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# On the Approximate Solution of Hosszú's Functional Equation

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**Abstract**

We show that every approximate solution of the Hosszú's functional equation

$$f(x+y+xy) = f(x) + f(y) + f(xy) \text{ for } any\ x,\ y \in \mathbb{R},$$

is an additive function and also we investigate the Hyers-Ulam stability of this equation in the following setting

$$|f(x+y+xy) - f(x) - f(y) - f(xy)| \leq \delta + \varphi(x,y)$$

for any $x, y \in \mathbb{R}$ and $\delta > 0$.

*Keywords:* Additive Function, Hosszú's Functional Equation, Hyers-Ulam Stability.
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## 1. Introduction

In the book, "A collection of Mathematical problems", S. M. Ulam posed the question of the stability of the Cauchy functional equation. Ulam asked: if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near to the solutions of the strict equation? [17] Originally, he had proposed the following more specific question during a lecture given before the University of Wisconsin's Mathematics Club in 1940.

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Given a group $G_1$, a metric group $(G_2, d)$, a number $\varepsilon > 0$ and a mapping $f$: $G_1 \longrightarrow G_2$ which satisfies the inequality $d(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G_1$, does there exist an homomorphism $h$: $G_1 \longrightarrow G_2$ and a constant $k > 0$, depending only on $G_1$ and $G_2$ such that $d(f(x), h(x)) \leq k\varepsilon$ for all $x$ in $G_1$? A partial and significant affirmative answer was given by D. H. Hyers [5] under the condition that $G_1$ and $G_2$ are Banach spaces. Furthermore many authors provided a generalization of Hyers's stability Theorem which allows the Cauchy difference to be unbounded (see [4, 12, 14]). The stability problems of various functional equations ha been investigated by many authors (see [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16]). The main purpose of this work is to study the Hyers-Ulam stability of the Hosszus functional equation

$$f(x + y + xy) = f(xy) + f(x) + f(y), \quad x, y \in \mathbb{R}. \tag{1.1}$$

Many investigations were used to establish Hyer's Ulam stability of this equation (see [1, 9, 11]). In this work we give an other way to establish this stability. Moreover we give the Hyers-Ulam-Rassias stability of this equation.

## 2. Notations

Throughout this paper we use following notations:

- $\delta$ is a positive number.
- $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is one application and $\varphi_0 = \varphi$, $\varphi_n(x, y) = \varphi_{n-1}(2^\epsilon x, 2^\epsilon y)$ with $n \in \mathbb{N}^*$ and $\epsilon \in \{-1, 1\}$.
- For some application $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define $\theta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ by

  $\theta(x, y) = 5\delta + 2|f(1)| + \varphi(x, 2y + 1) + \varphi(x + y + xy, 1) + 2\varphi(x, y) + \varphi(y, 1)$ if one of numbers $x$ or $y$ is non null and $\theta(0, 0) = \delta + \varphi(0, 0)$.
- $\widetilde{\varphi}(x, y) = \sum_{i=\frac{1-\epsilon}{2}}^{+\infty} \frac{\varphi_{i-1}(2^\epsilon x, 2^\epsilon y)}{2^{i\epsilon + \frac{1-\epsilon}{2}}}$ and consequently $\widetilde{\theta}(x, y) = 5\delta + 2|f(0)| + \widetilde{\varphi}(x, 2y + 1) + \widetilde{\varphi}(x + y + xy, 1) + 2\widetilde{\varphi}(x, y) + \widetilde{\varphi}(y, 1)$.

## 3. Preliminary Results

For later use we need the following lemmas

**Lemma 3.1.** *Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the functional inequality*

$$|f(x + y + xy) - f(x) - f(y) - f(xy)| \leq \delta + \varphi(x, y), \; x, \; y \in \mathbb{R}, \tag{3.1}$$

*for some $\delta$ and $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^+$. Then $f$ satisfies the following inequalities*

*i)* $|f(0)| \leq \frac{\delta + \varphi(0,0)}{2}$,

*ii)* $|f(x) + f(-x)| \leq \delta + \varphi(x, -1)$,

*iii) and* $|f(2x + 1) - 2f(x)| \leq \delta + |f(1)| + \varphi(x, 1)$.

**Proof .** i) By letting $x = y = 0$ in (3.1) we obtain $|-2f(0)| \leq \delta + \varphi(0,0)$ which implies that $|f(0)| \leq \frac{\delta+\varphi(0,0)}{2}$.
ii) Let $y = -1$ in (3.1) then we get

$$|f(x) + f(-x)| \leq \delta + \varphi(x,-1),\ x \in \mathbb{R}.$$

iii) For $y = 1$ in (3.1) we obtain that

$$|f(2x+1) - 2f(x)| \leq \delta + |f(1)| + \varphi(x,1),\ x \in \mathbb{R}.$$

□

**Lemma 3.2.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the functional inequality (3.1). Then $f$ satisfies the inequality*

$$|f(2st+t) - 2f(st) - f(t)| \leq \theta(t,s) \tag{3.2}$$

*for all* $s,\ t \in \mathbb{R}$.

