Int. J. Nonlinear Anal. Appl. 8 (2017) No. 2, 29-35
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2016.455

Perfect 2-colorings of the Platonic graphs

Mohammad Hadi Alaeiyan[a,*], Hamed Karami[b]

[a]School of Computer Engineering, Iran University of Science and Technology, Narmak, Tehran 16846, Iran
[b]School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16846, Iran

(Communicated by A. Ebadian)

Abstract

In this paper, we enumerate the parameter matrices of all perfect 2-colorings of the Platonic graphs consisting of the tetrahedral graph, the cubical graph, the octahedral graph, the dodecahedral graph, and the icosahedral graph.

Keywords: Perfect Coloring; Equitable Partition; Platonic Graph.
2010 MSC: Primary 05C15.

1. Introduction

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as "equitable partition" (see [8]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(6,3)$, $J(7,3)$, $J(8,3)$, $J(8,4)$, and $J(v,3)$ (v odd) (see [1, 2, 7]).

Fon-Der-Flass enumerated the parameter matrices of n-dimensional cube for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix (see [4, 5, 6]).

In this article we enumerate the parameter matrices of all perfect 2-colorings of the five Platonic graphs.

*Corresponding author
Email addresses: hadi_alaeiyan@comp.iust.ac.ir (Mohammad Hadi Alaeiyan), h_karami@iust.ac.ir (Hamed Karami)

Received: October 2015 *Revised:* February 2016

2. Preliminaries

A Platonic graph is a polyhedral graph corresponding to the skeleton of a Platonic solid. The Platonic graphs consist of five graphs; the tetrahedral graph, the cubical graph, the octahedral graph, the dodecahedral graph, and the icosahedral graph.

Now, we introduce two families of famous graphs.

Definition 2.1. The *Hypercube* graph H_n has vertices, respectively, edges given by

$$V(H_n) = \{a = (a_1, \ldots, a_n) : a_i \in \mathbb{Z}_2\},$$
$$E(H_n) = \{ab : a \text{ and } b \text{ differ in precisely one coordinate}\}.$$

Definition 2.2. The *generalized Petersen* graph $GP(n, k)$ has vertices, respectively, edges given by

$$V(GP(n,k)) = \{a_i, b_i : 0 \leq i \leq n-1\},$$
$$E(GP(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 0 \leq i \leq n-1\},$$

where the subscripts are expressed as integers modulo n (≥ 5), and k (≥ 1) is the "skip".

Note that the cubical graph is the graph H_3, and the dodecahedral graph is the graph $GP(10, 2)$. Next, we give a complete definition of perfect colorings.

Definition 2.3. For each graph G and each integer m, a mapping $T : V(G) \to \{1, \ldots, m\}$ is called a perfect m-coloring with matrix $A = (a_{ij})_{i,j\in\{1,\ldots,m\}}$, if it is surjective, and for all i, j, for every vertex of color i, the number of its neighbors of color j is equal to a_{ij}. The matrix A is called the *parameter matrix* of a perfect coloring. In the case $m = 2$, we call the first color *white*, and the second color *black*. Also, if λ is the eigenvalue of a parameter matrix obtained from a perfect m-coloring, we call it the eigenvalue of the perfect m-coloring.

Remark 2.4. In this paper, we consider all perfect 2-colorings, up to renaming the colors; i.e, we identify the perfect 2-coloring with the matrix

$$\begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix},$$

obtained by switching the colors with the original coloring.

Now, we first give some results concerning necessary conditions for the existence of perfect 2-colorings of a k-regular graph with a given parameter matrix $A = (a_{ij})_{i,j=1,2}$. The simplest condition for the existence of a perfect 2-colorings of a k-regular graph with the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is $a_{11} + a_{12} = a_{21} + a_{22} = k$. Also, when the graph is connected, it is clear that neither a_{12} nor a_{21} cannot be equal to zero, otherwise white and black vertices of the graph would not be adjacent, which is impossible.

The next proposition gives a formula for calculating the number of white vertices in a perfect 2-coloring.

