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Fixed points for Chatterjea contractions on a metric space with a graph

Kamal Fallahi*, Aris Aghanians

Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran

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Abstract

In this work, we formulate Chatterjea contractions using graphs in metric spaces endowed with a graph and investigate the existence of fixed points for such mappings under two different hypotheses. We also discuss the uniqueness of the fixed point. The given result here is a generalization of Chatterjea's fixed point theorem from metric spaces to metric spaces endowed with a graph.

Keywords: G-Chatterjea mapping; fixed point; orbitally G-continuous mapping.
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1. Introduction and preliminaries

Let (X, d) be a metric space. In [4], Chatterjea investigated the existence and uniqueness of fixed points for mappings $T : X \to X$ which satisfy

$$d(Tx, Ty) \leq \alpha\big[d(x, Ty) + d(y, Tx)\big] \tag{1.1}$$

for all $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$ (known as Chatterjea contractions), and proved that such mappings have a unique fixed point in complete metric spaces.

In 2008, Jachymski [5] studied the Banach contraction principle in metric spaces endowed with a graph and subsequently, Beg et al. [1] extended some results in [5] for set valued mappings. Recently in 2013, Bojor [2] followed Jachymski's idea for Kannan contractions using a new assumption called the weak T-connectivity of the graph.

The main purpose of this paper is to study Chatterjea contractions in metric spaces endowed with a graph by standard iterative techniques and avoid imposing the assumption of weak T-connectivity

*Corresponding author
Email addresses: fallahi1361@gmail.com (Kamal Fallahi), a.aghanians@dena.kntu.ac.ir (Aris Aghanians)

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on the graph. Our main result generalizes Chatterjea's fixed point theorem in metric spaces and also in metric spaces equipped with a partial order.

We begin by recalling some basic concepts related to graphs which are frequently used in this paper. For more details, it is referred to [3, 5].

An edge of an arbitrary graph with identical ends is called a loop and an edge with distinct ends is called a link. Two or more links with the same pairs of ends are said to be parallel edges.

Let (X, d) be a metric space and G be a directed graph with vertex set $V(G) = X$ such that the set $E(G)$ consisting of the edges of G contains all loops, that is, $(x, x) \in E(G)$ for all $x \in X$. Assume further that G has no parallel edges. Then G can be denoted by the ordered pair $(V(G), E(G))$, and also it is said that the metric space (X, d) is endowed with the graph G.

The metric space (X, d) can also be endowed with the graphs G^{-1} and $\widetilde{G}$, where the former is the conversion of G which is obtained from G by reversing the directions of the edges, and the latter is an undirected graph obtained from G by ignoring the directions of the edges. In other words,

$$V(G^{-1}) = V(\widetilde{G}) = X, \quad E(G^{-1}) = \big\{(x, y) : (y, x) \in E(G)\big\} \quad \text{and} \quad E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

It should be remarked that if both (x, y) and (y, x) belong to $E(G)$, then we will face with parallel edges in the graph $\widetilde{G}$. To avoid this problem, we delete either the edge (x, y) or the edge (y, x) (but not both of them) from G and consider the graph $\widetilde{G}$ obtained from the remaining graph.

A graph $G = (V(G), E(G))$ is said to be transitive if $(x, y), (y, z) \in E(G)$ implies $(x, z) \in E(G)$ for all $x, y, z \in V(G)$.

A graph H is called a subgraph of G if $V(H)$ and $E(H)$ are (nonempty) subsets of $V(G)$ and $E(G)$, respectively, and that $(x, y) \in E(H)$ implies $x, y \in V(H)$ for all $x, y \in V(G)$.

We also need a few notions about the connectivity of graphs.

Suppose that (X, d) is a metric space endowed with a graph G. If $x, y \in X$, then a finite sequence $(x_i)_{i=0}^N$ consisting of $N + 1$ vertices of G is called a path in G from x to y of length N whenever $x_0 = x$, $x_N = y$ and (x_{i-1}, x_i) is an edge of G for $i = 1, \dots, N$. The graph G is called connected if there exists a path in G between each two vertices of G, and weakly connected if the graph $\widetilde{G}$ is connected.

