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Coincidence point results for graph preserving hybrid pair of mappings

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Abstract

We analyze the existence of coincidence points for hybrid pair of mappings defined on b-metric spaces endowed with a digraph G. Our main result is an extension of the well-known Nadler's fixed point theorem. Finally, we present a coincidence point theorem for mappings satisfying a general contractive condition of integral type. We include some examples to examine the validity of our results.

Keywords: b-metric, digraph, lower semicontinuous function, coincidence point.
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1. Introduction

Banach contraction principle [7] is a very popular tool of mathematics in solving many problems in several branches of mathematics. Because of its importance, it has been extended and generalized in many ways(see [1, 2, 6, 14, 22, 23, 25, 27, 28, 29, 30] and references therein). Among all these, an interesting generalization was given by Nadler [28]. In fact, Nadler extended the Banach contraction principle from the single-valued mappings to the multi-valued mappings. Later on, hybrid fixed point theory for nonlinear single-valued and multi-valued mappings takes a vital role in many aspects. In 1989, Bakhtin [4] introduced the concept of b-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in metric spaces to b-metric spaces.

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In recent investigations, the study of fixed point theory combining a graph is a new development in the domain of contractive type multi-valued theory. Starting from these considerations, the study of fixed points and common fixed points of mappings satisfying a certain contractive type condition endowed with a graph attracted many researchers, see for examples [9, 10, 11, 16, 17, 21, 31]. Inspired and motivated by the results in [5, 14, 18], we introduce the concept of (g, T, G)-lower semicontinuous functions in b-metric spaces and obtain some coincidence point results for hybrid pair of single-valued and multi-valued mappings in b-metric spaces with a digraph. Our results extend, unify and generalize several well-known comparable results in the literature. Finally, some examples are provided to justify the validity of our results.

2. Some Basic Concepts

In this section, we collect some basic notations, definitions and results in b-metric spaces which will be used throughout the paper.

Definition 2.1. *[13] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$*.*

The pair (X, d) is called a b-metric space.

It is to be noted that the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above fact.

Example 2.2. *[24] Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that*

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.3. *[3] Let $p \in (0, 1)$. Then the space $L^p([0, 1])$ of all real functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 | f(t) |^p \, dt < \infty$ endowed with the functional $d : L^p([0, 1]) \times L^p([0, 1]) \to \mathbb{R}$ given by*

$$d(f, g) = \left(\int_0^1 | f(t) - g(t) |^p \, dt \right)^{\frac{1}{p}}$$

for all $f,\ g \in L^p([0, 1])$ is a b-metric space with $s = 2^{\frac{1}{p}}$.

Definition 2.4. *[12] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) (x_n) *converges to x if and only if* $\lim_{n\to\infty} d(x_n, x) = 0$*. We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x (n \to \infty)$*.*

(ii) (x_n) *is Cauchy if and only if* $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) (X, d) *is complete if and only if every Cauchy sequence in* X *is convergent.*

Remark 2.5. *[12] In a b-metric space* (X, d)*, the following assertions hold:*

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

Definition 2.6. *[20] Let* (X, d) *be a b-metric space. A subset* $A \subseteq X$ *is said to be open if and only if for any* $a \in A$*, there exists* $\epsilon > 0$ *such that the open ball* $B(a, \epsilon) \subseteq A$*. The family of all open subsets of* X *will be denoted by* τ*.*

Theorem 2.7. *[20]* τ *defines a topology on* (X, d)*.*

Theorem 2.8. *[20] Let* (X, d) *be a b-metric space and* τ *be the topology defined above. Then for any nonempty subset* $A \subseteq X$ *we have*

(i) A *is closed if and only if for any sequence* (x_n) *in* A *which converges to* x*, we have* $x \in A$*;*

(ii) if we define $\overline{A}$ *to be the intersection of all closed subsets of* X *which contains* A*, then for any* $x \in \overline{A}$ *and for any* $\epsilon > 0$*, we have* $B(x, \epsilon) \cap A \neq \emptyset$*.*

Definition 2.9. *[26] Let* (X, d) *be a b-metric space and* A *be a nonempty subset of* X*. The diameter of* A*, denoted by* $\delta(A)$*, is defined by* $\delta(A) = sup\{d(x, y) : x,\ y \in A\}$*. The subset* A *is said to be bounded if* $\delta(A)$ *is finite.*

Let (X, d) be a b-metric space. Let $CB(X)$ be the set of all nonempty closed bounded subsets of X and $CL(X)$ be the set of all nonempty closed subsets of X. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$, where 2^X denotes the collection of all nonempty subsets of X. For $A,\ B \in CL(X)$, define

$$
\begin{aligned}
H(A, B) &= max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},\ if\ the\ maximum\ exists;\\
&= \infty,\ otherwise
\end{aligned}
$$

where $d(x, B) = inf\{d(x, y) : y \in B\}$. Such a map H is called the generalized Hausdorff b-distance induced by d.

Definition 2.10. *Let* (X, d) *be a b-metric space and* $T : X \to CL(X)$ *and* $g : X \to X$ *be two mappings. If* $y = gx \in Tx$ *for some* x *in* X*, then* x *is called a coincidence point of* T *and* g *and* y *is called a point of coincidence of* T *and* g*.*

We next review some basic notions in graph theory.