Proposition 2.5. [1] If W is the set of white vertices in a perfect 2-coloring of a graph G with matrix $A = (a_{ij})_{i,j=1,2}$, then

$$|W| = |V(G)| \frac{a_{21}}{a_{12} + a_{21}}$$

The next theorem is useful to enumerate parameter matrices.

Theorem 2.6. [9] If T is a perfect coloring of a graph G in m colors, then any eigenvalue of T is an eigenvalue of G.

Corollary 2.7. It is easy to see that every perfect 2-coloring of a k-regular graph with parameter matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two eigenvalues: one is k, and the other is $a - c$ such that we obviously have $a - c \neq k$. So, from Theorem 2.6, we conclude that $a - c$ is an eigenvalue of a k-regular connected graph which is not equal to k.

The next proposition gives some constructions for perfect 2-colorings of Hypercube graphs.

Proposition 2.8. [[5]]

(a) For every $n = 2^k - 1$ and for every c, $1 \leq c \leq n$, there exists a perfect coloring of H_n with matrix $\begin{bmatrix} c-1 & n-c+1 \\ c & n-c \end{bmatrix}$.

(b) If there exists a perfect coloring with matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then, for every $k \geq 1$, there exists a perfect coloring with matrix $\begin{bmatrix} a+k & b \\ c & d+k \end{bmatrix}$.

(c) If there exists a perfect coloring with matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then, for every $k \geq 1$, there exists a perfect coloring with matrix $\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$.

(d) If there exists a perfect coloring with matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then, $\frac{b+c}{(b,c)}$ is a power of 2.

The next theorem gives a necessary condition for the existence of perfect 2-colorings of Hypercube graphs.

Theorem 2.9. [[4]] If T is a perfect 2-coloring of a hypercube graph with matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we have $a \geq \frac{3c-b}{4}$.

Finally, we end this section with the eigenvalues of the tetrahedral graph, the octahedral graph, the icosahedral graph, and the dodecahedral graph.

Theorem 2.10. [[3]] The distinct eigenvalues of the tetrahedral graph are the numbers $-1, 3$. The distinct eigenvalues of the octahedral graph are the numbers $-2, 0, 4$. The distinct eigenvalues of the icosahedral graph are the numbers $-\sqrt{5}, -1, \sqrt{5}, 5$. The distinct eigenvalues of the dodecahedral graph are the numbers $-\sqrt{5}, -2, 0, 1, \sqrt{5}, 3$.

3. Perfect 2-colorings of Platonic graphs

Theorem 3.1. *The graph tetrahedral has perfect 2-colorings only with the matrices* $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Proof . The tetrahedral graph is a 3-regular connected graph. Hence, a parameter matrix of a perfect 2-coloring of it must be one of the following matrices:

$$\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix},$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

By Theorem 2.10 and Corollary 2.7, it is clear that the tetrahedral graph can have perfect 2-colorings with the matrices $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Coloring one of the vertices white and the others black gives a perfect 2-coloring with the matrix $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$. Also, Coloring two of the vertices white and the others black gives a perfect 2-coloring with the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. □

Theorem 3.2. *The cubical graph has perfect 2-colorings only with the matrices* $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Proof . The cubical graph is a 3-regular connected graph. Hence, a parameter matrix of a perfect 2-coloring of it must be one of the following matrices:

$$\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix},$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

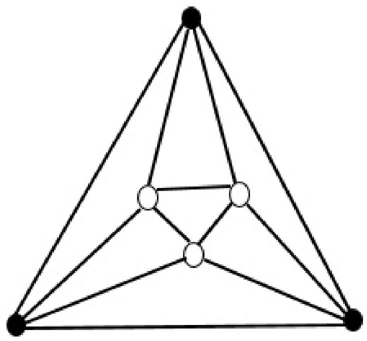
By Theorem 2.9, it is clear that there are no perfect 2-colorings with the matrices $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$. Also, from Proposition 2.8, we conclude that there are perfect 2-colorings with the other matrices. □

Theorem 3.3. *The octahedral graph has perfect 2-colorings only with the matrices* $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}$.