Definition 1.1. [6, Definition 3] Let $(X, \preceq)$ be a poset. A mapping $T : X \to X$ is called nondecreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

Definition 1.2. [7, Definitions 3.1 and 3.6] Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then

i) T is called a Picard operator if T has a unique fixed point $\widehat{x} \in X$ and $T^n x \to \widehat{x}$ for all $x \in X$;
ii) T is called a weakly Picard operator if the sequence $\{T^n x\}$ converges to a fixed point of T for all $x \in X$.

It is clear that a Picard operator is a weakly Picard one but the identity mapping of any metric space with more that one point shows that the converse is not generally true. In fact, the set of fixed points of a weakly Picard operator can have any arbitrary cardinality. Nevertheless, one can easily see that a weakly Picard operator is Picard if and only if it has a unique fixed point.

Definition 1.3. [5, Definition 2.4] Let (X, d) be a metric space endowed with a graph G. A mapping $T : X \to X$ is called orbitally G-continuous on X if for all $x, y \in X$ and all sequences $\{p_n\}$ of positive integers with $(T^{p_n} x, T^{p_{n+1}} x) \in E(G)$ for all $n \geq 1$, the convergence $T^{p_n} x \to y$ implies $T(T^{p_n} x) \to Ty$.

Trivially, a continuous mapping on a metric space is orbitally G-continuous for all graphs G but as we see in the next two examples, the converse is not generally true. The first example shows that a mapping on a metric space (X, d) can be orbitally G-continuous on X for all graphs G but fail to be continuous on X.

Example 1.4. Consider the set $\mathbb{R}^+$ of all nonnegative real numbers equipped with the usual Euclidean metric and define a mapping $T : \mathbb{R}^+ \to \mathbb{R}^+$ by the rule

$$Tx = \begin{cases} \dfrac{x}{2} & x \in \mathbb{R}^+ \cap \mathbb{Q}, \\ \dfrac{x}{3} & x \in \mathbb{R}^+ \cap \mathbb{Q}^c \end{cases} \qquad (x \in \mathbb{R}^+)$$

where $\mathbb{Q}$ is the set of all rationals. Then it is clear that T is only continuous at zero. In particular, T is not continuous on the whole $\mathbb{R}^+$.

Now, assume that $\mathbb{R}^+$ is endowed with any arbitrary graph G. To prove the orbital G-continuity of T on $\mathbb{R}^+$, suppose that $x, y \in \mathbb{R}^+$ and $\{p_n\}$ is a sequence of positive integers with $(T^{p_n}x, T^{p_{n+1}}x) \in E(G)$ for all $n \geq 1$ such that $T^{p_n}x \to y$. If $\{p_n\}$ is constant for sufficiently large indices n, then there is nothing to prove. Otherwise, if x is rational, then $T^{p_n}x = \frac{x}{2^{p_n}} \to 0$ which shows that $y = 0$. Therefore,

$$T(T^{p_n}x) = \frac{x}{2^{p_n+1}} \to 0 = Ty.$$

Finally, in the case that x is irrational, a similar argument shows that $T(T^{p_n}x) \to Ty$. Hence T is orbitally G-continuous on $\mathbb{R}^+$.

The second example shows better that how a graph plays an effective role to imply a weaker type of continuity.