Let (X, d) be a b-metric space. We assume that G is a digraph with the set of vertices $V(G) = X$ and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x, x) : x \in X\}$. We also assume that G has no parallel edges. So we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [8, 15, 19]. If x, y are vertices of the digraph G, then a path in G from x to y of length n $(n \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^{n}$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

Definition 2.11. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. Then the mapping $f : X \to X$ is called edge preserving if*

$$x, y \in X,\ (x, y) \in E(\tilde{G}) \Rightarrow (fx, fy) \in E(\tilde{G}).$$

Definition 2.12. *Let (X, d) be a b-metric space with a graph $G = (V(G), E(G))$ and let f, $g : X \to X$ be two mappings. Then f is called edge preserving w.r.t. g if*

$$x, y \in X,\ (gx, gy) \in E(\tilde{G}) \Rightarrow (fx, fy) \in E(\tilde{G}).$$

Definition 2.13. *Let (X, d) be a b-metric space with a graph $G = (V(G), E(G))$. Then the mapping $T : X \to CL(X)$ is called edge preserving if*

$$x, y \in X,\ x \neq y,\ (x, y) \in E(\tilde{G}) \Rightarrow (z_1, z_2) \in E(\tilde{G}),\ for\ all\ z_1 \in Tx,\ z_2 \in Ty.$$

Definition 2.14. *Let (X, d) be a b-metric space with a graph $G = (V(G), E(G))$. Let $T : X \to CL(X)$ be a multi-valued mapping and $g : X \to X$ be a single-valued mapping. Then T is called edge preserving w.r.t. g if*

$$x, y \in X,\ x \neq y,\ (gx, gy) \in E(\tilde{G}) \Rightarrow (z_1, z_2) \in E(\tilde{G}),\ for\ all\ z_1 \in Tx,\ z_2 \in Ty.$$

3. Main Results

Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Let $T : X \to CL(X)$ be a multi-valued mapping and $g : X \to X$ be a single-valued mapping. We define the function $f_{gT} : X \to \mathbb{R}$ as $f_{gT}(x) = d(gx, Tx)$. If $g = I$, the identity map on X, then f_{gT} reduces to f_T where $f_T(x) = d(x, Tx)$ for all $x \in X$. For a positive constant $\alpha \in (0, 1)$ and each $x \in X$, we define the set

$${}^{g}I_{\alpha}^{x} = \{y \in Tx : \alpha d(gx, y) \leq d(gx, Tx)\}.$$

If $g = I$, the identity map on X, then ${}^{g}I_{\alpha}^{x}$ reduces to I_{α}^{x} which is given by

$$I_{\alpha}^{x} = \{y \in Tx : \alpha d(x, y) \leq d(x, Tx)\}.$$

Definition 3.1. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to CL(X)$ be a multi-valued mapping. A function $f : X \to \mathbb{R}$ is called T-lower semicontinuous if, for each $(x_n) \subseteq X$ with $x_{n+1} \in Tx_n$ and $\lim_{n\to\infty} x_n = x \in X$, we have*

$$fx \leq \liminf_{n\to\infty} sfx_n.$$

Definition 3.2. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to CL(X)$ be a multi-valued mapping. Let ρ be a binary relation over X and let $S = \rho \cup \rho^{-1}$. A function $f : X \to \mathbb{R}$ is called (T,S)-lower semicontinuous if, for each $(x_n) \subseteq X$ with $x_{n+1} \in Tx_n$, $x_n S x_{n+1}$ and $\lim_{n\to\infty} x_n = x \in X$, we have*

$$fx \leq \liminf_{n\to\infty} sfx_n.$$

Definition 3.3. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Let $T : X \to CL(X)$ be a multi-valued mapping and $g : X \to X$ be a single-valued mapping. A function $f : X \to \mathbb{R}$ is called (g,T)-lower semicontinuous if, for each $(gx_n) \subseteq g(X)$ with $gx_{n+1} \in Tx_n$ and $\lim_{n\to\infty} gx_n = x(= gt,\ for\ some\ t \in X) \in g(X)$, we have*

$$ft \leq \liminf_{n\to\infty} sfx_n.$$

Definition 3.4. *Let $(X,d,\preceq)$ be a partially ordered b-metric space with the coefficient $s \geq 1$. Let $T : X \to CL(X)$ be a multi-valued mapping and $g : X \to X$ be a single-valued mapping. A function $f : X \to \mathbb{R}$ is called $(g,T,\preceq)$-lower semicontinuous if, for each $(gx_n) \subseteq g(X)$ with $gx_{n+1} \in Tx_n$, gx_n, gx_{n+1} are comparable and $\lim_{n\to\infty} gx_n = x(= gt,\ for\ some\ t \in X) \in g(X)$, we have*

$$ft \leq \liminf_{n\to\infty} sfx_n.$$

Definition 3.5. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. Let $T : X \to CL(X)$ be a multi-valued mapping and $g : X \to X$ be a single-valued mapping. A function $f : X \to \mathbb{R}$ is called (g,T,G)-lower semicontinuous if, for each $(gx_n) \subseteq g(X)$ with $gx_{n+1} \in Tx_n$, $(gx_n, gx_{n+1}) \in E(\tilde{G})$ and $\lim_{n\to\infty} gx_n = x(= gt,\ for\ some\ t \in X) \in g(X)$, we have*

$$ft \leq \liminf_{n\to\infty} sfx_n.$$

It is valuable to note that if $G = G_0$, where G_0 is the complete graph $(X, X \times X)$, then (g,T,G)-lower semicontinuity reduces to (g,T)-lower semicontinuity.

We now assume that (X,d) is a b-metric space endowed with a reflexive digraph G such that $V(G) = X$ and G has no parallel edges. Let $g : X \to X$ and $T : X \to CL(X)$ be such that $T(X) \subseteq g(X)$. Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq g(X)$, there exists an element $x_1 \in X$ such that $gx_1 \in Tx_0$. Continuing in this way, we can construct a sequence (gx_n) such that $gx_n \in Tx_{n-1}$, $n = 1, 2, 3, \cdots$.

Theorem 3.6. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. Let $T : X \to CL(X)$ and $g : X \to X$ be such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that T is edge preserving w.r.t. g and there exists $r \in (0, s^{-1}\alpha)$ with $\alpha \in (0,1)$ such that for any $x \in X$, there is $gy \in {}^{g}I^{x}_{\alpha}$ satisfying*

$$d(gy, Ty) \leq rd(gx, gy). \tag{3.1}$$

If f_{gT} is (g,T,G)-lower semicontinuous and there exists $x_0 \in X$ such that $(gx_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$, then g and T have a point of coincidence in $g(X)$.

