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New Approximation Techniques for Solving Variational Inclusions Problem via SP Iterative Algorithm with Mixed Errors for Accretive Lipschitzian Operators

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Abstract

Using different convergence techniques and under the lack of parametrical restrictions, the convergence and stability results of SP iterative algorithm with mixed errors for accretive Lipschitzian operators in Banach spaces are established. We propose numerical examples to verify effectiveness of new convergence techniques and to show that SP iterative algorithm with mixed errors converges more effectively than the Mann, Ishikawa and Noor iterative algorithms with mixed errors. Moreover, new iterative approximation of solution for variational inclusion problem in Banach spaces is investigated by using SP iterative algorithm with mixed errors for accretive Lipschitzian operators. Our results are improvement and generalization of results of Kim[15], Gu[10], Gu and Lu[11], Chugh and Kumar[7] and many others in the literature.

Keywords: Iterative Schemes; Fixed Point, Stability; Accretive Operators; Variational Inequality.
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1. Introduction and Preliminaries

Let X be a real Banach space with dual X^*. The normalized duality mapping J from X to 2^{X^*} is given by $J(x) = \{f \in X^* :< x, f >= \|x\|^2 = \|f\|^2\},\ x \in X$, where $< .,. >$ denotes the generalized duality pairing between X and X^*.

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Definition 1.1. *Let $T : X \to X$ be a mapping with domain $D(T)$ and range $R(T)$ and I denotes the identity operator on X.*

(i) T is said to be Lipschitizian, if there exists $L > 0$ such that for all $x, y \in X$, we have

$$\|Tx - Ty\| \leq L\|x - y\|.$$

(ii) T is said to be non-expansive, if for all $x, y \in X$, we have

$$\|Tx - Ty\| \leq \|x - y\|.$$

(iii) T is said to be accretive iff for all $r > 0$ and $x, y \in X$, we have

$$\|x - y\| \leq \|x - y + r(Tx - Ty)\|. \tag{1.1}$$

(iv) T is said to be pseudo-contractive if for all $r > 0$ and $x, y \in X$, we have

$$\|x - y\| \leq \|(1 - r)(x - y) + r(Tx - Ty)\|. \tag{1.2}$$

Hence, a mapping T is said to be pseudo-contractive iff $I - T$ is accretive. Moreover $(I + T)^{-1}$ is a non-expansive if T is accretive [2]. So, non-expansive and pseudo-contractive mappings are closely connected with accretive mappings.

Definition 1.2 (2). *Let (X, d) be a complete metric space, $T : X \to X$ a selfmap of X. Let $\{x_n\}_{n=1}^{\infty} \subset X$, be the sequence generated by an iterative algorithm involving T which is defined by $x_{n+1} = f(T, x_n)$ ...$(*)$, where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point p of T. Let $\{p_n\}_{n=1}^{\infty} \subset X$ be an arbitrary sequence in X and set $k_n = \|p_n - f(T, p_n)\|$. Then, the iterative procedure (*) is said to be T-stable if and only if $lim_{n\to\infty} k_n = 0$ implies $lim_{n\to\infty} p_n = p$. Moreover if $\sum_{n=0}^{\infty} k_n < \infty$ implies that $lim_{n\to\infty} p_n = p$, then the iterative algorithm defined by $x_{n+1} = f(T, x_n)$ is said to be almost T-stable. Stability implies almost stability but converse may not true[see [25] for details].*

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years[4, 9, 10, 12, 14, 23, 27]. One of the most interesting and important problems in the theory of variational inclusions is the development of an efficient and implementable iterative algorithm. Various kinds of iterative methods have been studied to find the approximate solutions for variational inclusions.
Mann iterative algorithm with errors due to Liu[19,20] :

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n + l_n, \tag{1.3}$$

where $0 \leq \alpha_n \leq 1$ and $\{l_n\}$ is a summable sequence in X.
Ishikawa iterative algorithm with errors due to Liu[19,20]:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n + a_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n + b_n, \end{aligned} \tag{1.4}$$

where $0 \leq \alpha_n, \beta_n \leq 1$ and $\{a_n\},\ \{b_n\}$ are summable sequences in X.
Noor iterative algorithm with errors due to Cho et al.[6]:

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \alpha_n^1 T y_n + a_n u_n \\ y_n &= \beta_n x_n + \beta_n^1 T z_n + b_n v_n \\ z_n &= \gamma_n x_n + \gamma_n^1 T x_n + c_n w_n, \end{aligned} \tag{1.5}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in X and $0 \leq \alpha_n, \beta_n, \gamma_n, a_n, b_n, c_n,$ $\alpha_n^1, \beta_n^1, \gamma_n^1 \leq 1$ with $\alpha_n + \alpha_n^1 + a_n = \beta_n + \beta_n^1 + b_n = \gamma_n + \gamma_n^1 + c_n = 1$.
SP iterative algorithm due to Phuengrattana and Suantai [24] :

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \end{aligned} \tag{1.6}$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$.