Example 1.5. Consider again the set $\mathbb{R}^+$ equipped with the usual Euclidean metric and define a mapping $T : \mathbb{R}^+ \to \mathbb{R}^+$ by the rule

$$Tx = \begin{cases} \dfrac{x}{2} & x \neq 0, \\ 1 & x = 0 \end{cases} \qquad (x \in \mathbb{R}^+).$$

Obviously, T is not continuous at $x = 0$, and in particular, on the whole $\mathbb{R}^+$. Now assume that $\mathbb{R}^+$ is endowed with a graph $G = (V(G), E(G))$, where $V(G) = \mathbb{R}^+$ and $E(G) = \{(x, x) : x \in \mathbb{R}^+\}$, that is, $E(G)$ contains nothing but all loops. If $x, y \in \mathbb{R}^+$ and $\{p_n\}$ is a sequence of positive integers with $(T^{p_n}x, T^{p_{n+1}}x) \in E(G)$ for all $n \geq 1$ such that $T^{p_n}x \to y$, then $\{T^{p_n}x\}$ is necessarily a constant sequence. Thus, $T^{p_n}x = y$ for all $n \geq 1$ and so $T(T^{p_n}x) \to Ty$. Hence T is orbitally G-continuous on $\mathbb{R}^+$.

2. The main results

Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be an arbitrary mapping. Throughout this section, we use $\mathrm{Fix}(T)$ to denote the set of all fixed points of T, and by X_T, it is meant the set of all points $x \in X$ such that $(x, Tx) \in E(G)$. In other words,

$$\mathrm{Fix}(T) = \{x \in X : Tx = x\} \quad \text{and} \quad X_T = \big\{x \in X : (x, Tx) \in E(G)\big\}.$$

Since $E(G)$ contains all loops, it follows that $\mathrm{Fix}(T) \subseteq X_T$.

Motivated by [5, Definition 2.1] and [2, Definition 4], we introduce G-Chatterjea mappings in metric spaces endowed with a graph as follows:

Definition 2.1. Let (X, d) be a metric space endowed with a graph G. We say that a mapping $T : X \to X$ is a G-Chatterjea mapping if

C1) T preserves the egdes of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

C2) there exists an $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha\big[d(x, Ty) + d(y, Tx)\big]$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

If $T : X \to X$ is a G-Chatterjea mapping, then we call the number α in (C2) the constant of T.

We now give some examples of G-Chatterjea mappings in metric spaces endowed with a graph.

Example 2.2. Let (X, d) be a metric space endowed with a graph G. Since $E(G)$ contains all loops, it follows that any constant mapping $T : X \to X$ preserves the edges of G, and since d vanishes on the diagonal of X, it follows that T satisfies (C2) for any constant $\alpha \in [0, \frac{1}{2})$. Hence each constant mapping with domain X is a G-Chatterjea mapping.

Example 2.3. Let (X, d) be a metric space and a mapping $T : X \to X$ satisfy (1.1). Consider the complete graph G_0 whose vertex set coincides with X, that is, $V(G_0) = X$ and $E(G_0) = X \times X$, and assume that (X, d) is endowed with the graph G_0. Then it is clear that T preserves the edges of G_0 and (1.1) ensures that T satisfies (C2). Therefore, T is a G_0-Chatterjea mapping with constant α. Thus, G_0-Chatterjea mappings in metric spaces endowed with the complete graph G_0 are precisely the Chatterjea contractions in metric spaces, and hence G-Chatterjea mappings are a generalization of Chatterjea contractions from metric spaces to metric spaces endowed with a graph. As stated before, the existence and uniqueness of fixed points for Chatterjea contractions in complete metric spaces were investigated by Chatterjea (see [4]) in 1972. Also, in 1977, Rhoades [8] compared Chatterjea contractions with a number of other well-known contractions in metric spaces.

Example 2.4. Let $(X, \preceq)$ be a poset and d be a metric on X. Consider the poset graphs G_1 and G_2 by

$$V(G_1) = X \quad \text{and} \quad E(G_1) = \big\{(x, y) \in X \times X : x \preceq y\big\}$$

and $G_2 = \widetilde{G_1}$. Since $\preceq$ is reflexive, it follows that both $E(G_1)$ and $E(G_2)$ contain all loops. Assume that (X, d) is endowed with one of the graphs G_1 and G_2. Then a mapping $T : X \to X$ preserves the edges of G_1 if and only if T is nondecreasing, and T satisfies (C2) for the graph G_1 if and only if

$$d(Tx, Ty) \leq \alpha\big[d(x, Ty) + d(y, Tx)\big] \tag{2.1}$$

for all comparable elements $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$. Moreover, T preserves the edges of G_2 if and only if T maps the comparable elements of $(X, \preceq)$ onto comparable elements, and T satisfies (C2) for the graph G_2 if and only if (2.1) holds. Thus, each G_1-Chatterjea mapping is a G_2-Chatterjea one.