**Proof .** Next, by setting in (3.1) $(x,y) = (t, 2s+1)$ *or* $(y,x) = (t, 2s+1)$ we get that
$|f(2st+t) - 2f(ts) - f(t)| \leq$
$|f(2st+t) + f(2s+1) + f(t) + f(t + (2s+1) + t(2s+1))| +$
$|f(2(t+s+ts)+1) - 2f(t+s+st)| +$
$|2f(t+s+st) - 2f(ts) - 2f(t) - 2f(s))| +$
$|f(2s+1) - 2f(s)| \leq$
$[\delta + \varphi(t, 2s+1)] + [\delta + |f(1)| + \varphi(t+s+st, 1)] +$
$2[\delta + \varphi(t,s)] + [\delta + f(1) + \varphi(s,1)] \leq$
$5\delta + 2|f(1)| + \varphi(t,2s+1) + \varphi(t+s+st,1) + 2\varphi(t,s) + \varphi(s,1) = \theta(t,s)$ for all $s,\ t \in \mathbb{R}$.□

**Lemma 3.3.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the functional inequality (3.1). Then, for all $s,\ t \in \mathbb{R}$ we have*

$$|f(2st+t) - f(2st) - f(t)| \leq \theta(t,s) + \theta(2st, \frac{-1}{2}) + |f(0)| + 2[\delta + \varphi(st,-1)] \tag{3.3}$$

$\leq 12\delta + 4|f(1)| + |f(0)| + \varphi(t, 2s+1) + \varphi(t+s+ts, 1) + 2\varphi(t,s) + \varphi(s,1) + \varphi(2st, 0) + \varphi(st - \frac{1}{2}, 1) + 2\varphi(2st, \frac{-1}{2}) + \varphi(\frac{-1}{2}, 1) + 2\varphi(st, -1)$.

**Proof .**By letting $s = -\frac{1}{2}$ in (3.3) we obtain that
$|f(t) + 2f(\frac{-t}{2})| \leq \theta(t, \frac{-1}{2}) + |f(0)|$.
Furthermore, for all $t \in \mathbb{R}$, we have $|f(t) - 2f(\frac{t}{2})| = |f(t) + 2f(-\frac{t}{2}) - (2f(-\frac{t}{2}) + 2f(\frac{t}{2}))|$
$\leq |f(t) + 2f(-\frac{t}{2})| + 2|f(-\frac{t}{2}) + f(\frac{t}{2})| \leq \theta(t, \frac{-1}{2}) + |f(0)| + 2(\delta + \varphi(\frac{t}{2}, -1))$
$\leq 7\delta + 2|f(1)| + |f(0)| + \varphi(t, 0) + \varphi(\frac{1}{2}t - \frac{1}{2}, 1) + 2\varphi(t, \frac{-1}{2}) + \varphi(\frac{-1}{2}, 1) + \varphi(\frac{t}{2}, -1)$.
Finnally for all $s,\ t \in \mathbb{R}$, we get from (3.1) and (3.3) that
$|f(2st+t) - f(2st) - f(t)|$
$\leq |f(2st+t) - 2f(st) - f(t)| + |f(2st) - 2f(st)|$
$\leq \theta(t,s) + \theta(2st, \frac{-1}{2}) + |f(0)| + 2(\delta + \varphi(st, -1))$
$\leq 5\delta + 2|f(1)| + |f(0)| + \varphi(t, 2s+1) + \varphi(t+s+ts, 1) + 2\varphi(t,s) + \varphi(s,1)$
$+5\delta + 2|f(1)| + \varphi(2st, 0) + \varphi(st - \frac{1}{2}, 1) + 2\varphi(2st, \frac{-1}{2}) + \varphi(\frac{-1}{2}, 1) + 2[\delta + \varphi(st, -1)]$
$\leq 12\delta + 4|f(1)| + |f(0)| + \varphi(t, 2s+1) + \varphi(t+s+ts, 1) + 2\varphi(t,s) + \varphi(s,1) + \varphi(2st, 0) + \varphi(st - \frac{1}{2}, 1) + 2\varphi(2st, \frac{-1}{2}) + \varphi(\frac{-1}{2}, 1) + 2\varphi(st, -1)$. □

