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Differential transform method for a nonlinear system of differential equations arising in HIV infection of $CD4^+T$ cells

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Abstract

In this paper, differential transform method (DTM) is described and is applied to solve systems of nonlinear ordinary differential equations which is arising in HIV infections of cell. Intervals of validity of the solution will be extended by using Pade approximation. The results also will be compared with those results obtained by Runge-Kutta method. The technique is described and is illustrated with one numerical example. The numerical results shown that the reliability and efficiency of the method.

Keywords: Differential transform method; Systems of nonlinear ordinary differential equations; Pade approximation; Fourth order Runge-Kutta method.
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1. Introduction

Many mathematical models have proposed Vivo dynamical of T cell and HIV (Human Immunodeficiency Virus) interaction. Our interesting model is the one that is presented in [1], which is explained by the following nonlinear system of ordinary differential equations:

$$\begin{cases} \frac{dT}{dt} = p - \alpha T + rT(1 - \frac{T+I}{T_{max}}) - kVT, \\ \frac{dI}{dt} = kVT - \beta I, \\ \frac{dV}{dt} = N\beta I - \gamma V. \end{cases} \tag{1.1}$$

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with the following initial conditions,

$$T(0) = r_1, I(0) = r_2, V(0) = r_3,$$

where $T(t)$, $I(t)$ and $V(t)$ denoted the concentration of susceptible $CD4^+T$ cells, infected $CD4^+T$ cells by the HIV viruses, and free HIV virus particles in the blood, respectively. Parameters α,β, and γ are natural turn-over rates of uninfected T cells, infected T cells, and virus particles, respectively. The logistic growth of the healthy $CD4^+T$ cells is now described by $rT(1-\frac{T+I}{T_{max}})$, and proliferation of infected $CD4^+T$ cells is neglected. The term kVT describes the incidence of HIV infection of healthy $CD4^+T$ cells, where $k > 0$ is the infection rate. Each infected $CD4^+T$ cell is assumed to produce N virus particles during its lifetime, including any of its daughter cells. The body is believed to produce $CD4^+T$ cells from precursors in the bone marrow and thymus at a constant rate . When stimulated by antigen or mitogen, cells multiply through mitosis with a rate p. T_{max} is the maximum level of cell concentration in the body [2, 5]. In this paper, a new kind of analytical approach for a non-linear system of ordinary differential equations called Differential transformation method (DTM) is addressed and used to approximate solutions for a well-known non-linear system. The differential transformation method is a kind of analytical technique based on the Taylor series expansion. This method constructs an analytic approximation to the solution, polynomial form. The concept of differential transform method was first proposed by Zhou and was applied to solve linear and nonlinear initial value problems in electric circuit analysis [6]. Chen and Liu applied this method to solve two-boundary-value problems [7]. Jang, Chen and Liu used two-dimensional differential transform method to solve partial differential equations [8]. Yu and Chen applied the differential transformation method for optimization of the rectangular fins with variable thermal parameters [9, 10]. Unlike the traditional high order Taylor series method that requires many symbolic computations, the differential transform method is an iterative procedure for obtaining Taylor series solutions. This method will not consume too much computer time when applying to non-linear or parameter varying systems.

2. Basic Idea of Differential Transform Method

As in Refs. [7, 8, 11, 12, 13], the differential transformation is based on some elementary definitions, and some statements, which will be stated as follows.

Definition 2.1. *The one-dimensional differential transform of a function $c(x)$ is defined as:*

$$C(k) = \frac{1}{k!}[\frac{\partial^k}{\partial x^k}c(x)]_{x=x_0},$$

where $c(x)$ is analytic and continuously differentiable with respect to x on the domain of interest and $C(k)$ is the transformed function, which is called T-function.

Definition 2.2. *The inverse differential transform of $C(k)$ is defined as follows:*

$$c(x) = \sum_{k=0}^{\infty} C(k)(x-x_0)^k,$$

when $x_0 = 0$, Definitions 2.1 and 2.2 turn to the followings:

$$C(k) = \frac{1}{k!}[\frac{\partial^k}{\partial x^k}c(x)]_{x=0}, \tag{2.1}$$

$$c(x) = \sum_{k=0}^{\infty} C(k)x^k, \tag{2.2}$$

where $c(x)$ is the original function and $C(k)$ is the T-function.

Substituting (2.1) into (2.2) leads to

$$c(x) = \sum_{k=0}^{\infty} \frac{1}{k!}[\frac{\partial^k}{\partial x^k} c(x)]_{x=0} x^k.$$

In real applications, finite form of the series (2.2) will be considered as follows:

$$c(x) = \sum_{k=0}^{n} C(k)x^k.$$

From Definitions 2.1 and 2.2, it is readily proved that the transformed functions comply with the basic mathematical operations [9, 10, 13]. These statements are illustrated in the Table 1.