Example 2.5. Let (X,d) be a metric space and $\varepsilon > 0$. Two elements $x, y \in X$ are called ε-close if $d(x,y) < \varepsilon$. Define the ε-graph G_3 by

$$V(G_3) = X \quad \text{and} \quad E(G_3) = \big\{(x,y) \in X \times X : d(x,y) < \varepsilon\big\}.$$

Since d vanishes on the diagonal of X, it follows that $E(G_3)$ contains all loops. Assume that (X,d) is endowed with the graph G_3. Then a mapping $T : X \to X$ preserves the edges of G_3 if and only if T maps the ε-close elements of X onto ε-close elements, and T satisfies (C2) for the graph G_3 if and only if

$$d(Tx,Ty) \le \alpha\big[d(x,Ty) + d(y,Tx)\big] \tag{2.2}$$

for all ε-close elements $x, y \in X$, where $\alpha \in [0, \frac{1}{2})$.

Remark 2.6. Assume (X,d) is a metric space and $T : X \to X$ is a mapping.

- It is clear that the set X_T related to the complete graph G_0 coincides with X, and T is orbitally G_0-continuous on X if and only if T is orbitally continuous on X, that is, $T^{p_n}x \to y$ implies $T(T^{p_n}x) \to Ty$ for all $x, y \in X$ and all sequences $\{p_n\}$ of positive integers (see [5, Definition 2.2]);

- If $\preceq$ is a partial order on X, then the set X_T related to the poset graph G_1 consists of all points $x \in X$ such that $x \preceq Tx$, and T is orbitally G_1-continuous on X if and only if $T^{p_n}x \to y$ implies $T(T^{p_n}x) \to Ty$ for all $x, y \in X$ and all sequences $\{p_n\}$ of positive integers such that $\{T^{p_n}x\}$ is nondecreasing;

- If $\preceq$ is a partial order on X, then the set X_T related to the poset graph G_2 is the set of all points $x \in X$ such that x and Tx are comparable, and T is orbitally G_2-continuous on X if and only if $T^{p_n}x \to y$ implies $T(T^{p_n}x) \to Ty$ for all $x, y \in X$ and all sequences $\{p_n\}$ of positive integers such that the successive terms of $\{T^{p_n}x\}$ are pairwise comparable;

- Finally, if $\varepsilon > 0$, then the set X_T related to the graph G_3 is the set of all points $x \in X$ such that x and Tx are ε-close, and T is orbitally G_3-continuous on X if and only if $T^{p_n}x \to y$ implies $T(T^{p_n}x) \to Ty$ for all $x, y \in X$ and all sequences $\{p_n\}$ of positive integers such that the successive terms $\{T^{p_n}x\}$ are pairwise ε-close.

The next example shows that contractions and Chatterjea contractions are independent.

Example 2.7. Consider the set $X = [0,1]$ equipped with the usual Euclidean metric and define a mapping $T : X \to X$ by the rule

$$Tx = \begin{cases} \dfrac{1}{4} & 0 \le x < 1, \\[2mm] \dfrac{1}{8} & x = 1 \end{cases} \qquad (x \in X).$$

Then T is not a contraction because given any $k \in [0,1)$, we have

$$|Tx - T1| = \frac{1}{8} \ge |x-1| > k|x-1|$$

for all $x \in [\frac{7}{8}, 1)$. On the other hand, T satisfies (1.1) for $\alpha = \frac{1}{7}$ and hence T is a G_0-Chatterjea mapping with constant $\alpha = \frac{1}{7}$. In particular, T is a Chatterjea contraction.

