Int. J. Nonlinear Anal. Appl. 7 (2016) No. 2, 69-76
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2016.454

(p, q)-Genuine Baskakov-Durrmeyer operators

Vijay Gupta[a,*], Th.M. Rassias[b]

[a]*Department of Mathematics, Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi-110078, India*
[b]*Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece*

(Communicated by M. Eshaghi)

Abstract

In the present article, we propose the (p, q) variant of genuine Baskakov Durrmeyer operators. We obtain moments and establish some direct results, which include weighted approximation and results in terms of modulus of continuity of second order.

Keywords: (p, q)-Beta function; (p, q)-Gamma function; Baskakov operators; Durrmeyer variant; Steklov mean; K-functional; direct estimates.
2010 MSC: Primary 41A25; Secondary 41A30.

1. Introduction

The quantum calculus has many applications in different areas of mathematics, physics and engineering sciences. In the last two decades it was used widely by researchers in the field of approximation theory. Several new generalizations to the well known operators have been proposed and their approximation behavior has been discussed, see for instance [2], [7], [8], [13], [14], [15], [11], [16]-[19] and references therein. The extension of quantum calculus based on two parameters (p, q) was discussed in special functions by Sahai and Yadav [22]. Very recently some approximation properties of certain (p, q) operators have been discussed in [9], [12] and [10]. One can capture the quantum calculus results from (p, q)-calculus, but not conversely. Some basic notations and definitions of the (p, q)-calculus are mentioned as follows:
The (p, q)-numbers are defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

*Corresponding author
Email addresses: vijaygupta2001@hotmail.com (Vijay Gupta), trassias@math.ntua.gr (Th.M. Rassias)

Received: December 2015 *Revised:* May 2016

The (p,q)-factorial is defined by $[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}$, $n \geq 1$, $[0]_{p,q}! = 1$. The (p,q)-binomial coefficient is given by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! \, [k]_{p,q}!}, 0 \leq k \leq n.$$

The (p,q)-derivative of the function f is defined as

$$D_{p,q} f\left(x\right) = \frac{f\left(px\right) - f\left(qx\right)}{\left(p-q\right)x}, x \neq 0$$

and $D_{p,q} f\left(0\right) = f^{'}\left(0\right)$, provided that f is differentiable at 0. Note also that for $p = 1$, the (p,q)-derivative reduces to the $q-$derivative.

Definition 1.1. ([20])Let n is a nonnegative integer, we define the (p,q)-Gamma function as

$$\Gamma_{p,q}\left(n+1\right) = \frac{\left(p \ominus q\right)_{p,q}^{n}}{\left(p-q\right)^{n}} = [n]_{p,q}!, \quad 0 < q < p,$$

where $(a \ominus b)_{p,q}^{n} = \prod_{i=0}^{n-1} (p^i a - q^i b)$.

Definition 1.2. The (p,q)-Beta function of second kind is defined as

$$B_{p,q}\left(m,n\right) = \int_{0}^{\infty} \frac{t^{m-1}}{\left(1 \oplus pt\right)_{p,q}^{m+n}} d_{p,q}t, m, n \in \mathbb{N},$$

where the (p,q)-power basis is defined as $(a \oplus b)_{p,q}^{n} = \prod_{i=0}^{n-1} (p^i a + q^i b)$.

Theorem 1.3. Let $m, n \in \mathbb{N}$. We have the following relation between (p,q)-Beta and (p,q)-Gamma function:

$$B_{p,q}\left(m,n\right) = q^{[2-m(m-1)]/2} p^{-m(m+1)/2} \frac{\Gamma_{p,q}\left(m\right) \Gamma_{p,q}\left(n\right)}{\Gamma_{p,q}\left(m+n\right)}.$$

In the present paper, we propose the (p,q)-genuine-Baskakov-Durrmeyer operators and estimate some approximation properties, which include weighted approximation result and direct results in terms of modulus of continuity of second order.

2. (p,q)-Baskakov-Durrmeyer Operators and Moments

The (p,q)-analogue of Baskakov operators for $x \in [0,\infty)$ and $0 < q < p \leq 1$ is defined as

$$B_{n,p,q}\left(f,x\right) = \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) f\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \tag{2.1}$$

where

$$b_{n,k}^{p,q}(x) = \left[\begin{array}{c} n+k-1 \\ k \end{array}\right]_{p,q} p^{k+n(n-1)/2} q^{k(k-1)/2} \frac{x^k}{(1 \oplus x)_{p,q}^{n+k}}.$$

In case $p = 1$, we get the q-Baskakov operators [1]. If $p = q = 1$, we get at once the well known Baskakov operators.

