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Strong Convergence Theorems for Weighted Resolvent Average of a Finite Family of Monotone Operators

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Abstract

This paper is devoted to finding a zero point of a weighted resolvent average of a finite family of monotone operators. A new proximal point algorithm and its convergence analysis is given. It is shown that the sequence generated by this new algorithm, for a finite family of monotone operators converges strongly to the zero point of their weighted resolvent average. Finally, our results are illustrated by some numerical examples.

Keywords: Weighted resolvent average, proximal point algorithm, projection algorithm, monotone operators.
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1. Introduction

Let H be a real Hilbert space with norm $\|.\|$ and inner product $\langle .,. \rangle$. For each $x, y \in H$, we have [9]

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle . \tag{1.1}$$

The operator $T : H \to H$ is called *nonexpansive* (resp. *firmly nonexpansive*) if $\|Tx - Ty\| \leq \|x-y\|$ (resp. $\|Tx-Ty\|^2+\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2 \leq \|x-y\|^2$) for all $x, y \in H$, where Id is the identity mapping on H. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$, i.e., $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

Let A be a set-valued mapping with the *domain* $\mathrm{Dom}A = \{x \in H : A(x) \neq \emptyset\}$ and the *range* $\mathrm{ran}A = \{u \in H : \exists x \in \mathrm{Dom}A \text{ such that } u \in A(x)\}$. The *graph* of A is the set $\mathrm{gra}A = \{(x, u) \in H \times H : u \in A(x)\}$.

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An operator $A : H \multimap H$ is said to be *monotone* if

$$\langle x - y, u - v\rangle \geq 0, \quad \forall (x, u), (y, v) \in \mathrm{gra}A.$$

A monotone operator A is called *maximal monotone* if there exists no monotone operator B such that gra$A$ is a proper subset of gra$B$.

The *resolvent* of A is the mapping $J_A = (A + \mathrm{Id})^{-1}$. It is well known that (see Proposition 23.7 in [2]) J_A is single-valued and firmly nonexpansive if A is monotone. In addition, if A is maximal monotone, then J_A is also maximal monotone and, in this case we have $\mathrm{Dom}J_A = H$. Moreover, $0 \in A(x)$ if and only if $x \in \mathrm{Fix}(J_A)$. For each $x, y \in \mathrm{ran}(\mathrm{Id} + A)$ we have (see [11])

$$\|J_{\lambda A}x - J_{\lambda A}y\|^2 \leq \|x - y\|^2 - \|\big(x - J_{\lambda A}x\big) - \big(y - J_{\lambda A}y\big)\|^2. \tag{1.2}$$

Let us consider the zero point problem for monotone operator A on a real Hilbert space H, i.e., finding a point $x \in \mathrm{Dom}A$ such that $0 \in A(x)$. It was first introduced by Martinet [8] in 1970. Rockafellar [10] defined the proximal point algorithm of Martinet by generalizing a sequence $\{x_n\}$ such that

$$x_{n+1} = J_{s_n A}x_n + e_n, \ n \in \mathbb{N}, \tag{1.3}$$

for arbitrary point $x_0 \in H$, where $\{e_n\}$ is a sequence of errors and $\{s_n\} \subseteq (0, \infty)$. The sequence $\{x_n\}$ is known to converge weakly to a zero of A, if $\liminf_{n\to\infty} s_n > 0$ and $\sum_{n=0}^{\infty} \|e_n\| < \infty$, see [10], but fails in general to converge strongly [6]. Recently, Xu [12] investigated a modified version of the initial proximal point algorithm studied by Rockafellar with $x_0 \in H$ chosen arbitrary,

$$x_{n+1} = \beta_n x_0 + (1 - \beta_n)J_{s_n A}x_n + e_n, \ n \in \mathbb{N}, \tag{1.4}$$

where $\{e_n\}$ is the error sequence. For $\{e_n\}$ summable, it was proved that (see [12]) $\{x_n\}$ is strongly convergent if $s_n \to \infty$ and $\beta_n \subseteq (0, 1)$ with $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$.

In this paper, we prove strong convergence of a proximal point algorithm to a zero point of weighted resolvent average of a finite family of monotone operators.

2. The main results

In this section, we present a new proximal point algorithm for a finite family of monotone operators and its convergence analysis.

First, we recall (see [1]) the definition of the proximal average and resolvent average. To this end, we assume that $m \in \mathbb{N}$ and $I = \{1, 2, \ldots, m\}$. For every $i \in I$, let $A_i : H \multimap H$ be a set-valued mapping and let $\lambda_i > 0$, $\sum_{i\in I} \lambda_i = 1$. We set $\boldsymbol{A} = (A_1, \ldots, A_m)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$.