Clearly, the mapping T in Example 2.7 is not continuous at $x = 1$, and in particular, on the whole metric space (X, d). Therefore, Example 2.7 shows that despite all contractions are (uniformly) continuous, a Chatterjea contraction need not be a continuous mapping.

Example 2.8. Consider the finite set $X = \{0, 1, 2\}$ equipped with the usual Euclidean metric and define a mapping $T : X \to X$ by $T = \{(0, 0), (1, 2), (2, 0)\}$. Given any $\alpha \in [0, \frac{1}{2})$, we have

$$|T1 - T2| = 2 > \alpha = \alpha\big[|1 - T2| + |2 - T1|\big].$$

Therefore, T does not satisfy (1.1). Now if a graph G is defined by $V(G) = X$ and $E(G) = \{(0, 0), (1, 1), (2, 2), (0, 2)\}$, since $T0 = T2 = 0$, it follows immediately that T is a G-Chatterjea mapping with any arbitrary constant $\alpha \in [0, \frac{1}{2})$.

The following proposition follows immediately from the definition of a G-Chatterjea mapping and it is a generalization of the result concluded form Example 2.4, which says that each G_1-Chatterjea mapping is a G_2-Chatterjea mapping.

Proposition 2.9. Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a mapping.

i) If T preserves the edges of G, then T preserves the edges of G^{-1} (respectively, $\widetilde{G}$);
ii) If T satisfies (C2) for the graph G, then T satisfies (C2) for the graph G^{-1} (respectively, $\widetilde{G}$);
iii) If T is a G-Chatterjea mapping, then T is a G^{-1}-Chatterjea mapping (respectively, a $\widetilde{G}$-Chatterjea mapping).

In order to prove our main theorem, we begin with an interesting and important property of G-Chatterjea mappings which is needed in the sequel.

Proposition 2.10. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Chatterjea mapping. Then* $\mathrm{Fix}(T)$ *does not contain both ends of any link of G.*

Proof . Suppose that x and y are two fixed points of T such that $(x, y) \in E(G)$. Then from (C2), we have

$$d(x, y) = d(Tx, Ty) \le \alpha\big[d(x, Ty) + d(y, Tx)\big] = 2\alpha d(x, y),$$

where $\alpha \in [0, \frac{1}{2})$ is the constant of T. Hence $d(x, y) = 0$, that is, $x = y$. □

According to Proposition 2.10, if (X, d) is a metric space endowed with a graph G, then no G-Chatterjea mapping can keep both vertices of any link of G fixed. In particular,

- if $G = G_0$, then no Chatterjea contraction can have more than one fixed point;
- if $\preceq$ is a partial order on X, then no G_1-Chatterjea mapping and no G_2-Chatterjea mapping can have two distinct comparable fixed points;
- if $\varepsilon > 0$, then no G_3-Chatterjea mapping can have two distinct ε-close fixed points.

The next useful lemma shows that in a metric space (X, d) endowed with a graph G, two successive iterates of any point of X_T under a G-Chatterjea mapping $T : X \to X$ are getting arbitrarily closer whenever the numbers of the iterates are getting sufficiently large.

Lemma 2.11. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Chatterjea mapping with constant α. Then*

$$d(T^n x, T^{n+1}x) \leq \big(\frac{\alpha}{1-\alpha}\big)^n \cdot d(x, Tx) \tag{2.3}$$

for all $x \in X_T$ and all $n \geq 0$. In particular, $d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$, for all $x \in X_T$.