Remark 1.3. *Putting $l_n = 0$ and $a_n = b_n = 0$ in (1.3) and (1.4), respectively, we can get Mann [21] and Ishikawa [13] iterative algorithms, respectively. Also if we put $\beta_n = \gamma_n = 0$, then SP iterative algorithm (1.6) becomes Mann iterative algorithm[21].*
The convergence and stability problems for iterative algorithms involving various type of operators have been studied extensively by many authors [1-3, 5-8, 11, 15-18, 24, 25, 28, 29]. Osilike [24] proved that certain Mann and Ishikawa iterative procedures are stable with respect to Lipschitz pseudo-contractions in an arbitrary Banach space. In 2006, Kim [15] studied the strong convergence of Ishikawa iterative algorithm with mixed errors for the accretive Lipschitzian operators in Banach spaces. Chugh and Kumar [7] studied the strong convergence and almost stability of SP iterative algorithm with mixed errors for the accretive Lipschitzian operators in Banach spaces using Lemma 1.5. Chugh et al. [8] studied some strong convergence results of random iterative algorithms with errors using accretive maps in Banach spaces.

We shall need the following important lemmas:

Lemma 1.4. *([2]). Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying: $a_{n+1} \leq \delta a_n + b_n, \quad n \geq 1$, where $b_n \geq 0, lim_{n\to\infty} b_n = 0$ and $0 \leq \delta < 1$. Then $a_n \to 0$ as $n \to \infty$.*

Lemma 1.5. *([7,15]) Let a_n, b_n and c_n be non-negative real sequences satisfying the condition: $a_{n+1} \leq (1-\lambda_n)a_n + b_n, +c_n, n \geq n_0$, where n_0 is some non-negative integer and λ_n is a sequence in [0,1] such that $\sum_{n=0}^{\infty} \lambda_n = \infty, b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} \lambda_n < \infty$. Then $a_n \to 0$ as $n \to \infty$.*

Let $T, A : X \to X$, $g : X \to X^*$ be three mappings on a real reflexive Banach space X and $\varphi : X^* \to R \cup \{\infty\}$ be a function with continuous subdifferential $\partial\varphi : X^* \to 2^{X^*}$ defined by $(\partial\varphi)x = \{x^* \in X^* : \varphi(y) - \varphi(x) \geq < y - x, x^* >, \forall y \in X\}$. If for any given $y \in X$, there exists a $x \in X$ such that

$$g(x) \in D(\partial\varphi), < Tx - Ax - y, f - g(x) > \geq \varphi(g(x) - \varphi(x)), \forall f \in X^* \tag{1.7}$$

holds, then, x is solution of a variational inclusion problem (1.7).

Lemma 1.6 (4). *Let $\partial\varphi \circ g : X \to 2^X$ be a mapping on a real reflexive Banach space X. Then the followings are equivalent:*

(i) $p \in X$ is a solution of variational inclusion problem (1.7);

(ii) $p \in X$ is a fixed point of the mapping $S : X \to 2^X$;
$Sx = y - \big(Tx - Ax + \partial\varphi(g(x))\big) + x$;

(iii) $p \in X$ is a solution of the equation $y = Tx - Ax + \partial\varphi(g(x))$.

Also, it is well known (see[22]) that if $T : X \to X$ is accretive and continuous, then T is m-accretive, so that for given $y \in X$, the equation $x + Tx = y$ has a unique solution.

Due to revolution in computer programming, the stability of iterative algorithms has extensively been studied . Also, numerically, it is of vital interest to know which of the given iterative algorithm converges faster to a desired solution. Hence in computational mathematics, a fixed point iterative algorithm is valuable and useful for applications if it satisfies the following conditions: (i) it converges to a fixed point of a given operator (ii) it is stable (iii) it is faster as compared to other iterative algorithms existing in the literature.
Motivated by above facts, in this paper, we improve results of Chugh and Kumar [7], Kim[15] and many other using Lemma 1.4 instead of Lemma 1.5 and using different convergence techniques instead of old convergence techniques as proposed in [7,10]. We support our results with two numerical examples and applications. Moreover, with the help of C++ programs, we show that using new convergence techniques, SP iterative algorithm with errors becomes more rapid and stable instead of almost stable as in [7,10].