TABLE 1

Transformation of some functions [9]

Original function	Transformed function
c(t)=u(t)± v(t)	C(k)=U(k)±V(k)
c(t)=au(t)	C(k)=aU(k)
c(t)=$\frac{\partial}{\partial t}$u(t)	C(k)=(k+1)U(k+1)
c(t)=u(t)v(t)	C(k)=$\sum_{r=0}^{k}$U(r)V(k-r)
c(t)=e^{lt}	C(k)=$\frac{l^k}{k!}$
c(t)=sin$(wt+a)$	C(k)=$\frac{w^k}{k!}\sin(\frac{k\pi}{2}+a)$
c(t)=cos$(wt+a)$	C(k)=$\frac{w^k}{k!}\cos(\frac{k\pi}{2}+a)$

3. Pade approximation

Here we will investigate the construction of the Pade approximants for the functions studied. Pade approximation of a function is given by the ratio of two polynomials. The coefficients of the polynomial in the numerator and denominator are determined by using the coefficients in the Taylor series expansion of the function. Suppose that we are given a power series $\sum_{i=0}^{\infty} c_i t^i$ representing a function so that $f(x) = \sum_{i=0}^{\infty} c_i t^i$.

The Pade approximation of a function is a rational fraction and a notation for such a Pade approximation is shown as:

$$[m,n] = \frac{P_m(x)}{Q_n(x)} = \frac{a_0 + a_1 t + a_2 t^2 + ... + a_m t^m}{b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n} = c_0 + c_1 t + c_2 t^2 + \cdots + c_{m+n} t^{m+n}, \tag{3.1}$$

where $P_m(x)$ and $Q_n(x)$ are polynomials of degree at most m and n. We impose the normalization condition $Q_n(0) = b_0 = 1$, therefore, there are $m+1$ independent numerator coefficients and n independent denominator coefficients, making $m+n+1$ unknown coefficients in all. This number suggests that normally the ought to fit the power series in equation (3.1) through the orders and finally we know that the Pade approximation is uniquely determined.

The construction of $[m,n]$ approximation involves only algebraic operations. Each choice of m degree of the numerator and n degree of the denominator, leads to an approximation. The major difficulty in applying this technique is how to direct the choice in order to obtain the best approximation. This requires a criterion which dictates the choice of approximation, depending on the shape of the solution. A criterion which has worked well here is the choice of [m,n] approximation such that $m = n$. Using the symbolic computation software MATHEMATICA, we directly employ the command "Pad Approximation" about the point $x = x_0$ to generate the Pad approximation of the function in the following sections.

4. Applications

Considering the following values, the differential transformation of the system (1.1) can be constructed as follows:

TABLE 2
Numerical values for variables of system (1.1)

Variable	Numerical Value	Variable	Numerical Value	Variable	Numerical Value
T(0)	0.1	α	0.02	k	0.0027
I(0)	0	β	0.3	T_{max}	1500
V(0)	0.1	r	3	N	10
p	0.1	γ	2.4		

$$\begin{cases} T(k+1) = \frac{1}{k+1}(0.1 - 0.02T(k) + 3T(k) - \frac{1}{500}\sum_{j=0}^{k} T(j)T(k-j) \\ \qquad\qquad -\frac{1}{500}\sum_{j=0}^{k} T(j)I(k-j) - 0.0027\sum_{j=0}^{k} V(j)T(k-j)), \\ I(k+1) = \frac{1}{k+1}(0.0027\sum_{j=0}^{k} V(j)T(k-j) - 0.3I(k)), \\ V(k+1) = \frac{1}{k+1}(3I(k) - 2.4V(k)). \end{cases}$$

Substituting the numerical values of $T(0)$,$I(0)$ and $V(0)$ from Table 2, into (3.1), results in the following values:

TABLE 3
Numerical values for $T(k)$, $I(k)$ and $V(k)$

k	$T(k)$	$I(k)$	$V(k)$
0	0.397953	0.0000027	-0.24
1	0.6428490535	0.00001727365500	0.2880405000
2	0.6417076047	-0.000003905153687	-0.2304151263
3	0.5250961402	0.000003311524422	0.1382461469
4	0.3325368902	-9.565406252$\times10^{-7}$	-0.06635616360
5	0.1813768628	4.831193843$\times10^{-7}$	0.02654198717
6	0.09109267420	-4.957048591$\times10^{-8}$	-0.009099902836
7	0.4810805574	7.924201926$\times10^{-8}$	0.002729952262
8	0.02614743888	2.9052940$\times10^{-8}$	-0.0007279608559
9	0.01764151747	3.044379004$\times10^{-8}$	0.0001747193213
10	0.01377794034	2.2917438$\times10^{-8}$	-0.00003811227634