Proof . Suppose that $x \in X_T$ is given. Then $(x, Tx) \in E(G)$ and since T preserves the edges of G, it follows by induction that $(T^n x, T^{n+1}x) \in E(G)$ for all $n \geq 0$. Now, let $n \geq 0$ be fixed. If $n = 0$, then (2.3) holds trivially. Otherwise, from (C2), we have

$$d(T^n x, T^{n+1}x) \leq \alpha\big[d(T^{n-1}x, T^{n+1}x) + \underbrace{d(T^n x, T^n x)}_{=0}\big] \leq \alpha\big[d(T^{n-1}x, T^n x) + d(T^n x, T^{n+1}x)\big],$$

and so

$$d(T^n x, T^{n+1}x) \leq \frac{\alpha}{1-\alpha} \cdot d(T^{n-1}x, T^n x).$$

Using induction, we get

$$d(T^n x, T^{n+1}x) \leq \frac{\alpha}{1-\alpha} \cdot d(T^{n-1}x, T^n x) \leq \cdots \leq \big(\frac{\alpha}{1-\alpha}\big)^n \cdot d(x, Tx).$$

In particular, because $\alpha < \frac{1}{2}$, we have $\frac{\alpha}{1-\alpha} < 1$, and hence $d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$. $\Box$

Our main theorem shows that a G-Chatterjea mapping T defined on a complete metric space (X, d) endowed with a graph G has a fixed point in X whenever T is orbitally G-continuous on X or the triple (X, d, G) has a suitable property.

Theorem 2.12. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to X$ be a G-Chatterjea mapping. Then the restriction of T to the set X_T is a weakly Picard operator if one of the following statements holds:*

1) *T is orbitally G-continuous on X;*
2) *The triple (X, d, G) has the following property:*
   (∗) *If $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 1$.*

In particular, whenever (1) *or* (2) *holds, then* $\mathrm{Fix}(T) \neq \emptyset$ *if and only if $X_T \neq \emptyset$.*

Proof . If $X_T = \emptyset$, then there is nothing to prove. Otherwise, if $x \in X_T$, then $(x, Tx) \in E(G)$ and since T preserves the edges of G, it follows that $(Tx, T^2x) \in E(G)$, that is, $Tx \in X_T$. Thus, X_T is T-invariant, that is, T maps X_T into itself.

Now, suppose that x is an arbitrary point of X_T. From Lemma 2.11, we have

$$\begin{aligned} d(T^n x, T^m x) &\leq d(T^n x, T^{n+1}x) + \cdots + d(T^{m-1}x, T^m x) \\ &\leq (\lambda^n + \cdots + \lambda^{m-1}) d(x, Tx) \\ &\leq \frac{\lambda^n}{1-\lambda} \cdot d(x, Tx), \end{aligned}$$

for all $m \geq n \geq 0$, where $\lambda = \frac{\alpha}{1-\alpha} \in [0, 1)$, and $\alpha \in [0, \frac{1}{2})$ is the constant of T. Therefore, letting $m, n \to \infty$ yields $d(T^n x, T^m x) \to 0$. Hence $\{T^n x\}$ is a Cauchy sequence and because (X, d) is complete, there exists an $\widehat{x} \in X$ (depending on x) such that $T^n x \to \widehat{x}$ as $n \to \infty$.

We next show that $\widehat{x}$ is a fixed point for T. To this end, note first that since X_T is T-invariant, it follows by induction that $T^n x \in X_T$ for all $n \geq 0$. Thus, $(T^n x, T^{n+1}x) \in E(G)$ for all $n \geq 0$. Now, if T is orbitally G-continuous on X, then $T^{n+1}x \to T\widehat{x}$ as $n \to \infty$. Because the limit of a convergent sequence in a metric space is unique, we get $T\widehat{x} = \widehat{x}$. Otherwise, if the triple (X, d, G) has Property $(*)$, then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $(T^{n_k}x, \widehat{x}) \in E(G)$ for all $k \geq 1$. Therefore, from (C2), we have

