{\alpha-1}\frac{dt}{t-a}=-\frac{1}{\alpha}\ln^\alpha\frac{2(b-a)}{t-a}\bigg|_x^b$$
$$=-\frac{1}{\alpha}\ln^\alpha 2+\frac{1}{\alpha}\ln^\alpha\frac{2(b-a)}{x-a}$$

By similar way

$$G\left(\frac{b-x}{b-a}\right) = -\frac{1}{\alpha}\ln^{\alpha} 2 + \frac{1}{\alpha}\ln^{\alpha}\frac{2(b-a)}{b-x}$$

So

$$\begin{aligned}
&\frac{1}{2(b-a)\Gamma(\alpha)}\int_a^b \left[\ln^{\alpha-1}\frac{b-a}{x-a} + \ln^{\alpha-1}\frac{b-a}{b-x}\right] f(x)dx \\
&\leqslant \frac{1}{4(b-a)\Gamma(\alpha)}\int_a^b \left[-\frac{2}{\alpha}\ln^{\alpha} 2 + \frac{1}{\alpha}\ln^{\alpha}\frac{2(b-a)}{b-x} + \frac{1}{\alpha}\ln^{\alpha}\frac{2(b-a)}{x-a}\right] f(x)dx \\
&+ \frac{f(a)+f(b)}{2\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}(-\ln t)^{\alpha-1}dt \leqslant \frac{f(a)+f(b)}{2} \qquad \Rightarrow \\
&\frac{1}{2(b-a)\Gamma(\alpha)}\int_a^b \left[\ln^{\alpha-1}\frac{b-a}{x-a} + \ln^{\alpha-1}\frac{b-a}{b-x}\right] f(x)dx \\
&\leqslant \frac{1}{4(b-a)\Gamma(\alpha+1)}\int_a^b \left[\ln^{\alpha}\frac{2(b-a)}{b-x} + \ln^{\alpha}\frac{2(b-a)}{x-a} - 2\ln^{\alpha} 2\right] f(x)dx \\
&+ \frac{f(a)+f(b)}{2\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}(-\ln t)^{\alpha-1}dt \leqslant \frac{f(a)+f(b)}{2}.
\end{aligned}$$

□ In the following corollary we obtain a new refinement of Hermite-Hadamard inequality, when f is a positive function.

Corollary 2.6. *Let f be a positive convex function on $[a, b]$. Then*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{1}{2(b-a)}\int_a^b \left[\ln\frac{b-a}{x-a} + \ln\frac{b-a}{b-a}\right] f(x)dx \\
\leqslant \frac{f(a)+f(b)}{2}
\end{aligned}$$

Proof . we have

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^{b} f(x)dx\right]$$

By change of variable $x = \dfrac{a+t}{2}$ and using Hermite-Hadamard inequality and Fubini' Theorem we obtain

$$\begin{aligned}
\int_a^{\frac{a+b}{2}} f(x)dx &= \frac{1}{2}\int_a^b f\left(\frac{a+t}{2}\right)dt \leqslant \frac{1}{2}\int_a^b \left(\frac{1}{t-a}\int_a^t f(x)dx\right)dt \\
&= \frac{1}{2}\int_a^b \left(f(x)\int_x^b \frac{dt}{t-a}\right)dx = \frac{1}{2}\int_a^b f(x)\ln\frac{b-a}{x-a}dx
\end{aligned}$$

By similar way

$$\int_{\frac{a+b}{2}}^{b} f(x)dx \leqslant \frac{1}{2}\int_a^b f(x)\ln\frac{b-a}{b-x}dx$$

So

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{1}{2(b-a)}\int_a^b \left[\ln\frac{b-a}{x-a} + \ln\frac{b-a}{b-x}\right] f(x)dx$$
$$\leqslant \frac{f(a)+f(b)}{2}$$

The last inequality is obvious by Corollary 2.2 (ii) for $\alpha = 2$. □

3. Examples

1. Let $f(x) = x^n$ and $\alpha = n(n \in \mathbb{N})$. By change of variable $x = tb + (1-t)a$ we have

$$J_{a^+}^n f(b) = \frac{1}{\Gamma(n)}\int_a^b (b-x)^{n-1}x^n dx = \frac{1}{\Gamma(n)}\int_0^1 (1-t)^{n-1}(b-a)^{n-1}$$
$$(tb+(1-t)a)^n\,(b-a)dt = \frac{(b-a)^n}{\Gamma(n)}\int_0^1 (1-t)^{n-1}\sum_{k=0}^{n}\binom{n}{k}(tb)^{n-k}((1-t)a)^k$$
$$= \frac{(b-a)^n}{\Gamma(n)}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\int_0^1 t^{n-k}(1-t)^{n+k-1}dt$$
$$= \frac{(b-a)^n}{\Gamma(n)}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k B(n-k+1, k+n)$$

Where B is Beta function and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

So

$$J_{a^+}^n f(b) = \frac{(b-a)^n}{\Gamma(n)}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\frac{(n-k)!(n+k-1)!}{(2n)!}$$
$$= \frac{(b-a)^n}{2n\Gamma(n)}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\frac{1}{\binom{2n-1}{n-k}}$$

By similar way we obtian

$$J_{b^-}^n f(a) = \frac{(b-a)^n}{2n\Gamma(n)}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\frac{1}{\binom{2n-1}{k}}$$

Hence

$$J_{a^+}^n f(b) + J_{b^-}^n f(a) = \frac{(b-a)^n}{2n\Gamma(n)}\sum_{k=0}^{n}b^{n-k}a^k\left[\frac{\binom{n}{k}}{\binom{2n-1}{n-k}} + \frac{\binom{n}{k}}{\binom{2n-1}{k}}\right]$$