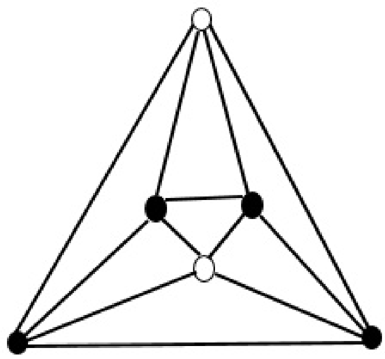
Proof . The octahedral graph is a 4-regular connected graph. Hence, a parameter matrix of a perfect 2-coloring of it must be one of the following matrices:

$$\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix},$$
$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix},$$
$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

By Theorem 2.10 and Collorary 2.7, it is clear that the octahedral graph may have perfect 2-colorings only with the matrices $\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$, and $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. On the other hand, from Proposition 2.5, it follows that there are no perfect 2-colorings with the matrix $\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$. Also, if there existed a perfect 2-coloring with the matrix $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, we would have $|W| = 3$, by Proposition 2.5. However, it is not possible to find a subset of size 3 that each element have exactly one adjacent vertex in that subset. Hence, the metrix $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is not a parameter matrix. Finally, we show perfect 2-colorings of the octahedral graph with the matrices $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}$ in Figure 1. □



(a) with the matrix $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$



(b) with the matrix $\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}$

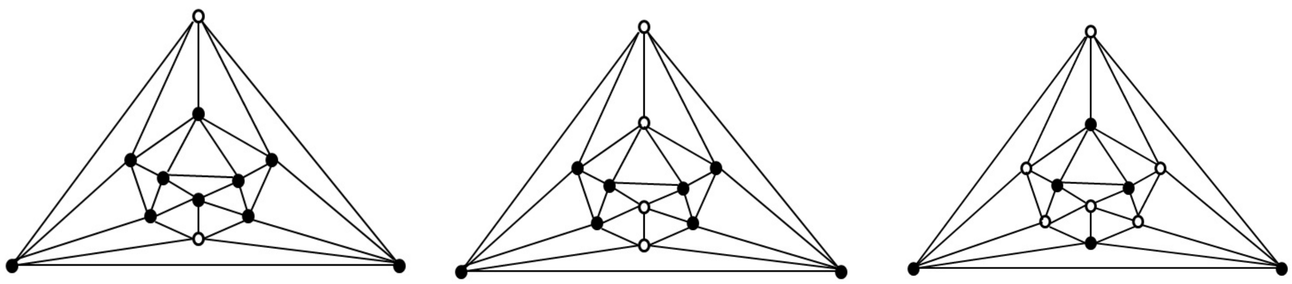
Figure 1: perfect 2-colorings of the octahedral graph

Theorem 3.4. *The icosahedral graph has perfect* 2*-colorings only with the matrices* $\begin{bmatrix} 0 & 5 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, *and* $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

Proof . The icosahedral graph is a 5-regular connected graph. Hence, a parameter matrix of a perfect 2-coloring of it must be one of the following matrices:

$$\begin{bmatrix}0 & 5\\5 & 0\end{bmatrix}, \begin{bmatrix}0 & 5\\4 & 1\end{bmatrix}, \begin{bmatrix}0 & 5\\3 & 2\end{bmatrix}, \begin{bmatrix}0 & 5\\2 & 3\end{bmatrix}, \begin{bmatrix}0 & 5\\1 & 4\end{bmatrix},$$
$$\begin{bmatrix}1 & 4\\4 & 1\end{bmatrix}, \begin{bmatrix}1 & 4\\3 & 2\end{bmatrix}, \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}, \begin{bmatrix}1 & 4\\1 & 4\end{bmatrix},$$
$$\begin{bmatrix}2 & 3\\3 & 2\end{bmatrix}, \begin{bmatrix}2 & 3\\2 & 3\end{bmatrix}, \begin{bmatrix}2 & 3\\1 & 4\end{bmatrix},$$
$$\begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}, \begin{bmatrix}3 & 2\\1 & 4\end{bmatrix},$$
$$\begin{bmatrix}4 & 1\\1 & 4\end{bmatrix}.$$

By Theorem 2.10 and Corollary 2.7, it follows that the icosahedral graph can have perfect 2-colorings with the matrices $\begin{bmatrix}0 & 5\\1 & 4\end{bmatrix}$, $\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$, and $\begin{bmatrix}2 & 3\\3 & 2\end{bmatrix}$. Also, there exist perfect 2-colorings with the above matrices that has been shown in Figure 2. □