Proof . We first note that ${}^{g}I_{\alpha}^{x}$ is nonempty for any constant $\alpha \in (0,1)$ because Tx is a nonempty closed set for any $x \in X$. Suppose there exists $x_0 \in X$ such that $(gx_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$. If $gx_0 \in Tx_0$, then there is nothing to prove. So, we assume that $gx_0 \notin Tx_0$. Then, by using condition (3.1), for $x_0 \in X$, there exists $gx_1 \in {}^{g}I_{\alpha}^{x_0}$ such that

$$d(gx_1, Tx_1) \leq rd(gx_0, gx_1).$$

As $gx_1 \in Tx_0$, it follows that $(gx_0, gx_1) \in E(\tilde{G})$ and $gx_0 \neq gx_1$ which implies that $x_0 \neq x_1$. T being edge preserving w.r.t. g, it must be the case that $(z_1, z_2) \in E(\tilde{G})$ for all $z_1 \in Tx_0,\ z_2 \in Tx_1$. If $gx_1 \in Tx_1$, then the theorem is proved. So, we assume that $gx_1 \notin Tx_1$. By an argument similar to that used above, for $x_1 \in X$, there exists $gx_2 \in {}^{g}I_{\alpha}^{x_1}$ such that

$$d(gx_2, Tx_2) \leq rd(gx_1, gx_2),$$

$(gx_1, gx_2) \in E(\tilde{G})$ and $gx_1 \neq gx_2$. Continuing this process, we can construct a sequence (gx_n) in $g(X)$ such that $gx_{n+1} \in {}^{g}I_{\alpha}^{x_n},\ gx_n \neq gx_{n+1},\ (gx_n, gx_{n+1}) \in E(\tilde{G})$ for $n = 0,\ 1,\ 2, \cdots$ and

$$d(gx_{n+1}, Tx_{n+1}) \leq rd(gx_n, gx_{n+1}) \tag{3.2}$$

for all $n \in \mathbb{N} \cup \{0\}$.

On the other hand $gx_{n+1} \in {}^{g}I_{\alpha}^{x_n}$ implies that

$$\alpha d(gx_n, gx_{n+1}) \leq d(gx_n, Tx_n) \tag{3.3}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Using conditions (3.2) and (3.3), we obtain

$$d(gx_{n+1}, gx_{n+2}) \leq \frac{1}{\alpha} d(gx_{n+1}, Tx_{n+1}) \leq \frac{r}{\alpha} d(gx_n, gx_{n+1}) = kd(gx_n, gx_{n+1}) \tag{3.4}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $k = \frac{r}{\alpha} < s^{-1}$.

We now show that (gx_n) is a Cauchy sequence in $g(X)$.

For $m, n \in \mathbb{N}$ with $m > n$, we obtain by repeated use of condition (3.4) that

$$\begin{aligned}
d(gx_n, gx_m) \quad &\leq \quad sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\
&\qquad + s^{m-n-1} d(gx_{m-2}, gx_{m-1}) + s^{m-n-1} d(gx_{m-1}, gx_m) \\
&\leq \quad [sk^n + s^2 k^{n+1} + \cdots + s^{m-n-1} k^{m-2} + s^{m-n-1} k^{m-1}] d(gx_0, gx_1) \\
&\leq \quad [sk^n + s^2 k^{n+1} + \cdots + s^{m-n-1} k^{m-2} + s^{m-n} k^{m-1}] d(gx_0, gx_1) \\
&= \quad sk^n [1 + (ks) + (ks)^2 + \cdots + (ks)^{m-n-1}] d(gx_0, gx_1) \\
&< \quad sk^n [1 + (ks) + (ks)^2 + \cdots] d(gx_0, gx_1) \\
&= \quad \frac{sk^n}{1 - ks} d(gx_0, gx_1) \\
&\to 0 \quad as\ n \to \infty.
\end{aligned}$$

This gives that (gx_n) is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists $u \in g(X)$ such that $\lim_{n\to\infty} gx_n = u = gt$ for some $t \in X$.
Again, using conditions (3.2) and (3.3), we get

$$d(gx_{n+1}, Tx_{n+1}) \leq \frac{r}{\alpha} d(gx_n, Tx_n) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

This implies that

$$d(gx_n, Tx_n) \leq \left(\frac{r}{\alpha}\right)^n d(gx_0, Tx_0) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Therefore,

$$\liminf_{n\to\infty} sf_{gT}(x_n) = \lim_{n\to\infty} sf_{gT}(x_n) = \lim_{n\to\infty} sd(gx_n, Tx_n) = 0.$$

Since $gx_{n+1} \in Tx_n$, $(gx_n, gx_{n+1}) \in E(\tilde{G})$, $\lim_{n\to\infty} gx_n = gt$ and f_{gT} is (g, T, G)-lower semicontinuous, we get

$$f_{gT}(t) = d(gt, Tt) = 0.$$

Since Tt is closed, it follows that $u = gt \in Tt$, i.e., u is a point of coincidence of g and T. □

Corollary 3.7. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Let $T : X \to CL(X)$ and $g : X \to X$ be such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that there exists $r \in (0, s^{-1}\alpha)$ with $\alpha \in (0, 1)$ such that for any $x \in X$, there is $gy \in {}^gI^x_\alpha$ satisfying*

$$d(gy, Ty) \leq rd(gx, gy).$$

If f_{gT} is (g, T)-lower semicontinuous, then g and T have a point of coincidence in $g(X)$.