The $\boldsymbol{\lambda}$-*weighted resolvent average* of $\boldsymbol{A}$ is defined by

$$R(\boldsymbol{A}, \boldsymbol{\lambda}) = \Big(\sum_{i\in I} \lambda_i (A_i + \mathrm{Id})^{-1}\Big)^{-1} - \mathrm{Id}. \tag{2.1}$$

The equation (2.1) is equivalent to the following equation:

$$J_{R(\boldsymbol{A},\boldsymbol{\lambda})} = \sum_{i\in I} \lambda_i J_{A_i}. \tag{2.2}$$

Here, we consider some useful lemmas.

Lemma 2.1. *Let for each $i \in I$, $A_i : H \multimap H$ be a monotone operator. Then $(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) = \mathrm{Fix}(J_{R(\boldsymbol{A},\boldsymbol{\lambda})})$.*

Proof . It follows from (2.1) and (2.2) that

$$\begin{aligned} x \in (R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) &\Leftrightarrow 0 \in R(\boldsymbol{A},\boldsymbol{\lambda})(x) \\ &\Leftrightarrow 0 \in \Big(\Big(\sum_{i\in I} \lambda_i (A_i + \mathrm{Id})^{-1}\Big)^{-1} - \mathrm{Id}\Big)(x) \\ &\Leftrightarrow x \in \Big(\sum_{i\in I} \lambda_i (A_i + \mathrm{Id})^{-1}\Big)^{-1}(x) \\ &\Leftrightarrow x \in \Big(\sum_{i\in I} \lambda_i (A_i + \mathrm{Id})^{-1}\Big)(x) \\ &\Leftrightarrow x \in \sum_{i\in I} \lambda_i J_{A_i}(x) \Leftrightarrow x \in J_{R(\boldsymbol{A},\boldsymbol{\lambda})}(x) \\ &\Leftrightarrow x \in \mathrm{Fix}(J_{R(\boldsymbol{A},\boldsymbol{\lambda})}). \end{aligned}$$

□

Lemma 2.2. *Let $\{A_i : H \multimap H\}_{i\in I}$ be a finite family of monotone operators with $(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) \neq \emptyset$, where $\lambda_i > 0$ and $\sum_{i\in I} \lambda_i = 1$. Let K ba a nonempty closed and convex subset of H such that*

$$\overline{\mathrm{Dom} R(\boldsymbol{A},\boldsymbol{\lambda})} \subseteq K \subseteq \mathrm{ran}(\mathrm{Id} + R(\boldsymbol{A},\boldsymbol{\lambda})). \tag{2.3}$$

Assume that f is a k-contraction mapping on K into itself. Let $\{x_n\}$ be the sequence generated by $x_1 \in K$ and

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n) J_{R(\boldsymbol{A},\lambda)} x_n + e_n, \ n \in \mathbb{N}, \tag{2.4}$$

where $\{\beta_n\} \subseteq (0,1)$ and $\{e_n\}$ is a sequence of errors such that $e_n \in H$ and $\sum_{n\in\mathbb{N}} \|e_n\| < \infty$. Then $\big\{\|x_n - z\| : n \in \mathbb{N}\big\}$ is bounded for each $z \in (R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0)$. Consequently, $\{x_n\}$ is bounded.

Proof . By using our assumption, nonexpansivity of the resolvent and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\beta_n f(x_n) + (1-\beta_n) J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n + e_n - z\| \\ &= \|\beta_n (f(x_n) - z) + (1-\beta_n)(J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n - z) + e_n\| \\ &\leq \beta_n \|f(x_n) - z\| + (1-\beta_n)\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})} z\| + \|e_n\| \\ &\leq \beta_n \|f(x_n) - z\| + (1-\beta_n)\|x_n - z\| + \|e_n\| \\ &\leq \beta_n \|f(x_n) - f(z)\| + \beta_n \|f(z) - z\| + (1-\beta_n)\|x_n - z\| + \|e_n\| \\ &\leq k\beta_n \|x_n - z\| + \beta_n \|f(z) - z\| + (1-\beta_n)\|x_n - z\| + \|e_n\| \\ &\leq (1-(1-k)\beta_n)\|x_n - z\| + \beta_n (1-k)\frac{1}{1-k}\|f(z) - z\| + \|e_n\| \\ &\leq \max\Big\{\|x_n - z\|, \frac{1}{1-k}\|f(z) - z\|\Big\} + \|e_n\|. \end{aligned}$$

This shows by induction that

$$\|x_{n+1}-z\| \le \max\left\{\|x_1-z\|, \frac{1}{1-k}\|f(z)-z\|\right\} + \sum_{i=1}^{n}\|e_i\|.$$

Therefore, $\{\|x_n-z\| : n\in\mathbb{N}\}$ is bounded for each $z\in R(\boldsymbol{A},\boldsymbol{\lambda})^{-1}(0)$. Hence $\{x_n\}$ is bounded. □