2. Main Results

Theorem 2.1. *Let T be an accretive Lipschitzian self map with Lipschitz constant $L \geq 1$ on a real Banach space X. For any given operator $S : X \to X$ defined by $Sx = f - Tx,\ x \in X$, where $f \in X$ is any given point,the SP iterative algorithm with mixed errors[7] is given by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n S y_n + u_n \\ y_n &= (1-\beta_n)z_n + \beta_n S z_n + v_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n S x_n + w_n, \end{aligned} \tag{2.1}$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are sequences in X with following conditions:

(i) $0 < \alpha < \alpha_n - \alpha_n^2 L^3(1+L) - \beta_n(L-1) - \beta_n\gamma_n(L-1)^2 - \gamma_n L < 1, (n \geq 0)$;
(ii) $u_n = u_n^{'} + u_n^{''}, \|u_n^{'}\| = o(\alpha_n), (n \geq 0)$ *and* $\sum_{n=0}^{\infty} \|u_n^{''}\| < \infty$;
(iii) $\sum_{n=0}^{\infty} \|v_n\| < \infty, \sum_{n=0}^{\infty} \|w_n\| < \infty$.

Then for any given $x_0 \in X$,
(1) the SP iterative algorithm with mixed errors generated from x_0 by (2.1) converges strongly to a unique fixed point p of S.
(2) the SP iterative algorithm with mixed errors generated from x_0 by (2.1) is S-stable, that is, for any sequence $\{p_n\} \subset X, lim_{n\to\infty} p_n = p$ if and only if $lim_{n\to\infty} k_n = 0$, where $k_n = \|p_{n+1} - (1-\alpha_n)q_n - \alpha_n S q_n - u_n\|$, $q_n = (1-\beta_n)r_n + \beta_n S r_n + v_n$, $r_n = (1-\gamma_n)p_n + \gamma_n S p_n + w_n$.

Proof .(1) From (2.1), we have

$$(x_{n+1} - p) - \alpha_n(Sx_{n+1} - Sp) = (1-\alpha_n)(y_n - p) - \alpha_n(Sx_{n+1} - Sy_n) + u_n \tag{2.2}$$

As T is an accretive Lipschitzian mapping, so the mapping (S) will be accretive Lipschitzian and hence using (1.1) and (2.2), we get

$$\begin{aligned} \|x_{n+1} - p\| &\leq \|x_{n+1} - p - \alpha_n(Sx_{n+1} - Sp)\| \\ &= \|(1-\alpha_n)(y_n - p) - \alpha_n(Sx_{n+1} - Sy_n) + u_n\| \\ &\leq (1-\alpha_n)\|(y_n - p)\| + \alpha_n\|(Sy_n - Sx_{n+1})\| + \|u_n\| \end{aligned} \tag{2.3}$$

Now, using Lipschitz condition on S, (2.1) implies

$$\begin{aligned}\|Sy_n - Sx_{n+1}\| &\leq L\|x_{n+1} - y_n\| \\ &\leq L\alpha_n\|y_n - Sy_n\| + L\|u_n\| \\ &\leq L\alpha_n\|y_n - p\| + L\alpha_n\|Sy_n - p\| + L\|u_n\| \\ &\leq (1+L)L\alpha_n\|y_n - p\| + L\|u_n\| \end{aligned} \tag{2.4}$$

Also, from (2.1), we have the following estimates:

$$\begin{aligned}\|y_n - p\| &\leq (1-\beta_n)\|z_n - p\| + \beta_n\|Sz_n - p\| + \|v_n\| \\ &\leq (1-\beta_n)\|z_n - p\| + \beta_n L\|z_n - p\| + \|v_n\| \\ &= [1 + \beta_n(L-1)]\|z_n - p\| + \|v_n\| \end{aligned} \tag{2.5}$$