Remark 2.1. It can easily be verified

$$B_{n,p,q}(1,x)=1, B_{n,p,q}(t,x)=x, B_{n,p,q}\left(t^2,x\right)=x^2+\frac{p^{n-1}x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right).$$

Definition 2.2. Using (p,q)-Beta function of second kind, we propose below for $x\in[0,\infty), 0<q<p\leq 1$ the (p,q) genuine Baskakov-Durrmeyer operators

$$\begin{aligned} D_n^{p,q}(f,x) &= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1\oplus pt)_{p,q}^{k+n+1}}f(qp^{n+k}t)d_{p,q}t \\ &\quad +b_{n,0}^{p,q}(x)f(0) \end{aligned} \tag{2.2}$$

where $b_{n,k}^{p,q}(x)$ is as defined in (2.1).

Lemma 2.3. *For $x\in[0,\infty)$, $0<q<p\leq 1$, we have*

1. $D_n^{p,q}(1,x)=1$,
2. $D_n^{p,q}(t,x)=x$,
3. $D_n^{p,q}(t^2,x)=\frac{[n+1]_{p,q}x^2+[2]_{p,q}p^{n-2}qx}{q^2[n-1]_{p,q}}$.

Proof . Using (2.2) and Remark 2.1, we have

$$\begin{aligned} D_n^{p,q}(1,x) &= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1\oplus pt)_{p,q}^{k+n+1}}d_{p,q}t+b_{n,0}^{p,q}(x) \\ &= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)+b_{n,0}^{p,q}(x)=\sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)=1. \end{aligned}$$

Next, applying Remark 2.1, we have

$$\begin{aligned} D_n^{p,q}(t,x) &= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^k qp^{n+k}}{(1\oplus pt)_{p,q}^{k+n+1}}d_{p,q}t \\ &= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}qp^{n+k}B_{p,q}(k+1,n) \\ &= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x).\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}} \\ &= B_{n,p,q}(t,x)=x. \end{aligned}$$

Also, using $[k+1]_{p,q}=q^k+p[k]_{p,q}$ and $[n+1]_{p,q}=p^n+q[n]_{p,q}$, we

$$\begin{aligned}
D_n^{p,q}\left(t^2,x\right) &= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k+1}q^2p^{2n+2k}}{(1\oplus pt)_{p,q}^{k+n+1}}d_{p,q}t \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}q^2p^{2n+2k}B_{p,q}(k+1,n) \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x).\frac{p^{2n-3}[k+1]_{p,q}[k]_{p,q}}{q^{2k-1}[n]_{p,q}[n-1]_{p,q}} \\
&= \sum_{k=1}^{\infty} b_{n,k}^{p,q}(x).\frac{p^{2n-3}(q^k+p[k]_{p,q})[k]_{p,q}}{q^{2k-1}[n]_{p,q}[n-1]_{p,q}} \\
&= \frac{p^{n-2}}{[n-1]_{p,q}}B_{n,p,q}\left(t,x\right)+\frac{[n]_{p,q}}{q[n-1]_{p,q}}B_{n,p,q}\left(t^2,x\right) \\
&= \frac{p^{n-2}x}{[n-1]_{p,q}}+\frac{[n]_{p,q}}{q[n-1]_{p,q}}\left[x^2+\frac{p^{n-1}x}{[n]_{p,q}}\left(1+\frac{p}{q}x\right)\right] \\
&= \frac{[n+1]_{p,q}x^2+[2]_{p,q}p^{n-2}qx}{q^2[n-1]_{p,q}}.
\end{aligned}$$

□

Remark 2.4. For $x\in[0,\infty)$, $0<q<p\le 1$, we have

1. $D_n^{p,q}(t-x,x)=0$,
2. $D_n^{p,q}((t-x)^2,x)=\frac{[2]_{p,q}p^{n-2}x(px+q)}{q^2[n-1]_{p,q}}$.