(a) with the matrix $\begin{bmatrix}0 & 5\\1 & 4\end{bmatrix}$ (b) with the matrix $\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$ (c) with the matrix $\begin{bmatrix}2 & 3\\3 & 2\end{bmatrix}$

Figure 2: perfect 2-colorings of the icosahedral graph

Theorem 3.5. *The dodecahedral graph has perfect* 2*-colorings with the matrices* $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$ *and* $\begin{bmatrix}0 & 3\\2 & 1\end{bmatrix}$.

Proof . The dodecahedral graph is a 3-regular connected graph. Hence, a parameter matrix must be one of the following matrices:

$$\begin{bmatrix}0 & 3\\3 & 0\end{bmatrix}, \begin{bmatrix}0 & 3\\2 & 1\end{bmatrix}, \begin{bmatrix}0 & 3\\1 & 2\end{bmatrix},$$
$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix},$$
$$\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}.$$

By Theorem 2.10 and Corollary 2.7, it is clear that there are no perfect 2-colorings of the dodecahedral graph with the matrices $\begin{bmatrix}0 & 3\\3 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 3\\1 & 2\end{bmatrix}$, and $\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$.

Also, by Proposition 2.5, it follows that there are no perfect 2-colorings with the matrix $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.
Now, consider the mapping $T : V(GP(10,2)) \to \{1,2\}$ by $T(a_i) = 1$ and $T(b_i) = 2$, for $i = 0, \ldots, 9$.
It is easy to see that the given mapping gives a perfect 2-coloring with the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.
Finally, consider the mapping $T : V(GP(10,2)) \to \{1,2\}$ by

$$T(a_{5i}) = T(a_{5i+2}) = T(a_{5i+3}) = T(b_{5i}) = T(b_{5i+1}) = T(b_{5i+4}) = 2,$$
$$T(a_{5i+1}) = T(a_{5i+4}) = T(b_{5i+2}) = T(b_{5i+3}) = 1,$$

for $i = 0, 1$. It can be easily checked that the given mapping gives a perfect 2-coloring with the matrix $\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$. □

Finally, we summarize the results of this paper in the following table.

Graphs	Parameter Matrices
The tetrahedral graph	$\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
The cubical graph	$\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
The octahedral graph	$\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$
The icosahedral graph	$\begin{bmatrix} 0 & 5 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$
The dodecahedral graph	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$

Table 1: Parameter matrices of Platonic graphs.

References

[1] S.V. Avgustinovich and I. Yu. Mogilnykh, *Perfect 2-colorings of Johnson graphs $J(6,3)$ and $J(7,3)$*, Lecture Notes Comput. Sci. 5228 (2008) 11–19.
[2] S.V. Avgustinovich and I. Yu. Mogilnykh, *Perfect colorings of the Johnson graphs $J(8,3)$ and $J(8,4)$ with two colors*, J. Appl. Ind. Math. 5 (2011) 19–30.
[3] D. Cvetkovic, M. Doob and H. Sachs, *Spectra of graphs*, Johan Ambrosius Barth, Heidelberg, 1995.
[4] D.G. Fon-Der-Flaass, *A bound on correlation immunity*, Siberian Electron. Math. Rep. J. 4 (2007) 133–135.
[5] D.G. Fon-Der-Flaass, *Perfect 2-colorings of a hypercube*, Siberian Math. J. 4 (2007) 923–930.
[6] D.G. Fon-der-Flaass, *Perfect 2-colorings of a 12-dimensional Cube that achieve a bound of correlation immunity*, Siberian Math. J. 4 (2007) 292–295.
[7] A.L. Gavrilyuk and S.V. Goryainov, *On perfect 2-colorings of Johnson graphs $J(v,3)$*, J. Comb. Designs. 21 (2013) 232–252.
[8] C. Godsil, *Compact graphs and equitable partitions*, Linear Algebra Appl. 2258 (1997) 259–266
[9] C. Godsil and R. Gordon, *Algebraic graph theory*, Springer Science+Business Media, LLC, 2004.