Proof . The proof follows from Theorem 3.6 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. □

Corollary 3.8. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. Assume that $T : X \to CL(X)$ is edge preserving and there exists $r \in (0, s^{-1}\alpha)$ with $\alpha \in (0, 1)$ such that for any $x \in X$, there is $y \in I^x_\alpha$ satisfying*

$$d(y, Ty) \leq rd(x, y).$$

If f_T is (T, G)-lower semicontinuous and there exists $x_0 \in X$ such that $(x_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$, then T has a fixed point in X.

Proof . The proof follows from Theorem 3.6 by taking $g = I$, the identity map on X. □

Corollary 3.9. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and let $T : X \to CL(X)$ be a multivalued mapping. Assume that there exists $r \in (0, s^{-1}\alpha)$ with $\alpha \in (0, 1)$ such that for any $x \in X$, there is $y \in I^x_\alpha$ satisfying*

$$d(y, Ty) \leq rd(x, y).$$

If f_T is T-lower semicontinuous, then T has a fixed point in X.

Proof . The proof follows from Theorem 3.6 by taking $g = I$ and $G = G_0$. □

Corollary 3.10. *Let $(X, d, \preceq)$ be a partially ordered b-metric space with the coefficient $s \geq 1$. Let $T : X \to CL(X)$ and $g : X \to X$ be such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that if $x, y \in X$, $x \neq y$ and gx, gy are comparable, then z_1, z_2 are comparable for all $z_1 \in Tx$, $z_2 \in Ty$. Suppose also that there exists $r \in (0, s^{-1}\alpha)$ with $\alpha \in (0,1)$ such that for any $x \in X$, there is $gy \in {}^{g}I_{\alpha}^{x}$ satisfying*

$$d(gy, Ty) \leq rd(gx, gy).$$

If f_{gT} is $(g, T, \preceq)$-lower semicontinuous and there exists $x_0 \in X$ such that gx_0, z are comparable for all $z \in Tx_0$, then g and T have a point of coincidence in $g(X)$.

Proof . The proof can be obtained from Theorem 3.6 by taking $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

Corollary 3.11. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Let ρ be a binary relation over X and let $S = \rho \cup \rho^{-1}$. Suppose $T : X \to CL(X)$ is such that if $x, y \in X$, $x \neq y$ and xSy, then $z_1 S z_2$ for all $z_1 \in Tx$, $z_2 \in Ty$. Suppose also that there exists $r \in (0, s^{-1}\alpha)$ with $\alpha \in (0,1)$ such that for any $x \in X$, there is $y \in I_{\alpha}^{x}$ satisfying*

$$d(y, Ty) \leq rd(x, y).$$

If f_T is (T, S)-lower semicontinuous and there exists $x_0 \in X$ such that $x_0 S z$ for all $z \in Tx_0$, then T has a fixed point in X.

Proof . The proof follows from Theorem 3.6 by taking $g = I$ and $G = (V(G), E(G))$, where $V(G) = X$, $E(G) = \{(x, y) \in X \times X : xSy\} \cup \triangle$. □

As an application of Theorem 3.6, we obtain the following theorems.

Theorem 3.12. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to CL(X)$ and $g : X \to X$ be a hybrid pair of mappings such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that there exists $r \in (0, s^{-1})$ such that*

$$H(Tx, Ty) \leq rd(gx, gy) \tag{3.5}$$

for all $x, y \in X$. Then g and T have a point of coincidence in $g(X)$.

Proof . We take $G = G_0 = (X, X \times X)$. By using condition (3.5), we obtain

$$d(gy, Ty) \leq H(Tx, Ty) \leq rd(gx, gy)$$

for all $x \in X$ and $gy \in Tx$. Hence condition (3.1) of Theorem 3.6 holds trivially for each $x \in X$ and $gy \in {}^{g}I_{\alpha}^{x}$ with $\alpha \in (0,1)$ such that $r < \alpha s^{-1}$. We now show that $f_{gT} : X \to \mathbb{R}$ defined by $f_{gT}(x) = d(gx, Tx)$ is (g, T, G_0)-lower semicontinuous. In fact, if $(gx_n) \subseteq g(X)$ with $gx_{n+1} \in Tx_n$ and $\lim_{n\to\infty} gx_n = x(= gt,\ for\ some\ t \in X) \in g(X)$, then

$$\begin{aligned} d(gt, Tt) &\leq s[d(gt, gx_{n+1}) + d(gx_{n+1}, Tt)] \\ &\leq s[d(gt, gx_{n+1}) + H(Tx_n, Tt)] \\ &\leq s[d(gt, gx_{n+1}) + rd(gx_n, gt)]. \end{aligned}$$

Taking limit as $n \to \infty$, we get $f_{gT}(t) = 0$. Consequently, it follows that

$$f_{gT}(t) \leq \liminf_{n\to\infty} s f_{gT}(x_n).$$

Thus, all the hypotheses of Theorem 3.6 hold true and the conclusion of Theorem 3.12 can be obtained from Theorem 3.6. □

The following is the Nadler's fixed point theorem in b-metric spaces.

Corollary 3.13. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and let $T : X \to CL(X)$ be a multivalued mapping. Assume that there exists $r \in (0, s^{-1})$ such that*

$$H(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then T has a fixed point in X.

Proof . The proof follows from Theorem 3.12 by taking $g = I$. □

Remark 3.14. *It is worth mentioning that Theorem 3.6 is a generalization of the above version of Nadler's fixed point theorem in the setting of b-metric spaces.*

The theorem stated below is a generalization Nadler's fixed point theorem in metric spaces which can be obtained from Theorem 3.12 by taking $s = 1$.

Theorem 3.15. *Let (X, d) be a metric space and let $T : X \to CL(X)$ and $g : X \to X$ be a hybrid pair of mappings such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that there exists $r \in (0, 1)$ such that*

$$H(Tx, Ty) \leq rd(gx, gy) \tag{3.6}$$

for all $x, y \in X$. Then g and T have a point of coincidence in $g(X)$.