$$\begin{aligned}
d(T\widehat{x}, \widehat{x}) &\leq d(T\widehat{x}, T^{n_k+1}x) + d(T^{n_k+1}x, \widehat{x}) \\
&= d(TT^{n_k}x, T\widehat{x}) + d(T^{n_k+1}x, \widehat{x}) \\
&\leq \alpha\big[d(T^{n_k}x, T\widehat{x}) + d(\widehat{x}, T^{n_k+1}x)\big] + d(T^{n_k+1}x, \widehat{x}) \\
&= \alpha d(T^{n_k}x, T\widehat{x}) + (1+\alpha) d(T^{n_k+1}x, \widehat{x}) \\
&\leq \alpha\big[d(T^{n_k}x, \widehat{x}) + d(\widehat{x}, T\widehat{x})\big] + (1+\alpha) d(T^{n_k+1}x, \widehat{x})
\end{aligned}$$

for all $k \geq 1$. Hence

$$d(T\widehat{x}, \widehat{x}) \leq \frac{\alpha}{1-\alpha} \cdot d(T^{n_k}x, \widehat{x}) + \frac{1+\alpha}{1-\alpha} \cdot d(T^{n_k+1}x, \widehat{x}) \to 0$$

as $k \to \infty$. So $d(T\widehat{x}, \widehat{x}) = 0$ or equivalently, $T\widehat{x} = \widehat{x}$.

Finally, because $\mathrm{Fix}(T) \subseteq X_T$, it follows that $\widehat{x} \in X_T$ and consequently, in both Cases (1) and (2), the restriction of T to X_T is a weakly Picard operator. □

If we set $G = G_0$ in Theorem 2.12, then as mentioned before, the set X_T related to any mapping $T : X \to X$ coincides with the whole set X. Therefore, combining Theorem 2.12 and Proposition 2.11 yields Chatterjea's fixed point theorem [4] in complete metric spaces as follows:

Corollary 2.13. Let (X, d) be a complete metric space and $T : X \to X$ be a mapping which satisfies (1.1). Then T is a Picard operator.

If $\preceq$ is a partial order on X and we set $G = G_1$ in Theorem 2.12, since the poset graph G_1 is transitive, it is seen that one has $(x_n, x) \in E(G_1)$ (or equivalently, $x_n \preceq x$) for all $n \geq 1$ in Property $(*)$ (for the details, see [5, Remark 3.1]). Thus, the following ordered version of Chatterjea's fixed point theorem in complete metric spaces equipped with a partial order is obtained:

Corollary 2.14. Let $(X, \preceq)$ be a poset, d be a metric on X such that (X, d) is a complete metric space, and $T : X \to X$ be a nondecreasing mapping which satisfies (2.1). Then the restriction of T to the set of all points $x \in X$ with $x \preceq Tx$ is a weakly Picard operator if one of the following statements holds:

1) T is orbitally G_1-continuous on X;
2) The triple $(X, d, \preceq)$ has the following property:

   If $x_n \to x$ and $\{x_n\}$ is nondecreasing, then $x_n \preceq x$ for all $n \geq 1$.

In particular, whenever (1) or (2) holds, then $\mathrm{Fix}(T) \neq \emptyset$ if and only if there exists an $x \in X$ such that $x \preceq Tx$.

If $\preceq$ is a partial order on X and we set $G = G_2$ in Theorem 2.12, then another ordered version of Chatterjea's fixed point theorem in complete metric spaces equipped with a partial order is obtained as follows:

Corollary 2.15. Let $(X, \preceq)$ be a poset, d be a metric on X such that (X, d) is a complete metric space, and $T : X \to X$ be a mapping which maps comparable elements of X onto comparable elements and satisfies (2.1). Then the restriction of T to the set of all points $x \in X$ such that x and Tx are comparable is a weakly Picard operator if one of the following statements holds:

1) T is orbitally G_2-continuous on X;
2) The triple $(X, d, \preceq)$ satisfies the following property:

   If $x_n \to x$ and the successive terms of $\{x_n\}$ are pairwise comparable, then $\{x_n\}$ has a subsequence whose terms are all comparable to x.