Therefore, the solution of the system (1.1) is given by:

$$\begin{aligned} T(t) &= 0.1 + 0.397953t + 0.6428490535t^2 + 0.6417076047t^3 + 0.5250961402t^4 + 0.3325368902t^5 \\ &+ 0.1813768628t^6 + 0.09109267420t^7 + 0.4810805574t^8 + 0.02614743888t^9 \\ &+ 0.01764151747t^{10} + 0.01377794034t^{11} + ..., \end{aligned}$$

$$\begin{aligned} I(t) &= 0 + 0.0000027t + 0.00001727365500t^2 - 0.000003905153687t^3 + 0.000003311524422t^4 \\ &- 9.565406252 \times 10^{-7}t^5 + 4.831193843 \times 10^{-7}t^6 - 4.957048591 \times 10^{-8}t^7 \\ &+ 7.924201926 \times 10^{-8}t^8 + 2.9052940 \times 10^{-8}t^9 + 3.044379004 \times 10^{-8}t^{10} + 2.2917438 \times 10^{-8}t^{11} + ..., \end{aligned}$$

$$\begin{aligned} V(t) &= 0.1 - 0.24t + 0.2880405000t^2 - 0.2304151263t^3 + 0.1382461469t^4 - 0.06635616360t^5 \\ &+ 0.02654198717t^6 - 0.009099902836t^7 + 0.002729952262t^8 - 0.0007279608559t^9 \\ &+ 0.0001747193213t^{10} - 0.00003811227634t^{11} + ... \end{aligned}$$

In the above results, six terms approximations are considered, because the rest of the terms are too small therefore,

$$T(t) = 0.1 + 0.397953t + 0.6428490535t^2 + 0.6417076047t^3 + 0.5250961402t^4 + 0.3325368902t^5 + 0.1813768628t^6 +,$$

$$I(t) = 0 + 0.0000027t + 0.00001727365500t^2 - 0.000003905153687t^3 + 0.000003311524422t^4 - 9.565406252 \times 10^{-7}t^5 + 4.831193843 \times 10^{-7}t^6,$$

$$V(t) = 0.1 - 0.24t + 0.2880405000t^2 - 0.2304151263t^3 + 0.1382461469t^4 - 0.06635616360t^5 + 0.02654198717t^6.$$

Now, the Pade approximation $[4, 4]$ are calculated as follows:

$$T_{pade}(t) = \frac{0.1 - 1.224115531t - 3.460113984t^2 - 1.802983875t^3 - 0.490931507t^4}{1 - 16.22068531t + 23.52107342t^2 - 14.07521016t^3 + 3.603023454t^4},$$

$$I_{pade}(t) = \frac{0.000027t + 0.000005102627t^2 - 0.0000242556t^3 - 0.00000336726t^4}{1 - 0.4507788263t + 0.4653299318t^2 - 0.01485902265t^3 + 0.3291816479t^4},$$

$$V_{pade}(t) = \frac{0.1 - 0.120287067t - 0.06212670482t^2 - 0.0166266799t^3 - 0.002041025824t^4}{1 - 1.197129325t + 0.6139724293t^2 - 0.163201268t^3 + 0.01950991258t^4},$$

5. Fourth Order Runge-Kutta Method

The system (1.1) in the vector form can be written as follows,

$$\{Y\} = \frac{dY}{dt} = F(t, Y), \qquad t \geq 0,$$

with initial condition

$$Y(0) = Y_0,$$

where

$$Y = \begin{pmatrix} T \\ I \\ V \end{pmatrix},$$

$$F(t, Y) = \begin{pmatrix} f_1(t, T, I, V) \\ f_2(t, T, I, V) \\ f_3(t, T, I, V) \end{pmatrix} = \begin{pmatrix} p - \alpha T + rT(1 - \frac{T+I}{T_{max}}) - kVT, \\ kVT - \beta I \\ N\beta I - \gamma V \end{pmatrix}$$