Theorem 3.16. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Let $T : X \to CL(X)$ and $g : X \to X$ be such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that there exists $r \in (0, s^{-1})$ such that for any $x \in X$, $gy \in Tx$,*

$$d(gy, Ty) \leq rd(gx, gy).$$

If f_{gT} is (g, T)-lower semicontinuous, then g and T have a point of coincidence in $g(X)$.

Proof . As ${}^gI^x_\alpha \subseteq Tx$, the proof follows from Theorem 3.6 by taking $G = G_0$. □

Now, we present the following theorem which can be seen as an extension of Theorem 3.3 of [18]. The proof is based on an argument similar to that used by Branciari in Theorem 2.1 of [5].

Theorem 3.17. *Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a graph. Let $T : X \to CL(X)$ and $g : X \to X$ be such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that T is edge preserving w.r.t. g and there exists a constant $r \in (0, 1)$ such that for any $x \in X$, $gy \in Tx$ with $(gx, gy) \in E(\tilde{G})$, there is $gz \in Ty$ satisfying*

$$\int_0^{d(gy,gz)} \varphi(t)dt \leq r \int_0^{d(gx,gy)} \varphi(t)dt, \tag{3.7}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable(i.e., with finite integral) on each compact subset of $[0, \infty)$, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$. If f_{gT} is (g, T, G)-lower semicontinuous and there exists $x_0 \in X$ such that $(gx_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$, then g and T have a point of coincidence in $g(X)$.

Proof . Suppose there exists $x_0 \in X$ such that $(gx_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$. If $gx_0 \in Tx_0$, then there is nothing to prove. So, we assume that $gx_0 \notin Tx_0$. Now, by using condition (3.7), for $x_0 \in X$, $gx_1 \in Tx_0$ with $(gx_0, gx_1) \in E(\tilde{G})$, there exists $gx_2 \in Tx_1$ such that

$$\int_0^{d(gx_1,gx_2)} \varphi(t)dt \leq r \int_0^{d(gx_0,gx_1)} \varphi(t)dt.$$

As $gx_1 \in Tx_0$, it follows that $gx_1 \neq gx_0$ and so $x_0 \neq x_1$. Since T is edge preserving w.r.t. g, it must be the case that $(z_1, z_2) \in E(\tilde{G})$ for all $z_1 \in Tx_0,\ z_2 \in Tx_1$. This gives that $(gx_1, gx_2) \in E(\tilde{G})$. If $gx_1 \in Tx_1$, then the theorem is proved. So, we assume that $gx_1 \notin Tx_1$.
Again, by using condition (3.7), for $x_1 \in X,\ gx_2 \in Tx_1$ with $(gx_1, gx_2) \in E(\tilde{G})$, there exists $gx_3 \in Tx_2$ such that

$$\int_0^{d(gx_2,gx_3)} \varphi(t)dt \leq r \int_0^{d(gx_1,gx_2)} \varphi(t)dt.$$

As $gx_2 \in Tx_1$, it follows that $gx_2 \neq gx_1$ and so $x_1 \neq x_2$. Continuing this process, we can construct a sequence (gx_n) in $g(X)$ such that $gx_{n+1} \in Tx_n,\ gx_n \neq gx_{n+1},\ (gx_n, gx_{n+1}) \in E(\tilde{G})$ for $n = 0,\ 1,\ 2, \cdots$ and

$$\int_0^{d(gx_{n+1},gx_{n+2})} \varphi(t)dt \leq r \int_0^{d(gx_n,gx_{n+1})} \varphi(t)dt, \tag{3.8}$$

for $n = 0,\ 1,\ 2,\ \cdots$.

We now prove that (gx_n) converges to a point of coincidence of g and T in three steps.

Step 1. $f_{gT}(x_n) \to 0$ as $n \to \infty$.

Let us put $u_n = d(gx_n, gx_{n+1}),\ n = 0,\ 1,\ 2,\ \cdots$. Then, it is easy to verify that $(u_n)_{n=0}^{\infty}$ is decreasing. By repeated use of condition (3.8), we obtain

$$\int_0^{d(gx_n,gx_{n+1})} \varphi(t)dt \leq r^n \int_0^{d(gx_0,gx_1)} \varphi(t)dt,\ n = 1,\ 2,\ 3,\ \cdots .$$

Therefore,

$$\int_0^{u_n} \varphi(t)dt \leq r^n \int_0^{u_0} \varphi(t)dt,\ n = 1,\ 2,\ 3,\ \cdots .$$

As a consequence, we have

$$\lim_{n\to\infty} \int_0^{u_n} \varphi(t)dt = 0.$$

As $(u_n)_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers, it is convergent. We shall show that $\lim\limits_{n\to\infty} u_n = 0$. If possible, suppose that $\lim\limits_{n\to\infty} u_n = c,\ where\ c > 0$. This implies that the sequence $(u_n)_{n=0}^{\infty}$ is eventually in every neighbourhood of c. So, there exists $n_0 \in \mathbb{N}$ such that $u_n \geq \frac{c}{2}$ for all $n \geq n_0$. Therefore,

$$\lim_{n\to\infty} \int_0^{u_n} \varphi(t)dt \geq \int_0^{\frac{c}{2}} \varphi(t)dt > 0,$$

which contradicts the fact that

$$\lim_{n\to\infty} \int_0^{u_n} \varphi(t)dt = 0.$$

Thus, $\lim_{n\to\infty} u_n = 0$.
As $0 \leq f_{gT}(x_n) = d(gx_n, Tx_n) \leq d(gx_n, gx_{n+1}) = u_n$, we have $f_{gT}(x_n) \to 0$ as $n \to \infty$.

Step 2. (gx_n) is a Cauchy sequence in $g(X)$.