and

$$\|z_n - p\| \leq [1 + \gamma_n(L-1)]\|x_n - p\| + \|w_n\| \tag{2.6}$$

Using inequalities (2.3)-(2.6) and condition (i), we arrive at

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\alpha_n)[1+\beta_n(L-1)][1+\gamma_n(L-1)]\|x_n - p\| \\ &+ \alpha_n^2 L(L+1)[1+\beta_n(L-1)][1+\gamma_n(L-1)]\|x_n - p\| \\ &+ (\alpha_n L + 1)\|u_n\| + (1-\alpha_n)\|v_n\| + \alpha_n^2 L(L+1)\|v_n\| \\ &+ \alpha_n^2 L(L+1)[1+\beta_n(L-1)]\|w_n\| + [1+\beta_n(L-1)](1-\alpha_n)\|w_n\| \\ &\leq [1 - \alpha_n + \alpha_n^2 L(L+1)][1+\beta_n(L-1)][1+\gamma_n(L+1)]\|x_n - p\| \\ &+ (1+L)\|u_n\| + [1 + L(L+1)]\|v_n\| + L[1+L(L+1)]\|w_n\| \\ &\leq [1 - \{\alpha_n - \alpha_n^2 L^2(L+1) - \beta_n(L-1)\}][1+\gamma_n(L+1)]\|x_n - p\| \\ &+ (1+L)\|u_n\| + [1 + L(L+1)]\|v_n\| + L[1+L(L+1)]\|w_n\| \\ &\leq [1 - \{\alpha_n - \alpha_n^2 L^3(L+1) - \beta_n(L-1) - \beta_n\gamma_n(L-1)^2 - \gamma_n L\}]\|x_n - p\| \\ &+ (1+L)\|u_n\| + [1 + L(L+1)]\|v_n\| + L[1+L(L+1)]\|w_n\| \\ &\leq [1-\alpha]\|x_n - p\| + (1+L)\|u_n\| \\ &+ [1 + L(L+1)]\|v_n\| + L[1+L(L+1)]\|w_n\|. \end{aligned} \tag{2.7}$$

Also, by condition (ii) we have $u_n^{'} = \alpha_n\delta_n$, where $\{\delta_n\}$ is a sequence of non negative numbers tending to 0. Hence

$$\|u_n\| \leq \alpha_n\delta_n + \|u_n^{''}\|. \tag{2.8}$$

Set $[1-\alpha] = \delta$ and $(1+L)(\alpha_n\delta_n + \|u_n^{''}\|) + [1+L(L+1)]\|v_n\| + L[1+L(L+1)]\|w_n\| = \sigma_n$. Then using (2.8), (2.7) yields

$$\|x_{n+1} - p\| \leq \delta\|x_n - p\| + \sigma_n. \tag{2.9}$$

By conditions(ii)-(iii) and Lemma 1.4, (2.9) yields $lim_{n\to\infty} x_p = 0$. Therefore, SP iterative algorithm with mixed errors (2.1) converges strongly to a fixed point p of S.
To prove uniqueness of fixed point p, let q be an another fixed point of S. Since $(-S)$ is accretive, so we have

$$\|q-p\| \leq \|q-p-\alpha_n(Sq-Sp)\| = \|q-p-\alpha_n(q-p)\| \leq (1-\alpha_n)\|q-p\|,$$

which is possible only when $p=q$.
(2)Suppose that $\{p_n\}$ is an arbitrary sequence in X and $lim_{n\to\infty}k_n = 0$. Then

$$\begin{aligned}\|p_{n+1}-Sp\| &= \|p_{n+1}-(1-\alpha_n)q_n-\alpha_n Sq_n-u_n\| \\ &\quad + \|(1-\alpha_n)q_n+\alpha_n Sq_n+u_n-Sp\| = k_n+\|S_n-Sp\|,\end{aligned} \tag{2.10}$$

where

$$s_n = (1-\alpha_n)q_n+\alpha_n Sq_n+u_n \tag{2.11}$$

Using (2.11), we have

$$(s_n-p)-\alpha_n(Ss_n-Sp) = (1-\alpha_n)(q_n-p)-\alpha_n(Ss_n-Sq_n)+u_n$$

which further implies

$$\begin{aligned}\|s_n-p\| &\leq \|s_n-p-\alpha_n(Ss_n-Sp)\| \\ &= \|(1-\alpha_n)(q_n-p)-\alpha_n(Ss_n-Sq_n)+u_n\| \\ &\leq (1-\alpha_n)\|q_n-p\|+\alpha_n\|Ss_n-Sq_n\|+\|u_n\|\end{aligned} \tag{2.12}$$