3. Direct Approximation

We consider the following class of functions:

Let $H_{x^2}\left[0,\ \infty\right)$ be the set of all functions f defined on $[0,\ \infty)$ satisfying the condition $|f\left(x\right)|\le M_f\left(1+x^2\right)$, where M_f is a constant depending only on f. By $C_{x^2}\left[0,\ \infty\right)$, we denote the subspace of all continuous functions belonging to $H_{x^2}\left[0,\ \infty\right)$. Also, let $C_{x^2}^*\left[0,\ \infty\right)$ be the subspace of all functions $f\in C_{x^2}\left[0,\ \infty\right)$, for which $\lim\limits_{|x|\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^*\left[0,\ \infty\right)$ is $\|f\|_{x^2}=\sup_{x\in[0,\ \infty)}\frac{|f(x)|}{1+x^2}$.

We discuss below the weighted approximation theorem:

Theorem 3.1. *Let $p=p_n$ and $q=q_n$ satisfies $0<q_n<p_n\le 1$ and for n sufficiently large $p_n\to 1$, $q_n\to 1$ and $q_n^n\to 1$ and $p_n^n\to 1$. For each $f\in C_{x^2}^*\left[0,\ \infty\right)$, we have*

$$\lim_{n\to\infty}\|D_n^{p_n,q_n}\left(f\right)-f\|_{x^2}=0.$$

Proof . Using the Theorem in [6] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\|D_n^{p_n,q_n}\left(t^{\nu},x\right)-x^{\nu}\|_{x^2}=0,\quad \nu=0,1,2. \tag{3.1}$$

Since $D_n^{p_n,q_n}\left(1,x\right)=1$ and $D_n^{p_n,q_n}\left(t,x\right)=x$ the first condition of (3.1) is fulfilled for $\nu=0,1$.

For $n>1$, we can write

$$\begin{aligned}\left\|D_n^{p_n,q_n}\left(t^2,x\right)-x^2\right\|_{x^2} &\leq \left(\frac{[n+1]_{p_n,q_n}}{q_n^2[n-1]_{p_n,q_n}}\right)\sup_{x\in[0,\ \infty)}\frac{x^2}{1+x^2}\\ &+\left(\frac{[2]_{p_n,q_n}p_n^{n-2}q_n}{q_n^2[n-1]_{p_n,q_n}}\right)\sup_{x\in[0,\ \infty)}\frac{x}{1+x^2},\end{aligned}$$

which implies that for $v=2$

$$\lim_{n\to\infty}\left\|D_n^{p_n,q_n}\left(t^v,x\right)-x^v\right\|_{x^2}=0.$$

Thus the proof is completed. □

Let $C_B[0,\infty)$ denote the space of all real valued continuous and bounded functions on $[0,\infty)$. In this space we consider the norm

$$\|f\|_{C_B}=\sup_{x\in[0,\infty)}|f(x)|.$$

For $\delta\geq 0$, the second order modulus of continuity of function $f\in C_B[0,\infty)$ (see [5]) is defined as

$$\widetilde{\omega}_2(f,\delta)=\sup_{\substack{x,u,v\geq 0\\|u-v|\leq\delta}}|f(x+2u)-2f(x+u+v)+f(x+2v)|.$$

The Steklov mean function for $f\in C_B$ is considered as

$$f_h(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\left[2f(x+u+v)-f(x+2(u+v))\right]dudv. \tag{3.2}$$

Since $f_h\in C_B$ we can write

$$f_h(x)-f(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\left[2f(x+u+v)-f(x+2(u+v))-f(x)\right]dudv.$$

It is obvious that

$$\|f_h-f\|_{C_B}\leq\widetilde{\omega}_2(f,h). \tag{3.3}$$

If f is continuous, then $f_h''\in C_B$ and

$$\|f_h''\|_{C_B}\leq\frac{9}{h^2}\widetilde{\omega}_2(f,h). \tag{3.4}$$

Theorem 3.2. *Let $q\in(0,1)$ and $p\in(q,1]$. The operator $D_n^{p,q}$ maps space C_B into C_B and*

$$\|D_n^{p,q}(f)\|_{C_B}\leq\|f\|_{C_B}.$$

Proof . Let $q\in(0,1)$ and $p\in(q,1]$. From Lemma 2.3 we have

$$\begin{aligned}\left|D_n^{p,q}(f,x)\right| &\leq \sum_{k=1}^{\infty}b_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1\oplus pt)_{p,q}^{k+n+1}}|f(qp^{n+k}t)|d_{p,q}t+b_{n,0}^{p,q}(x)|f(0)|\\ &= \sup_{x\in[0,\infty)}|f(x)|\,D_n^{p,q}(1,x)=\|f\|_{C_B}.\end{aligned}$$

□

We study in the following theorem the degree of approximation in terms of second order modulus of continuity. We apply linear approximate method viz. Steklov mean to prove the theorem.