If possible, suppose (gx_n) is not a Cauchy sequence in $g(X)$. Then there exists an $\epsilon > 0$ such that for each $i \in \mathbb{N}$, there are m_i, $n_i \in \mathbb{N}$ with $m_i > n_i > i$ such that

$$d(gx_{n_i}, gx_{m_i}) \geq \epsilon.$$

Therefore, we can choose the sequences (m_i), (n_i) in $\mathbb{N}$ such that for each $i \in \mathbb{N}$, m_i is the smallest positive integer in the sense that $d(gx_{n_i}, gx_{m_i}) \geq \epsilon$ but $d(gx_{n_i}, gx_p) < \epsilon$ for each $p \in \{n_i+1, \cdots, m_i-1\}$.

We now show that $d(gx_{n_i}, gx_{m_i}) \to \epsilon+$ as $i \to \infty$. As $\lim_{n\to\infty} u_n = 0$, by the triangular inequality, we have

$$\begin{aligned} \epsilon &\leq d(gx_{n_i}, gx_{m_i}) \\ &\leq d(gx_{n_i}, gx_{m_i-1}) + d(gx_{m_i-1}, gx_{m_i}) \\ &< \epsilon + d(gx_{m_i-1}, gx_{m_i}) \\ &\to \epsilon+, \ as\ i \to \infty. \end{aligned}$$

Next we shall show that there exists $n_0 \in \mathbb{N}$ such that for each natural number $i > n_0$, we have $d(gx_{n_i+1}, gx_{m_i+1}) < \epsilon$. If possible, suppose there exists a subsequence $(i_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $d(gx_{n_{i_k}+1}, gx_{m_{i_k}+1}) \geq \epsilon$. Then, we obtain

$$\begin{aligned} \epsilon &\leq d(gx_{n_{i_k}+1}, gx_{m_{i_k}+1}) \\ &\leq d(gx_{n_{i_k}+1}, gx_{n_{i_k}}) + d(gx_{n_{i_k}}, gx_{m_{i_k}}) + d(gx_{m_{i_k}}, gx_{m_{i_k}+1}) \\ &\to \epsilon, \ as\ k \to \infty. \end{aligned}$$

By using condition (3.8), we get

$$\int_0^{d(gx_{n_{i_k}+1}, gx_{m_{i_k}+1})} \varphi(t)dt \leq r \int_0^{d(gx_{n_{i_k}}, gx_{m_{i_k}})} \varphi(t)dt.$$

Taking limit as $k \to \infty$, we obtain

$$\int_0^{\epsilon} \varphi(t)dt \leq r \int_0^{\epsilon} \varphi(t)dt,$$

which is a contradiction since $r \in (0, 1)$ and $\int_0^{\epsilon} \varphi(t)dt > 0$. This ensures that for a certain $n_0 \in \mathbb{N}$, we have $d(gx_{n_i+1}, gx_{m_i+1}) < \epsilon$ for all $i > n_0$. We now prove that there exist a $\sigma_\epsilon \in (0, \epsilon)$ and an $i_\epsilon \in \mathbb{N}$ such that for each natural number $i > i_\epsilon$, we have $d(gx_{n_i+1}, gx_{m_i+1}) < \epsilon - \sigma_\epsilon$. In fact, if there exists a subsequence $(i_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $d(gx_{n_{i_k}+1}, gx_{m_{i_k}+1}) \to \epsilon-$ as $k \to \infty$, then by using condition (3.8), we get

$$\int_0^{d(gx_{n_{i_k}+1}, gx_{m_{i_k}+1})} \varphi(t)dt \leq r \int_0^{d(gx_{n_{i_k}}, gx_{m_{i_k}})} \varphi(t)dt.$$

Taking limit as $k \to \infty$, we obtain

$$\int_0^{\epsilon} \varphi(t)dt \leq r \int_0^{\epsilon} \varphi(t)dt,$$

which is again a contradiction. Therefore, for each natural number $i > i_\epsilon$,

$$\begin{aligned}\epsilon &\leq d(gx_{n_i}, gx_{m_i})\\ &\leq d(gx_{n_i}, gx_{n_i+1}) + d(gx_{n_i+1}, gx_{m_i+1}) + d(gx_{m_i+1}, gx_{m_i})\\ &< d(gx_{n_i}, gx_{n_i+1}) + (\epsilon - \sigma_\epsilon) + d(gx_{m_i+1}, gx_{m_i})\\ &\to \epsilon - \sigma_\epsilon, \ as\ i \to \infty.\end{aligned}$$

This gives that $\epsilon \leq \epsilon - \sigma_\epsilon$, a contradiction. Therefore, (gx_n) is a Cauchy sequence in $g(X)$.

Step 3. Existence of a coincidence point.

Since (gx_n) is a Cauchy sequence in $g(X)$ and $g(X)$ is complete, there exists $u \in g(X)$ such that $\lim_{n\to\infty} gx_n = u(= gt,\ for\ some\ t \in X)$. By using (g, T, G)-lower semicontinuity of f_{gT}, we have

$$0 \leq f_{gT}(t) \leq \liminf_{n\to\infty} f_{gT}(x_n) = \lim_{n\to\infty} f_{gT}(x_n) = 0,$$

which implies that $f_{gT}(t) = 0$ and so $d(gt, Tt) = 0$. As Tt is closed, it follows that $u = gt \in Tt$. Therefore, u is a point of coincidence of g and T in $g(X)$. □

The following corollary is the Theorem 3.3 of [18].

Corollary 3.18. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a multi-valued mapping. Assume that there exists a constant $r \in (0, 1)$ such that for any $x \in X$, $y \in Tx$, there is $z \in Ty$ satisfying*

$$\int_0^{d(y,z)} \varphi(t)dt \leq r \int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable(i.e., with finite integral) on each compact subset of $[0, \infty)$, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$. If f_T is T-lower semicontinuous, then T has a fixed point in X.