Now, by Corollary 2.5 (i) we obtion

$$\left(\frac{a+b}{2}\right)^n \leqslant \frac{\Gamma(n+1)}{2(b-a)^n}\frac{(b-a)^n}{2n\Gamma(n)}\sum_{k=0}^{n}b^{n-k}a^k\left[\frac{\binom{n}{k}}{\binom{2n-1}{n-k}} + \frac{\binom{n}{k}}{\binom{2n-1}{k}}\right]$$
$$\leqslant \frac{n}{(1+2^n(n-1))(b-a)}\frac{1}{n+1}(b^{n+1}-a^{n+1}) + \frac{(n-1)(2^n-1)}{1+2^n(n-1)}\cdot\frac{a^n+b^n}{2} \leqslant \frac{a^n+b^n}{2}$$

Via means' notations we get

$$A^n(a,b) \leqslant \frac{1}{4}\sum_{k=0}^{n} b^{n-k}a^k \left[\frac{\binom{n}{k}}{\binom{2n-1}{n-k}} + \frac{\binom{n}{k}}{\binom{2n-1}{k}}\right]$$
$$\leqslant \frac{n}{1+2^n(n-1)}L_n^n(a,b) + \frac{(n-1)(2^n-1)}{1+2^n(n-1)}A(a^n,b^n) \leqslant A(a^n,b^n)$$

where $A(a,b) = \dfrac{a+b}{2}$, $L_n(a,b) = \left[\dfrac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}}$ $(a,b>0)$

2. In Corollary 2.6 Let $f(x) = x^n (b > a > 0)$. Then

$$\left(\frac{a+b}{2}\right)^n \leqslant \frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} \leqslant \frac{1}{b-a}\int_a^b \left(\ln\frac{b-a}{x-a} + \ln\frac{b-a}{b-x}\right)x^n dx \leqslant \frac{a^n+b^n}{2} \tag{3.1}$$

Integrating by parts we have

$$\int_a^b x^n \ln(x-a)dx = \frac{1}{n+1}x^{n+1}\ln(x-a)\Big|_a^b - \frac{1}{n+1}\int_a^b \frac{x^{n+1}-a^{n+1}+a^{n+1}}{x-a}dx$$
$$= \frac{1}{n+1}b^{n+1}\ln(b-a) - \frac{1}{n+1}\int_a^b \left[\left(x^n + x^{n-1}a + \ldots + a^n\right) + \frac{a^{n+1}}{x-a}\right]dx$$
$$= \frac{1}{n+1}b^{n+1}\ln(b-a) - \frac{1}{n+1}\int_a^b \left(\sum_{k=0}^{n} x^{n-k}a^k + \frac{a^{n+1}}{x-a}\right)dx$$
$$= \frac{1}{n+1}b^{n+1}\ln(b-a) - \frac{1}{n+1}\left[\sum_{k=0}^{n} a^k \frac{1}{n-k+1}\left(b^{n-k+1}-a^{n-k+1}\right) + a^{n+1}\ln(b-a)\right]$$
$$= \frac{\ln(b-a)}{n+1}(b^{n+1}-a^{n+1}) - \frac{1}{n+1}\sum_{k=0}^{n}\frac{a^k\left(b^{n-k+1}-a^{n-k+1}\right)}{n-k+1}$$

Notice that $\lim\limits_{x\to a^+}\left[x^{n+1}-a^{n+1}\right]\ln(x-a) = 0$. By similar way we get

$$\int_a^b x^n \ln(b-x)dx = \frac{\ln(b-a)}{n+1}(b^{n+1}-a^{n+1}) - \frac{1}{n+1}\sum_{k=0}^{n}\frac{b^k(b^{n-k+1}-a^{n-k+1})}{n-k+1}$$

So

$$\frac{1}{b-a}\int_a^b \left(\ln\frac{b-a}{x-a} + \ln\frac{b-a}{b-x}\right)x^n dx = \frac{2(b^{n+1}-a^{n+1})\ln(b-a)}{(b-a)(n+1)}$$
$$-\frac{1}{b-a}\int_a^b \left(\ln(x-a)+\ln(b-x)\right)x^n dx$$

$$= \frac{2(b^{n+1}-a^{n+1})\ln(b-a)}{(b-a)(n+1)} - \frac{1}{b-a}\left[\frac{2\ln(b-a)}{n+1}(b^{n+1}-a^{n+1})\right.$$
$$\left. -\frac{1}{n+1}\sum_{k=0}^{n}\frac{\left(b^{n-k+1}-a^{n-k+1}\right)\left(b^k+a^k\right)}{n-k+1}\right] = \frac{1}{(b-a)(n+1)}\sum_{k=0}^{n}\frac{(b^{n-k+1}-a^{n-k+1})(a^k+b^k)}{n-k+1}$$

Now by (3.1) we have

$$\left(\frac{a+b}{2}\right)^n \leqslant \frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} \leqslant \frac{1}{n+1}\sum_{k=0}^{n}\frac{b^{n-k+1}-a^{n-k+1}}{(n-k+1)(b-a)}(a^k+b^k) \leqslant \frac{a^n+b^n}{2}$$

Finally, via means' notations we obtain

$$A^n(a,b) \leqslant L_n^n(a,b) \leqslant \frac{1}{n+1}\sum_{k=0}^{n}(a^k+b^k)L_{n-k+1}(a,b) \leqslant A(a^n,b^n)$$

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