Now, similar to estimates (2.4)-(2.6), we have the following estimates:

$$\|Ss_n-Sq_n\| \leq L\alpha_n(1+L)\|q_n-p\|+L\|u_n\| \tag{2.13}$$

$$\|q_n-p\| \leq [1+\beta_n(L-1)]\|r_n-p\|+\|v_n\| \tag{2.14}$$

and

$$\|r_n-p\| \leq [1+\gamma_n(L-1)]\|p_n-p\|+\|w_n\| \tag{2.15}$$

Using estimates (2.12)-(2.15), we arrive at

$$\begin{aligned}\|s_n-p\| &\leq [1-\{\alpha_n-\alpha_n^2L^3(L+1)-\beta_n(L-1)-\beta_n\gamma_n(L-1)^2-\gamma_nL\}]\|p_n-p\| \\ &\quad +(1+L)\|u_n\|+[1+L(L+1)]\|v_n\|+L[1+L(L+1)]\|w_n\|\end{aligned} \tag{2.16}$$

Substituting (2.16) in (2.10), we obtain

$$\begin{aligned}\|p_{n+1}-p\| &\leq k_n+[1-\{\alpha_n-\alpha_n^2L^3(L+1)-\beta_n(L-1)-\beta_n\gamma_n(L-1)^2-\gamma_nL\}] \\ &\quad \|p_n-p\|+(1+L)\|u_n\|+[1+L(L+1)]\|v_n\|+L[1+L(L+1)]\|w_n\| \\ &\leq \delta\|p_{n+1}-p\|+\sigma_n,\end{aligned} \tag{2.17}$$

where $[1-\alpha]=\delta$ and $k_n+(1+L)(\alpha_n\delta_n+\|u_n''\|)+[1+L(L+1)]\|v_n\|+L[1+L(L+1)]\|w_n\|=\sigma_n$. Using Lemma 1.4 and conditions (ii)-(iii) together with $lim_{n\to\infty}k_n=0$, (2.17) yields $lim_{n\to\infty}p_n = p$

Conversely, let $lim_{n\to\infty}p_n = p$, then using (2.11),(2.16) and conditions (ii)-(iii), we have

$$\begin{aligned} k_n &= \|p_{n+1} - (1-\alpha_n)q_n - \alpha_n Sq_n - u_n\| \\ &= \|p_{n+1} - s_n\| \\ &\leq \|p_n - p\| + \|s_n - p\| \\ &\leq \|p_n - p\| + [1 - \{\alpha_n - \alpha_n^2 L^3(L+1) - \beta_n(L-1) - \beta_n\gamma_n(L-1)^2 - \gamma_n L\}]\|p_n - p\| \\ &+ (1+L)\|u_n\| + [1 + L(L+1)]\|v_n\| + L[1 + L(L+1)]\|w_n\| \end{aligned}$$

which implies $lim_{n\to\infty}k_n = 0$. Therefore, the iterative algorithm (2.1) is S-stable. This completes the proof of Theorem 2.1. □

Corollary 2.2. *Let T be an accretive Lipschitzian self map with Lipschitz constant $L \geq 1$ on a real Banach space X. For any given operator $S : X \to X$ defined by $Sx = f - Tx,\ x \in X$, where $f \in X$ is any given point,the Mann iterative algorithm with mixed errors is given by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sx_n + u_n, \tag{2.18}$$

where $0 \leq \alpha_n \leq 1$ and $\{u_n\}$ is a sequence in X with the following conditions:

(i) $0 < \alpha < \alpha_n < \frac{1}{(1+L)L^3}, (n \geq 0)$;

(ii) $u_n = u_n^{'} + u_n^{''}, \|u_n^{'}\| = o(\alpha_n), (n \geq 0)$ *and* $\sum_{n=0}^{\infty} \|u_n^{''}\| < \infty$.

Then for any given $x_0 \in X$,
(1) the Mann iterative algorithm with mixed errors generated from x_0 by (2.18) converges strongly to a unique fixed point p of S.
(2) the Mann iterative algorithm with mixed errors generated from x_0 by (2.18) is S-stable, that is, for any sequence $\{p_n\} \subset X$, $lim_{n\to\infty}p_n = p$ if and only if $lim_{n\to\infty}k_n = 0$, where $k_n = \|p_{n+1} - (1 - \alpha_n)p_n - \alpha_n Sp_n - u_n\|$.

Proof . Taking $\beta_n = 0, \gamma_n = 0, v_n = 0$ and $w_n = 0$, in Theorem 2.1, the proof is obvious. □

The following examples and their numerical simulations show verification of Theorem 2.1 and display effectiveness of new convergence technique of SP iterative algorithm with mixed errors.