Lemma 2.3. [7] *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ with $t_{n_i} < t_{n_i+1}$ for all $i\in\mathbb{N}$. Then there exists a nondecreasing sequence $\{s(n)\}\subseteq\mathbb{N}$ such that $s(n)\to\infty$ and the following properties are satisfied by all (sufficiently large) numbers $n\in\mathbb{N}$:*

$$t_{s(n)} \le t_{s(n)+1}.$$

In fact

$$s(n) = \max\{k\le n : t_k < t_{k+1}\}.$$

Lemma 2.4. *Let $x\in H$ and $\{\alpha_n\}$ be a bounded sequence in H. Then there exists a constant $L>0$ such that $\|x+\alpha_n\|^2 \le \|x\|^2 + L\|\alpha_n\|$.*

Proof . By Cauchy-Schwarz inequality and for $L\ge 2\|x\| + \sup_{n\in\mathbb{N}}\|\alpha_n\|$, we have

$$\begin{aligned}\|x+\alpha_n\|^2 &= \|x\|^2 + 2\langle x,\alpha_n\rangle + \|\alpha_n\|^2\\ &\le \|x\|^2 + 2\|x\|\|\alpha_n\| + \|\alpha_n\|^2\\ &\le \|x\|^2 + \|\alpha_n\|(2\|x\|+\|\alpha_n\|)\\ &\le \|x\|^2 + L\|\alpha_n\|.\end{aligned}$$

□

Lemma 2.5. *Let $\{A_i : H \multimap H\}_{i\in I}$ be a finite family of monotone operators with $(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0)\neq\emptyset$, where $\lambda_i>0$ and $\sum_{i\in I}\lambda_i = 1$. Let K be the same as in Lemma 2.2. Assume that f is a k-contraction mapping on K into itself. Let $\{x_n\}$ be the sequence generated by* (2.4) *satisfy the following conditions:*

(i) $e_n\in H$ and $\sum_{n\in\mathbb{N}}\|e_n\|<\infty$,

(ii) $\lim_{n\to\infty}\beta_n = 0$.

Then $\lim_{n\to\infty}\|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\| = 0$.

Proof. Let $z\in R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0)$ be arbitrary. By using Lemma 2.1 and (1.2), for some appropriate constant $L>0$ obtained from Lemma 2.4, we get

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\beta_n f(x_n) + (1-\beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n + e_n - z\|^2\\ &\le \|\beta_n f(x_n) + (1-\beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - z\|^2 + L\|e_n\|\\ &\le \beta_n\|f(x_n)-z\|^2 + (1-\beta_n)\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}z\|^2 + L\|e_n\|\\ &\le \beta_n\|f(x_n)-z\|^2 + (1-\beta_n)(\|x_n-z\|^2 - \|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\|^2) + L\|e_n\|.\end{aligned}$$

Therefore,

$$
\begin{aligned}
(1-\beta_n)\|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\|^2 &\le (1-\beta_n)\|x_n - z\|^2 - \|x_{n+1}-z\|^2 + \beta_n\|f(x_n)-z\|^2 + L\|e_n\| \\
&\le \|x_n - z\|^2 - \|x_{n+1}-z\|^2 + \beta_n\|f(x_n)-z\|^2 + L\|e_n\|. \qquad (2.5)
\end{aligned}
$$

We consider two cases:

Case 1. Suppose that $\{\|x_n - z\|\}$ is a monotone sequence. It follows from Lemma 2.2 that $\{\|x_n - z\|\}$ is bounded and hence $\{\|x_n - z\|\}$ is convergent. Clearly,

$$\|x_{n+1} - z\|^2 - \|x_n - z\|^2 \to 0.$$

Since $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\|e_n\| = 0$ and $\{f(x_n)\}$ is a bounded sequence, from (2.5) we obtain that $\lim_{n\to\infty}(1-\beta_n)\|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\|^2 = 0$. Then

$$\lim_{n\to\infty}\|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\| = 0.$$

Case 2. Assume that $\{\|x_n - z\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N} : k \le n,\ \|x_k - z\| < \|x_{k+1} - z\|\}.$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge n_0$,

$$\|x_{\tau(n)} - z\| < \|x_{\tau(n)+1} - z\|.$$

From Case 1, we obtain that $\lim\limits_{n\to\infty}\|x_{\tau(n)} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{\tau(n)}\| = 0$. Now, from Lemma 2.3, we have

$$
\begin{aligned}
0 \le \|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\| &\le \max\{\|x_{\tau(n)} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{\tau(n)}\|, \|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\|\} \\
&\le \|x_{\tau(n)+1} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{\tau(n)+1}\|.
\end{aligned}
$$

Hence $\lim_{n\to\infty}\|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\| = 0$. □