In particular, whenever (1) or (2) holds, then $\mathrm{Fix}(T) \neq \emptyset$ if and only if there exists an $x \in X$ such that x and Tx are comparable.

Finally, if $\varepsilon > 0$ and we set $G = G_3$ in Theorem 2.12, then we get the following version of Chatterjea's fixed point theorem in complete metric spaces:

Corollary 2.16. Let (X, d) be a complete metric space, $\varepsilon > 0$ and $T : X \to X$ be a mapping which maps ε-close elements of X onto ε-close elements and satisfies (2.2). Then the restriction of T to the set of all points $x \in X$ such that x and Tx are ε-close is a weakly Picard operator if one of the following statements holds:

1) T is orbitally G_3-continuous on X;
2) The metric space (X, d) satisfies the following property:

   If $x_n \to x$ and the successive trems of $\{x_n\}$ are pairwise ε-close, then $\{x_n\}$ has a subsequence whose terms are all ε-close to x.

In particular, whenever (1) or (2) holds, then $\mathrm{Fix}(T) \neq \emptyset$ if and only if there exists an $x \in X$ such that $d(x, Tx) < \varepsilon$.

Now we give two sufficient conditions guaranteeing the uniqueness of the fixed point for a G-Chatterjea mapping in metric spaces endowed with a graph.

Theorem 2.17. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Chatterjea mapping. Then T has at most one fixed point in X if one of the following statements holds:*

a) *For all $x, y \in X$, there exists a path in G from x to y of length 2;*
b) *The subgraph of G with the vertices $\mathrm{Fix}(T)$ is weakly connected.*

Proof . Suppose that $\widehat{x}, \widehat{y} \in X$ are two fixed points for T. If (a) holds, then there exists a $z \in X$ such that $(\widehat{x}, z), (z, \widehat{y}) \in E(G)$. Thus, from (C2), we have

$$\begin{aligned} d(T^n z, \widehat{x}) &= d(T^n z, T^n \widehat{x}) \\ &= d(T^n \widehat{x}, T^n z) \\ &\leq \alpha\big[d(T^{n-1}\widehat{x}, T^n z) + d(T^{n-1} z, T^n \widehat{x})\big] \\ &= \alpha\big[d(T^n z, \widehat{x}) + d(T^{n-1} z, \widehat{x})\big] \end{aligned}$$

for all $n \geq 1$, where $\alpha \in [0, \frac{1}{2})$ is the constant of T. Therefore,

$$d(T^n z, \widehat{x}) \leq \frac{\alpha}{1-\alpha} \cdot d(T^{n-1} z, \widehat{x})$$

for all $n \geq 1$, and by induction, we get

$$d(T^n z, \widehat{x}) \leq \frac{\alpha}{1-\alpha} \cdot d(T^{n-1} z, \widehat{x}) \leq \cdots \leq \big(\frac{\alpha}{1-\alpha}\big)^n \cdot d(z, \widehat{x}),$$

for all $n \geq 0$. Now letting $n \to \infty$, we find $T^n z \to \widehat{x}$. Similarly, one can show that $T^n z \to \widehat{y}$. Hence $\widehat{x} = \widehat{y}$.

On the other hand, if (b) holds, then there exists a path $(x_i)_{i=0}^N$ in $\widetilde{G}$ from $\widehat{x}$ to $\widehat{y}$ such that $x_1, \ldots, x_{N-1} \in \mathrm{Fix}(T)$, that is, $x_0 = \widehat{x}$, $x_N = \widehat{y}$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$ for $i = 1, \ldots, N$. Since T is a G-Chatterjea mapping, it follows from Proposition 2.9 that T is also a $\widetilde{G}$-Chatterjea mapping. Therefore, from Proposition 2.10, we find

$$\widehat{x} = x_0 = x_1 = \cdots = x_{N-1} = x_N = \widehat{y}.$$

Consequently, in both Cases (a) and (b), T has at most one fixed point in X. □

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