Example 2.3. *Let $X = [0,3]$. Define an operator S from X to X as $Sx = 3 - x$, with fixed point 1.5. It is easy to see that the operator(–S) is a Lipschitz accretive operator with Lipschitz constant $L = 1$. Put $\alpha = 0.008$, $\alpha_n = \frac{1}{(1+L)^3}$, $\|u_n\| = \frac{1}{(n+1)(1+L)^3} + \frac{1}{(n+1)^2}$, $\beta_n = \gamma_n = \frac{1}{(1+L)^6}$, $\|v_n\| = \frac{1}{(n+2)^2}$ and $\|w_n\| = \frac{1}{(n+3)^2}$. All the conditions of Theorem 2.1 are satisfied. Therefore, the sequence $\{x_n\}$ defined by equation (2.1) converges strongly to the fixed point 1.5 and is S-stable. Taking initial value $x_0 = 2$, convergence comparison of different iterative algorithms to the fixed point 1.5 is shown in the Table 1.*

No of iterations n	*Mann iterative algorithm with mixed errors* x_{n+1}	*Ishikawa iterative algorithm with mixed errors* x_{n+1}	*Noor iterative algorithm with mixed errors* x_{n+1}	*SP iterative algorithm with mixed errors* x_{n+1}
1	1.96296	1.99868	1.99863	1.96043
2	1.92867	1.99736	1.99726	1.92398
3	1.89692	1.99605	1.9959	1.89043
4	1.86751	1.99474	1.99454	1.85953
5	1.84029	1.99343	1.99319	1.83107
-	-	-	-	-
138	1.50001	1.84708	1.8424	1.50001
139	1.50001	1.84616	1.84146	1.50001
140	1.50001	1.84525	1.84053	1.50001
141	1.50001	1.84433	1.8396	1.5
142	1.50001	1.84342	1.83867	1.5
143	1.50001	1.84252	1.83774	1.5
144	1.50001	1.84161	1.83681	1.5
145	1.50001	1.84071	1.83589	1.5
146	1.50001	1.83981	1.83497	1.5
147	1.50001	1.83891	1.83405	1.5
148	1.50001	1.83802	1.83314	1.5
149	1.50001	1.83712	1.83222	1.5
150	1.5	1.83623	1.83131	1.5
151	1.5	1.83534	1.83041	1.5
152	1.5	1.83446	1.8295	1.5
-	-	-	-	-
4194	1.5	1.50001	1.50001	1.5
4195	1.5	1.50001	1.50001	1.5
4196	1.5	1.50001	1.50001	1.5
4197	1.5	1.50001	1.5	1.5
4198	1.5	1.50001	1.5	1.5
-	-	-	-	-
4350	1.5	1.50001	1.5	1.5
4351	1.5	1.50001	1.5	1.5
4352	1.5	1.50001	1.5	1.5
4353	1.5	1.5	1.5	1.5
4354	1.5	1.5	1.5	1.5

Example 2.4. *Let* $X = [0, 1]$. *Define an operator* S *from* X *to* X *as* $Sx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$, *with fixed point 0.351708. It is easy to see that the operator(–S) is a Lipschitz accretive operator with Lipschitz constant* $L = 2$. *If we choose* $\alpha_n = \frac{1}{(1+L)^3}$, $\beta_n = \gamma_n = \frac{1}{(1+L)^7}$, $\|u_n\| = \frac{1}{(n+1)(1+L)^3} + \frac{1}{(n+1)^2}$, $\|v_n\| = \frac{1}{(n+2)^2}$, $\|w_n\| = \frac{1}{(n+3)^2}$ *and* $\alpha = 0.0026$. *All the conditions of Theorem 2.1 are satisfied. Therefore, the sequence* $\{x_n\}$ *defined by equation (2.1) converges strongly to the fixed point 0.351708 and is S-stable. Taking initial value* $x_0 = 2$, *convergence comparison of different iterative algorithms to the*

fixed point 0.351708 is shown in the Table 2.