Theorem 3.3. *Let $q \in (0,1)$ and $p \in (q,1]$. If $f \in C_B$, then*

$$\left|D_n^{p,q}(f,x) - f(x)\right| \leq 9\widetilde{\omega}_2\left(f, \frac{1}{\sqrt{[n-1]_{p,q}}}\right)\left[1 + \frac{xp^{n-2}(px+q)}{q^2}\right].$$

Proof . We use the Steklov function f_h defined by (3.2). For $x \geq 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\left|D_n^{p,q}(f,x) - f(x)\right| &\leq D_n^{p,q}(|f-f_h|,x) + \left|D_n^{p,q}(f_h - f_h(x),x)\right| \\ &\quad + |f_h(x) - f(x)|.\end{aligned}$$

By (3.3) we can write

$$D_n^{p,q}(|f-f_h|,x) \leq \|D_n^{p,q}(f-f_h)\|_{C_B} \leq \|f-f_h\|_{C_B} \leq \widetilde{\omega}_2(f,h).$$

By Taylor's expansion, we have

$$\left|D_n^{p,q}(f_h - f_h(x),x)\right| \leq |f_h'(x)|\, D_n^{p,q}(t-x,x) + \frac{1}{2}\|f_h''\|_{C_B} D_n^{p,q}\left((t-x)^2,x\right).$$

By Lemma 2.3 and (3.4) we have

$$\left|D_n^{p,q}(f_h - f_h(x),x)\right| \leq \frac{9}{2h^2}\widetilde{\omega}_2(f,h)\, D_n^{p,q}\left((t-x)^2,x\right),$$

where $D_n^{p,q}\left((t-x)^2,x\right)$ is as defined in Remark 2.4 for $x \geq 0$, $h > 0$. Setting $h = \sqrt{\frac{1}{[n-1]_{p,q}}}$, we get the desired result. □

For $f \in C_B[0,\infty)$ the $K-$functional are defined as

$$K_2(f,\delta) = \inf\{\, \|f-g\|_{C_B} + \delta\, \|g''\|_{C_B}\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [3, p. 177, Theorem 2.4] there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\,\omega_2\left(f,\sqrt{\delta}\right), \tag{3.5}$$

where

$$\omega_2\left(f,\sqrt{\delta}\right) = \sup_{0<h\leq\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$.

The following theorem is also in terms of modulus of continuity, here we use K-functional.

Theorem 3.4. *Let $f \in C_B[0,\infty)$, with $q \in (0,1)$. Then for every $x \in [0,\infty)$ and $n \geq 1$, we have*

$$\left|D_n^{p,q}(f,x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\frac{[2]_{p,q}p^{n-2}x(px+q)}{q^2[n-1]_{p,q}}}\right),$$

where C is an absolute constant.

Proof . Let $g \in W^2$, by Taylor's theorem, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\ g''(u)\ du, \quad t \in [0, \infty).$$

Using Remark 2.4, we get

$$D_n^{p,q}(g, x) \ = \ g(x) + D_n^{p,q}\left(\int_x^t (t-u)\ g''(u)\ du, x\right).$$

Obviously, we have

$$|\int_x^t (t-u)\ g''(u)| \leq (t-x)^2 ||g''||_{C_B}.$$

Thus

$$\begin{aligned} |D_n^{p,q}(g, x) - g(x)| &\leq D_n^{p,q}(t-x)^2, x)||g''||_{C_B} \\ &= \frac{[2]_{p,q} p^{n-2} x(px+q)}{q^2[n-1]_{p,q}} ||g''||_{C_B}. \end{aligned}$$