,

$$Y_0 = \begin{pmatrix} T(0) \\ I(0) \\ V(0) \end{pmatrix}$$

By using Fourth order Runge-kutta method for the system (1.1) we have

$$Y_{i+1} = Y_i + \frac{1}{6}(K_1 + 2K_2 + 3K_3 + K_4),$$

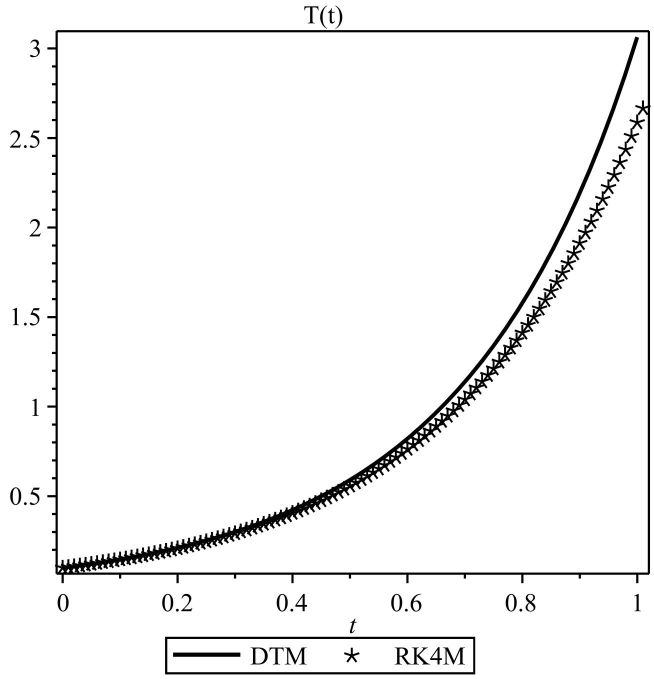

Figure 1: Numerical comparison for determination of $T(t)$ between DTM and Runge-Kutta method

where

$$K_1=\begin{pmatrix}k_{11}\\k_{21}\\k_{31}\end{pmatrix},\ K_2=\begin{pmatrix}k_{12}\\k_{22}\\k_{32}\end{pmatrix},\ K_3=\begin{pmatrix}k_{13}\\k_{23}\\k_{33}\end{pmatrix},\ K_4=\begin{pmatrix}k_{14}\\k_{24}\\k_{34}\end{pmatrix},$$

$$\begin{aligned}
k_{j1}&=h(t_i+T_i+I_i+V_i),\quad j=1,2,3,\\
k_{j2}&=hf_j(t_i+\frac{h}{2},T_i+\frac{k_{11}}{2},I_i+\frac{k_{21}}{2},V_i+\frac{k_{31}}{2}),\ j=1,2,3,\\
k_{j3}&=hf_j(t_i+\frac{h}{2},T_i+\frac{k_{12}}{2},I_i+\frac{k_{22}}{2},V_i+\frac{k_{32}}{2}),\ j=1,2,3,\\
k_{j4}&=hf_j(t_i+h,T_i+k_{13},I_i+k_{23},V_i+k_{33}),\ j=1,2,3,
\end{aligned}$$

where h is the step size.

The above equations can be expressed an explicit form as the following:

$$\begin{pmatrix}T_{i+1}\\I_{i+1}\\V_{i+1}\end{pmatrix}=\begin{pmatrix}T_i\\I_i\\V_i\end{pmatrix}+\frac{1}{6}\{\begin{pmatrix}k_{11}\\k_{21}\\k_{31}\end{pmatrix}+2\begin{pmatrix}k_{12}\\k_{22}\\k_{32}\end{pmatrix}+2\begin{pmatrix}k_{13}\\k_{23}\\k_{33}\end{pmatrix}+\begin{pmatrix}k_{14}\\k_{24}\\k_{34}\end{pmatrix}\},\ i=0,1,2,...$$

Considering $h=0.01$ the solutions of the Runge-kutta method are in good agreement with those of DTM. The numerical results are shown in Tables 4, 5 and 6. In addition, the solutions of DTM and Runge-kutta method are plotted in Figure 1, 2, 3.

TABLE 4
Numerical comparison for determination of $T(t)$ for different values of t