No of iterations n	*Mann iterative algorithm with mixed errors* x_{n+1}	*Ishikawa iterative algorithm with mixed errors* x_{n+1}	*Noor iterative algorithm with mixed errors* x_{n+1}	*SP iterative algorithm with mixed errors* x_{n+1}
1	0.962963	0.999545	0.999543	0.962086
2	0.927441	0.999089	0.999086	0.925764
3	0.893487	0.998634	0.998629	0.891089
4	0.861105	0.99818	0.998172	0.858062
5	0.830279	0.997725	0.997716	0.826666
-	-	-	-	-
177	0.351709	0.923137	0.922619	0.351709
178	0.351709	0.922724	0.922203	0.351709
179	0.351709	0.922311	0.921787	0.351708
180	0.351709	0.921899	0.921371	0.351708
181	0.351709	0.921487	0.9209557	0.351708
182	0.351709	0.921075	0.920539	0.351708
183	0.351708	0.920663	0.920124	0.351708
184	0.351708	0.920252	0.919709	0.351708
185	0.351708	0.919841	0.919294	0.351708
-	-	-	-	-
15316	0.351708	0.351709	0.351709	0.351708
15317	0.351708	0.351709	0.351709	0.351708
15318	0.351708	0.351709	0.351708	0.351708
15319	0.351708	0.351709	0.351708	0.351708
15320	0.351708	0.351709	0.351708	0.351708
-	-	-	-	-
15851	0.351708	0.351709	0.351708	0.351708
15852	0.351708	0.351709	0.351708	0.351708
15853	0.351708	0.351709	0.351708	0.351708
15854	0.351708	0.351708	0.351708	0.351708
15855	0.351708	0.351708	0.351708	0.351708

Remark 2.5. *Theorem 2.1 is an improvement of [Theorem 2.1,[7]], as "almost stability" of SP iterative algorithm with mixed errors is replaced by the "stability" using different convergence technique.*

It is shown that using new convergence technique,SP iterative algorithm with mixed errors has better convergence rate as compared to Mann, Ishikawa and Noor iterative algorithms with mixed errors and hence has good potential for further applications.

3. Applications

In this section, we investigate the solutions of nonlinear variational inclusion problem using iterative algorithms with mixed errors.

Theorem 3.1. *Suppose that X is a real reflexive Banach space, $T, A : X \to X,\ g : X \to X^*$ are three non-expansive mappings and $\varphi : X^* \to R\cup\{\infty\}$ is a function with non-expansive subdifferential $\partial\varphi$. Define an operator $R : X \to X$ by $Rx = f - (Tx - Ax + \partial\varphi(g(x))) + x$, where $f \in X$is any given point. Let $\{x_n\}$ be the iterative algorithm with mixed errors defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n R y_n + u_n \\ y_n &= (1-\beta_n)z_n + \beta_n R z_n + v_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n R x_n + w_n, \end{aligned} \tag{3.1}$$

where $0 \le \alpha_n, \beta_n, \gamma_n \le 1$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are sequences in X with following restrictions:

(i) $0 < \alpha < \alpha_n - \alpha_n^2 L^{*3}(1+L^*) - \beta_n(L^*-1) - \beta_n\gamma_n(L^*-1)^2 - \gamma_n L^* < 1, (n \ge 0);\ L^* = L+1$
(ii) $u_n = u_n^{'} + u_n^{''}, \|u_n^{'}\| = o(\alpha_n), (n \ge 0)$ *and* $\sum_{n=0}^{\infty} \|u_n^{''}\| < \infty;$
(iii) $\sum_{n=0}^{\infty} \|v_n\| < \infty, \sum_{n=0}^{\infty} \|w_n\| < \infty.$

Then the iterative algorithm (3.1) converges to $x^* \in X^*$ and x^* is the unique solution of nonlinear variational inclusion problem (1.7).

Proof . As $T,\ A,\ g$ and $\partial\varphi$ are non-expansive operators,so (–A) and $\partial\varphi \circ g$ are non-expansive operators. Hence, with ease we can show that

$$\|x - y\| = \|x - y + r[(T - A + \partial\varphi \circ g - I)x - (T - A + \partial\varphi \circ g - I)y\|.$$

Therefore, $T - A + \partial\varphi \circ g - I : X \to X$ is a Lipschitzian accretive operator with a Lipschitz constant say $L \ge 1$. Since $T - A + \partial\varphi\circ g - I$ is Lipschitzian accretive operator, so $T - A + \partial\varphi\circ g - I$ is m-accretive operator. Hence, for any $f \in X$, the equation $f = x + (T - A + \partial\varphi \circ g - I)x$ has a unique solution $x^* \in X$. Using Lemma 1.6, it is easy to see that $x^* \in X$ is a solution of nonlinear variational inclusion problem (1.7)and it is the fixed point of operator R . Again, since $T - A + \partial\varphi \circ g - I : X \to X$ is Lipschitzian accretive operator with Lipschitz constant $L \ge 1$, so $R : X \to X$ is Lipschitzian operator with Lipschitz constant $L^* = 1 + L$, such that (R) is an accretive. Replacing S by R in (2.1), L by L^* in condition (i) of Theorem 2.1 and following the procedure of the proof of Theorem 2.1, it is easy to prove that the iterative algorithm (3.1) converges to the unique solution $x^* \in X$ of nonlinear variational inclusion problem (1.7). □