Using Theorem 3.2, we have

$$\begin{aligned} |D_n^{p,q}(f, x) - f(x)| &\leq |D_n^q(f-g, x) - (f-g)(x)| + |D_n^{p,q}(g, x) - g(x)| \\ &\leq 2\ \|f-g\|_{C_B} + \frac{[2]_{p,q} p^{n-2} x(px+q)}{q^2[n-1]_{p,q}} ||g''||_{C_B}. \end{aligned}$$

Taking infimum on the right hand side over all $g \in W^2$, we get the desired result. This completes the proof of the theorem. □

Remark 3.5. For $q \in (0, 1)$ and $p \in (q, 1]$ it is seen that $\lim_{n\to\infty}[n]_{p,q} = 1/(q-p)$. In order to obtain convergence estimates of (p, q) genuine Baskakov operators, we assume $p = (p_n)$ and $q = (q_n)$ such that $0 < q_n < p_n \leq 1$ and for n sufficiently large $p_n \to 1, q_n \to 1$, $p_n^n \to a, q_n^n \to b$ and $[n]_{p_n,q_n} \to \infty$.

References

[1] A. Aral and V. Gupta, *Generalized q Baskakov operators*, Math. Slovaca 61 (2011) 619–634.
[2] A. Aral, V. Gupta and R.P. Agarwal, *Applications of q Calculus in Operator Theory*, Springer, 2013.
[3] R.A. DeVore and G.G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.
[4] Z. Ditzian and V. Totik, *Moduli of smoothness*, Springer, New York, 1987.
[5] M. Becker, *Global approximation theorems for Szász-Mirakyan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. 27 (1978) 127–142.
[6] A.D. Gadzhiev, *Theorems of the type of P.P. Korovkin type theorems*, Math. Zametki 20 (1976) 781–786, English Translation, Math. Notes 20 (5-6) (1976) 996–998.
[7] W. Gautschi, G. Mastroianni and Th.M. Rassias, *Approximation and Computation In Honor of Gradimir V. Milovanovic*, Springer, New York, 2011.
[8] V. Gupta, *Some approximation properties on q-Durrmeyer operators*, Appl. Math. Comput. 197 (2008) 172–178.
[9] V. Gupta, *(p, q)-Baskakov-Kantorovich operators*, Appl. Math. Inf. Sci. 10 (2016) 1551–1556.
[10] V. Gupta, *(p, q)-Szász-Mirakyan-Baskakov operators*, Complex Anal. Oper. Theory, in press.
[11] V. Gupta and R.P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer, Cham., 2014.
[12] V. Gupta and A. Aral, *Bernstein Durrmeyer operators based on two parameters*, Facta Universitatis, Series: Math. Informatics 31 (2016) 79–95.

[13] D.H. Hyers, G. Isac and Th.M. Rassias, *Topics in Nonlinear Analysis and Applications*, World Scientific Publishing Co., Singapore, New Jersey, London, 1997.

[14] G.V. Milovanovic, D.S. Mitrinovic and Th.M. Rassias, *Topics in Polynomials : Extremal Problems, Inequalities, Zeros*, World Scientific Publishing Co., Singapore, New Jersey, London, 1994.

[15] G.V. Milovanovic and Th.M. Rassias, *Analytic Number Theory, Approximation Theory and Special Functions*, Springer, New York, 2014.

[16] Th.M. Rassias, H.M. Srivastava and A. Yanushauskas, *Topics in Polynomials of One and Several Variables*, World Scientific Publishing Co., Singapore, New Jersey, London, 1993.

[17] Th.M. Rassias, *Approximation Theory and Applications*, Hadronic Press Inc., Florida, 1998.

[18] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.

[19] Th.M. Rassias, C.A. Floudas and S. Butenko, *Optimization in Science and Engineering In honor of the 60 th Birthday of Panos M. Pardalos*, Springer, New York, 2014.

[20] P.N. Sadjang, *On the* (p,q)*-Gamma and the* (p,q)*-Beta functions*, arXiv 1506.07394v1. 22 Jun 2015.

[21] P.N. Sadjang, *On the fundamental theorem of* (p,q)*-calculus and some* (p,q)*-Taylor formulas*, arXiv:1309.3934 [math.QA].

[22] V. Sahai and S. Yadav, *Representations of two parameter quantum algebras and* p,q*-special functions*, J. Math. Anal. Appl. 335 (2007) 268–279.