Proof . The proof follows from Theorem 3.17 by taking $g = I$ and $G = G_0$. □

Corollary 3.19. *Let (X, d) be a metric space. Let $T : X \to CL(X)$ and $g : X \to X$ be such that $T(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Assume that there exists a constant $r \in (0, 1)$ such that for any $x \in X$, $gy \in Tx$, there is $gz \in Ty$ satisfying*

$$d(gy, gz) \leq rd(gx, gy).$$

If f_{gT} is (g, T)-lower semicontinuous, then g and T have a point of coincidence in $g(X)$.

Proof . The proof follows from Theorem 3.17 by taking $G = G_0$ and $\varphi(t) = 1$ for each $t \geq 0$. □

Remark 3.20. *Several special cases of Theorem 3.17 can be obtained by restricting $T : X \to X$ and taking different φ and G.*

The following example shows that Theorem 3.6 is an extension of Theorem 3.12.

Example 3.21. *Let* $X = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0, 1\}$ *with* $d(x, y) =\mid x - y \mid^2$ *for all* $x, y \in X$*. Then* (X, d) *is a complete b-metric space with* $s = 2$*. Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \Delta \cup \{(0, \frac{1}{2^n}) : n = 0, 1, 2, \cdots\}$*. Let* $T : X \to CL(X)$ *be defined by*

$$Tx = \begin{cases} \{0, \frac{1}{2^{n+1}}\}, \ x = \frac{1}{2^n}, \ n \in \mathbb{N} \cup \{0\}, \\ \\ \{0\}, \ x = 0 \end{cases}$$

and $gx = \frac{x}{2}$ *for all* $x \in X$*. Obviously,* $T(X) = g(X) = X \setminus \{1\}$ *and* $g(X)$ *is a complete subspace of* (X, d)*.*

For $x = 1, \ y = 0$*, we have* $gx = \frac{1}{2}, \ gy = 0, \ Tx = \{0, \frac{1}{2}\}, \ Ty = \{0\}$*. Therefore,*

$$H(Tx, Ty) = \frac{1}{4} = d(gx, gy) > rd(gx, gy)$$

for any $r \in (0, s^{-1})$ *and hence condition (3.5) of Theorem 3.12 does not hold.*

For $x = \frac{1}{2^n}, \ n \in \mathbb{N} \cup \{0\}, \ y = 0$*, we have* $gx = \frac{1}{2^{n+1}}, \ gy = 0, \ Tx = \{0, \frac{1}{2^{n+1}}\}, Ty = \{0\}$ *and so* $(gx, gy) \in E(\tilde{G})$ *which implies that* $(z_1, z_2) \in E(\tilde{G})$ *for all* $z_1 \in Tx, \ z_2 \in Ty$*. Therefore,* T *is edge preserving w.r.t.* g*. Obviously,* $x_0 = 0 \in X$ *such that* $(gx_0, z) \in E(\tilde{G})$ *for all* $z \in Tx_0$*.*

Moreover, for $x = \frac{1}{2^n}, \ n \in \mathbb{N} \cup \{0\}$*, we have* $Tx = \{0, \frac{1}{2^{n+1}}\}$ *and so there exists* $gy = \frac{1}{2^{n+1}} \in {}^gI^x_\alpha$ *for any* $\alpha \in (0, 1)$ *such that*

$$d(gy, Ty) = d(\frac{1}{2^{n+1}}, \{0, \frac{1}{2^{n+1}}\}) = 0 = rd(gx, gy)$$

for any $r \in (0, \alpha s^{-1})$*.*
Also, for $x = 0$*, there exists* $gy = 0 \in {}^gI^x_\alpha$ *for any* $\alpha \in (0, 1)$ *such that*

$$d(gy, Ty) = 0 = rd(gx, gy)$$

for any $r \in (0, \alpha s^{-1})$*.*

Thus, condition (3.1) of Theorem 3.6 holds. Now, it is easy to compute that $f_{gT}(x) = 0$ *for all* $x \in X$*. Hence, it is obvious that* f_{gT} *is* (g, T, G)*-lower semicontinuous. Then the existence of a point of coincidence of* g *and* T *follows from Theorem 3.6.*

It should be noticed that Theorem 3.6 can not assure the uniqueness of a point of coincidence. It is obvious that g *and* T *have infinitely many points of coincidence in* $g(X)$*. In fact, if* $x \in X$*, then* $gx \in Tx$*. So, every element of* X *except* 1 *is a point of coincidence of* g *and* T*.*

We now examine the necessity of (g, T, G)-lower semicontinuity of f_{gT} in Theorem 3.6.

Example 3.22. *Let* $X = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0, 1\}$ *with* $d(x, y) =\mid x - y \mid^2$ *for all* $x, y \in X$*. Then* (X, d) *is a complete b-metric space with* $s = 2$*. Let* G *be a digraph such that* $V(G) = X$ *and* $E(G) = \{(\frac{1}{2^n}, \frac{1}{2^m}) : m \leq n, \ m, \ n = 0, 1, 2, \cdots\} \cup \{(0, 0), (0, 1)\}$*. Let* $T : X \to CL(X)$ *be defined by*

$$Tx = \begin{cases} \{\frac{1}{2^{n+1}}, \ \frac{1}{2^{n+2}}\}, \ x = \frac{1}{2^n}, \ n \in \mathbb{N} \cup \{0\}, \\ \\ \{1\}, \ x = 0 \end{cases}$$

and $gx = x$ for all $x \in X$. Obviously, $T(X) \subseteq g(X) = X$.

For $x = \frac{1}{2^n}$, $y = \frac{1}{2^m}$ $m \neq n$, $m, n \in \mathbb{N} \cup \{0\}$, we have $(gx, gy) \in E(\tilde{G})$ which implies that $(z_1, z_2) \in E(\tilde{G})$ for all $z_1 \in Tx$, $z_2 \in Ty$.
Again, for $x = 1$, $y = 0$, we have $(gx, gy) \in E(\tilde{G})$ which gives that $(z_1, z_2) \in E(\tilde{G})$ for all $z_1 \in Tx$, $z_2 \in Ty$. Therefore, T is edge preserving w.r.t. g. Obviously, $x_0 = 0 \in X$ such that $(gx_0, z) \in E(\tilde{G})$ for all $z \in Tx_0$.