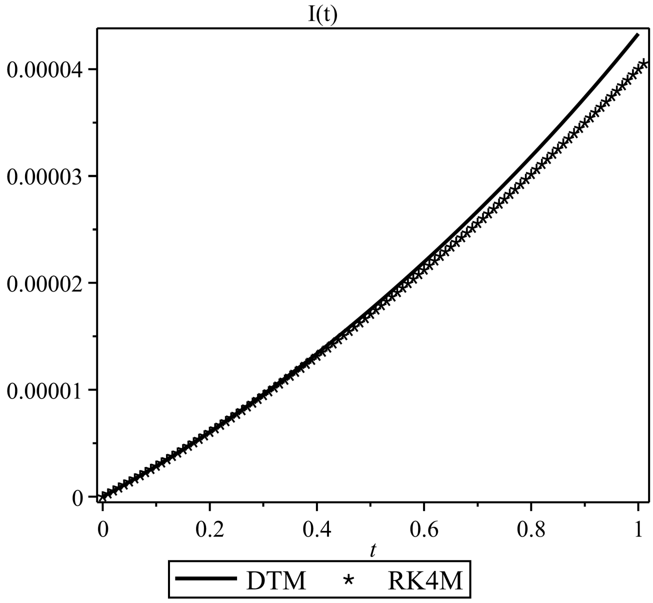

Figure 2: Numerical comparison for determination of $I(t)$ between DTM and Runge-Kutta method

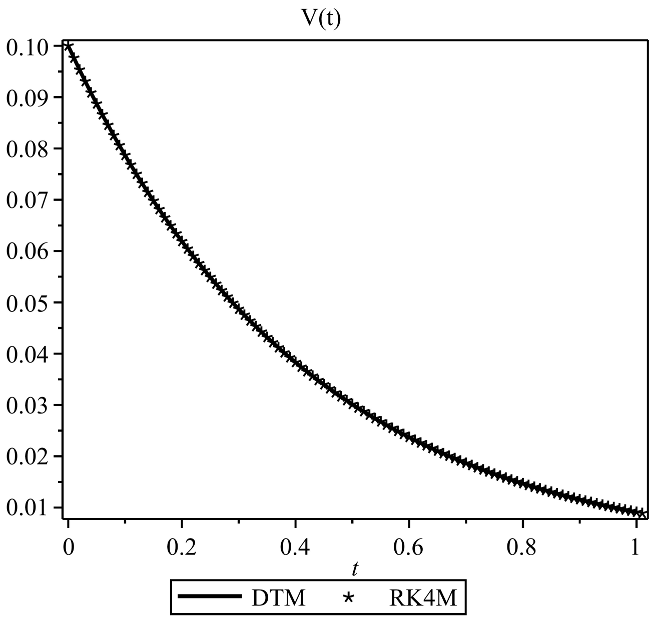

Figure 3: Numerical comparison for determination of $V(t)$ between DTM and Runge-Kutta method

t	DTM	$DTM - Pade$	$Runge - Kutta$
0	0.1	0.1	0.1
0.2	0.2116376961	0.2116378080	0.2088080121
0.4	0.4228059053	0.4228270719	0.4062401504
0.6	0.8214427381	0.8220867977	0.7644222857
0.8	1.580090941	1.589428602	1.414040889
1	3.068328771	3.167027048	2.591573918

TABLE 5
Numerical comparison for determination of $I(t)$ for different values of t

t	DTM	$DTM - Pade$	$Runge - Kutta$
0	0	0	0
0.2	0.000006064727822	0.000006064727837	0.00000603270115
0.4	0.00001339079678	0.00001339080558	0.00001315833510
0.6	0.00002195290254	0.00002195342310	0.00002122376663
0.8	0.00003183956342	0.00003185446433	0.00003017737081
1	0.00004335422355	0.00004394009567	0.0000400376991

TABLE 6
Numerical comparison for determination of $V(t)$ for different values of t

t	DTM	$DTM - Pade$	$Runge - Kutta$
0	0.1	0.1	0.1
0.2	0.06187984770	0.06187984749	0.06187984331
0.4	0.03829494797	0.03829494731	0.03829488788
0.6	0.02370481149	0.02370483059	0.02370455013
0.8	0.01468108240	0.01468128654	0.01468036373
1	0.009102264150	0.009103435566	0.009100845022

6. Conlcusion

In this paper, the HIV Infection of $CD4^{+}T$ cells model is solved by DTM and fourth order Runge-Kutta methods, successfully. The results obtained by these two methods are plotted in Figure 1. Comparison of these two methods can be resulted from Tables 4, 5, 6, or figure 1. Results of these two methods are close at the beginning of the intervals, and the solutions get to lose the common figures. The fourth part of Figure 1 and Tables 4,5,6 show that as the time passes, the concentration of number of healthy cells which is denoted by $T(t)$ and the HIV viruses which are denoted by $I(t)$ is the worth and the numbers of free HIV viruses which are denoted by $V(t)$ have more digits in common. Behavior of $T(t)$ and $I(t)$ are almost the same, and it increases as the time increases, but the rate of increasing of $I(t)$ is less than those of $T(t)$ and $V(t)$ decrease as the time increases. Computations are performed by using Maple 13 package.

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