## 4. Main Results

In this section we give our main result

**Theorem 4.1.** *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a function. Then, $f$ satisfies the functional equation*

$$f(x+y+xy)-f(x)-f(y)-f(xy)=0,\ x,\ y\in\mathbb{R} \tag{4.1}$$

*if only if $f$ is an additive function.*

**Proof .** The result is obtained by a similar calculation as in Lemmas 3.2 and 3.3.□

**Theorem 4.2.** *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ satisfies the functional inequality*

$$|f(x+y+xy)-f(x)-f(y)-f(xy)|\leq\delta+\varphi(x,y),\ x,\ y\in\mathbb{R} \tag{4.2}$$

*for some $\delta$ and $\varphi:\mathbb{R}\longrightarrow\mathbb{R}^+$ such that $\widetilde{\varphi}(x,y)<+\infty$. Then there exists a unique additive function $T:\mathbb{R}\longrightarrow\mathbb{R}$ such that for all $x\in\mathbb{R}$,*

$$|f(x)-T(x)|\leq\widetilde{\theta}(x,\frac{1}{2})+\widetilde{\theta}(x,\frac{-1}{2})+|f(0)|+2(\delta+\widetilde{\varphi}(\frac{x}{2},-1)),\ x\in\mathbb{R}.$$

**Proof .** By Lemmas 3.2 and 3.3 we have
$|f(x+y)-f(x)-f(y)|$

$$\leq\begin{cases}\theta(x,\frac{y}{2x})+\theta(y,\frac{-1}{2})+|f(0)|+2(\delta+\varphi(\frac{y}{2},-1)),\ x,\ y\in\mathbb{R}, & \text{if } x\neq 0;\\ \theta(y,\frac{x}{2y})+\theta(x,\frac{-1}{2})+|f(0)|+2(\delta+\varphi(\frac{x}{2},-1)),\ x,\ y\in\mathbb{R}, & \text{if } y\neq 0;\\ \theta(0,0)+\theta(0,\frac{-1}{2})+|f(0)|+2(\delta+\varphi(0,-1)), & \text{if } y=x=0.\end{cases}$$

Lemmas 3.2 and 3.3

$$(t,s)=\begin{cases}(x,\frac{y}{2x}) & \text{if } x\neq 0;\\ (y,\frac{x}{2y}) & \text{if } y\neq 0;\\ (0,0) & \text{if } x=y=0.\end{cases}$$

In view of [5], [12] and Theorem 4.1 we get the sought result. □

**Corollary 4.3.** *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ satisfies the functional inequality*

$$|f(x+y+xy)-f(x)-f(y)-f(xy)|\leq\delta,\ x,\ y\in\mathbb{R} \tag{4.3}$$

*for some real positive number $\delta$. Then there exists a unique additive function $T:\mathbb{R}\longrightarrow\mathbb{R}$ such that for all $x\in\mathbb{R}$,*

$$|f(x)-T(x)|\leq\frac{25}{2}\delta+4|f(1)|.$$

**Corollary 4.4.** *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ satisfies the functional inequality*

$$|f(x+y+xy)-f(x)-f(y)-f(xy)|\leq\delta(|x|^p+|y|^p),\ x,\ y\in\mathbb{R} \tag{4.4}$$

*for some real positive number $p\neq 1$. Then there exists a unique additive function $T:\mathbb{R}\longrightarrow\mathbb{R}$ such that for all $x\in\mathbb{R}$,*

$$|f(x)-T(x)|\leq\delta\frac{2^{\epsilon}}{2^{\epsilon}-2^{\epsilon p}}\{2|x|^p+2^{1-p}+2|\frac{x}{2}|^p+2+\frac{\varphi(0,0)}{2}\},$$

*where $\epsilon$ is the sign of $(1-p)$.*

**Corollary 4.5.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the functional inequality*

$$|f(x+y+xy) - f(x) - f(y) - f(xy)| \leq \delta(|x|^p|y|^q),\ x,\ y \in \mathbb{R} \tag{4.5}$$

*for some real positive numbers $p$ and $q$ such that $r = p+q \neq 1$. Then there exists a unique additive function $T : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$,*

$$|f(x) - T(x)| \leq \delta \frac{2^\epsilon}{2^\epsilon - 2^{\epsilon r}}\{|x|^p 2^{1-q} + 2|\frac{x}{2}|^p + \frac{\varphi(0,0)}{2}\},$$

*where $\epsilon$ is the sign of $(1-r)$.*

**Corollary 4.6.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the functional inequality*

$$|f(x+y+z-xy+yz-xyz) - f(x) - f(y) - f(z) + f(xy) - f(yz) + f(xyz)| \leq \delta \tag{4.6}$$

*for all $x,\ y,\ z \in \mathbb{R}$ and a some real positive number $\delta$. Then there exists a unique additive function $T : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}$,*

$$|f(x) - T(x)| \leq \frac{25}{2}(\delta + |f(0)|) + 4|f(1)|.$$

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