Further, for $x = \frac{1}{2^n}$, $n \in \mathbb{N} \cup \{0\}$, we have $Tx = \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\}$ and so there exists $gy = y = \frac{1}{2^{n+1}} \in {}^gI_\alpha^x$ for any $\alpha \in (0,1)$ such that

$$\begin{aligned} d(gy, Ty) &= d(\frac{1}{2^{n+1}}, \{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}\}) \\ &= d(\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}) \\ &= \mid \frac{1}{2^{n+1}} - \frac{1}{2^{n+2}} \mid^2 \\ &= \frac{1}{4}\, d(gx, gy). \end{aligned}$$

Also, for $x = 0$, there exists $gy = y = 1 \in {}^gI_\alpha^x$ for any $\alpha \in (0,1)$ such that

$$d(gy, Ty) = d(1, \{\frac{1}{2}, \frac{1}{2^2}\}) = d(1, \frac{1}{2}) = \frac{1}{4} = \frac{1}{4} d(gx, gy).$$

Therefore, for any $x \in X$, there is $gy \in {}^gI_\alpha^x$ for $\alpha = \frac{2}{3}$ such that

$$d(gy, Ty) = r\, d(gx, gy)$$

where $r = \frac{1}{4} < \alpha s^{-1}$.

Thus, condition (3.1) of Theorem 3.6 holds. But, it is easy to compute that

$$f_{gT}(x) = \begin{cases} \frac{1}{2^{2n+2}}, \ x = \frac{1}{2^n}, \ n \in \mathbb{N} \cup \{0\}, \\ \\ 1, \ x = 0. \end{cases}$$

This shows that f_{gT} is not (g, T, G)-lower semicontinuous. Thus, g and T have no point of coincidence in X due to lack of the (g, T, G)-lower semicontinuity of f_{gT}.

The following example shows that Theorem 3.17 is an extension of Theorem 3.15.

Example 3.23. *Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with $d(x, y) = \mid x - y \mid$ for all $x, y \in X$. Then (X, d) is a complete metric space. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{n}) : n = 1, 2, 3, \cdots\}$. Let $T : X \to CL(X)$ be defined by*

$$Tx = \begin{cases} \{0, \frac{1}{n+1}\}, \ x = \frac{1}{n}, \ n \in \mathbb{N}, \\ \\ \{0\}, \ x = 0 \end{cases}$$

and $gx = \frac{x}{x+1}$ for all $x \in X$. Obviously, $T(X) = g(X) = X \setminus \{1\}$ and $g(X)$ is a complete subspace of (X, d).

For $x = 1,\ y = 0$, *we have* $gx = \frac{1}{2},\ gy = 0,\ Tx = \{0, \frac{1}{2}\},\ Ty = \{0\}$. *Therefore,*

$$H(Tx, Ty) = \frac{1}{2} = d(gx, gy) > rd(gx, gy)$$

for any $r \in (0, 1)$ *and hence condition (3.6) of Theorem 3.15 does not hold.*

For $x = \frac{1}{n},\ n \in \mathbb{N},\ y = 0$, *we have* $gx = \frac{1}{n+1},\ gy = 0,\ Tx = \{0, \frac{1}{n+1}\},\ Ty = \{0\}$ *and so* $(gx, gy) \in E(\tilde{G})$ *which implies that* $(z_1, z_2) \in E(\tilde{G})$ *for all* $z_1 \in Tx,\ z_2 \in Ty$. *Therefore,* T *is edge preserving w.r.t.* g. *Obviously,* $x_0 = 0 \in X$ *is such that* $(gx_0, z) \in E(\tilde{G})$ *for all* $z \in Tx_0$.

We note that, for $x = \frac{1}{n},\ n \in \mathbb{N}$, *we have* $Tx = \{0, \frac{1}{n+1}\}$ *and* $gy = 0 = g0 \in Tx$ *with* $(gx, gy) \in E(\tilde{G})$. *So, for* $x \in X,\ gy = 0 = g0 \in Tx$ *with* $(gx, gy) \in E(\tilde{G})$, *there exists* $gz = g0 = 0 \in Ty$ *such that condition (3.7) of Theorem 3.17 holds for any* $r \in (0, 1)$ *and any Lebesgue-integrable mapping* $\varphi : [0, \infty) \to [0, \infty)$ *which is summable(i.e., with finite integral) on each compact subset of* $[0, \infty)$, *and such that for each* $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$. *Now, it is easy to compute that* $f_{gT}(x) = 0$ *for all* $x \in X$. *Hence, it is obvious that* f_{gT} *is* (g, T, G)*-lower semicontinuous. Then the existence of a point of coincidence of* g *and* T *follows from Theorem 3.17.*

It should be noticed that g *and* T *have infinitely many points of coincidence in* $g(X)$. *In fact, if* $x \in X$, *then* $gx \in Tx$. *So, every element of* X *except* 1 *is a point of coincidence of* g *and* T.

Remark 3.24. *It is valuable to note that* g *is not a Banach contraction. In fact, for* $x = \frac{1}{n},\ y = \frac{1}{m},\ n \neq m$, *we have*

$$\begin{aligned} \frac{d(gx, gy)}{d(x, y)} &= \frac{\mid \frac{1}{n+1} - \frac{1}{m+1} \mid}{\mid \frac{1}{n} - \frac{1}{m} \mid} \\ &= \frac{mn}{(n+1)(m+1)}. \end{aligned}$$

Therefore, $sup\{\frac{d(gx,gy)}{d(x,y)} : x,\ y \in X,\ x \neq y\} = 1$.

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