Putting $v_n = w_n = 0, \beta_n = 0$ and $\gamma_n = 0$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. *Suppose that X is a real reflexive Banach space, $T, A : X \to X,\ g : X \to X^*$ are three non-expansive mappings and $\varphi : X^* \to R\cup\{\infty\}$ is a function with non-expansive subdifferential $\partial\varphi$. Define an operator $R : X \to X$ by $Rx = f - (Tx - Ax) + x$, where $f \in X$is any given point. Let $\{x_n\}$ be the iterative algorithm with mixed errors defined by*

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n R y_n + u_n, \tag{3.2}$$

where $0 \le \alpha_n \le 1$ and $\{u_n\}$ are sequences in X with following restrictions:

(i) $0 < \alpha < \alpha_n < \frac{1}{L^{*3}(1+L^*)}, (n \ge 0);\ L^* = L+1$
(ii) $u_n = u_n^{'} + u_n^{''}, \|u_n^{'}\| = o(\alpha_n), (n \ge 0)$ *and* $\sum_{n=0}^{\infty} \|u_n^{''}\| < \infty.$

Then the iterative algorithm (3.2) converges to $x^* \in X^*$ and x^* is the unique solution of nonlinear variational inclusion problem (1.7).

Taking $\varphi \equiv 0$ and $u_n = v_n = w_n = 0$ in Theorem 3.1, we can obtain the following corollary:

Corollary 3.3. *Suppose X is a real reflexive Banach space, $T, A : X \to X$, $g : X \to X^*$ are three non-expansive mappings. Define an operator $R : X \to X$ by $Rx = f - (Tx - Ax) + x$, where $f \in X$ is any given point. Let $\{x_n\}$ be the iterative algorithm with mixed errors defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n R y_n \\ y_n &= (1-\beta_n)z_n + \beta_n R z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n R x_n, \end{aligned} \tag{3.3}$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$ with the following restrictions:

(i) $0 < \alpha < \alpha_n - \alpha_n^2 L^{*3}(1+L^*) - \beta_n(L^*-1) - \beta_n\gamma_n(L^*-1)^2 - \gamma_n L^* < 1, (n \geq 0),\ L^* = L+1$

Then the iterative algorithm (3.3) converges to $x^* \in X^*$ and x^* is the unique solution of nonlinear variational inequality $\big\langle Tx - Ax - y, f - g(x)\big\rangle \geq 0,\ \forall f \in X^*$.

Remark 3.4. *Theorem 3.1 extends and improves [Theorem 2 of [10]] as the Mann iterative algorithm with mixed errors is replaced by more general and faster SP iterative algorithm with mixed errors.*

Remark 3.5. *Corollary 3.2 improves the results of [10] as instead of almost stability, stability of Mann iterative algorithm with mixed errors is proved.*

Remark 3.6. *Theorem 3.1 extends and improves [Theorems 1,2 of[11]] as SP iterative algorithm with mixed errors is used which is more general and faster as compared to Mann iterative algorithm with mixed errors and has better convergence rate as compared to Ishikawa iterative algorithm with mixed errors.*

Remark 3.7. *Theorem 3.1 generalizes the results in [4] as the sequence $\{\alpha_n\}$ need not converge to zero and bounded condition on domain or range of mapping R is omitted. Theorem 3.1 also extends and improves some results of [9,12,20].*

4. Conclusions

1. Theorem 2.1 guarantees the convergence of SP iterative algorithm with mixed errors (2.1) using new convergence technique instead of old convergence technique as in [7,10].
2. Theorem 2.1 proves that SP iterative algorithm mixed errors (2.1) becomes stable instead of almost stable as in [7,10].
3. Examples 2.3 and Example 2.4 are examples of accretive maps in Banach spaces for supporting Theorem 2.1.
4. Table 1 and Table 2 show that using new convergence technique, SP iterative algorithm with mixed errors convergences faster than Mann, Ishikawa and Noor iterative algorithms with mixed errors.
5. In Section 3, we have shown applications of iterative algorithms with mixed errors to solve variational inclusion problem.

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