Lemma 2.6. [12, Lemma 2.5] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n + \beta_n,\ n \ge 0,$$

where $\{\gamma_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ satisfy the conditions:

(i) $\gamma_n \subset [0,1], \sum_{n=1}^{\infty}\gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\delta_n| < \infty$,

(iii) $\beta_n \ge 0$ *for all* $n \ge 0$ *with* $\sum_{n=0}^{\infty}\beta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Let K be a closed convex subset of H. Then for every point $x \in H$, there exists a unique *nearest point* in K, denoted by $P_K(x)$, such that

$$\|x - P_K(x)\| \le \|x - y\|,\ \forall y \in K.$$

The operator P_K is called *metric projection* of H onto K. It is well known that $P_K(x)$ is nonexpansive. The metric projection $P_K(x)$ is characterized by $P_K(x) \in K$ and

$$\langle u - P_K(x), x - P_K(x)\rangle \leq 0, \ \forall u \in K.$$

Theorem 2.7. *Suppose that $\{A_i : H \multimap H\}_{i\in I}$ is a finite family of monotone operators with $Z = (R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(0) \neq \emptyset$, where $\lambda_i > 0$ and $\sum_{i\in I} \lambda_i = 1$. Let K be the same as in Lemma 2.2. Assume that f is k-contraction on K into itself. Let $\{\beta_n\}$ be a real sequence in $(0, 1)$ and $\{e_n\}$ be a sequence of errors. Let $\{x_n\}$ be the sequence generated by (2.4). Assume that the following conditions are satisfied:*

(i) $e_n \in K$, satisfies $\sum_{n\in\mathbb{N}} \|e_n\| < \infty$,

(ii) $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$.

Then $\{x_n\}$ converges strongly to $z = P_Z f(z)$.

Proof . First, we show that there exists a unique $z \in Z$ such that $z = P_Z f(z)$. Since $Z = R(\boldsymbol{A}, \boldsymbol{\lambda})^{-1}(0)$ is closed and convex, the projection P_Z is well defined. It is enough to show that $P_Z f$ is contraction on K. Since P_Z is nonexpansive and f is k-contraction, we get

$$\|P_Z f(x) - P_Z f(y)\| \leq \|f(x) - f(y)\| \leq k\|x - y\|, \ x, y \in H.$$

It follows from Banach Contraction Theorem that there exists a unique element $z \in Z$ such that $z = P_Z f(z)$. Lemma 2.5 implies that $\lim\limits_{n\to\infty} \|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\| = 0$.

Next, we show that there exists a unique $z \in Z$ such that $\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \leq 0$, where $z = P_Z f(z)$. To show this inequality, we choose a subsequence $\{x_{n_\alpha}\}$ of $\{x_n\}$ such that

$$\lim_{\alpha\to\infty} \langle f(z) - z, x_{n_\alpha} - z\rangle = \limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle .$$

By Lemma 2.2, the sequence $\{x_{n_\alpha}\}$ is bounded, so there exists a subsequence $\{x_{n_{\alpha_j}}\}$ of $\{x_{n_\alpha}\}$ which converges weakly to u. Without loss of generality, we can assume that $\{x_{n_\alpha}\} \rightharpoonup u$. We show that $u \in Z$. To see this,

$$\begin{aligned}\|x_{n_\alpha} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}u\| &\leq \|x_{n_\alpha} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{n_\alpha}\| + \|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{n_\alpha} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}u\| \\ &\leq \|x_{n_\alpha} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{n_\alpha}\| + \|x_{n_\alpha} - u\|,\end{aligned}$$

which implies that

$$\limsup_{\alpha\to\infty} \|x_{n_\alpha} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}u\| \leq \limsup_{\alpha\to\infty} \|x_{n_\alpha} - u\|.$$

By the Opial property of Hilbert space H, we obtain $u = J_{R(\boldsymbol{A},\boldsymbol{\lambda})}u$. Hence $u \in Z$.

Therefore, we have

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{\alpha\to\infty} \langle f(z) - z, x_{n_\alpha} - z\rangle = \langle f(z) - z, u - z\rangle \leq 0.$$

Finally, we show that $x_n \to P_Z f(z)$. In fact, using Lemma 2.4, (1.1) and Lemma 2.2 of [5], for some appropriate constant $L > 0$, we have

$$\|x_{n+1} - z\|^2 = \|\beta_n f(x_n) + (1 - \beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n + e_n - z\|^2$$

$$
\begin{aligned}
&= \|\beta_n f(x_n) + (1-\beta_n) J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n + e_n - \beta_n z - (1-\beta_n) z\|^2 \\
&\le \|(1-\beta_n) J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n + e_n - (1-\beta_n) z\|^2 + 2\beta_n \langle f(x_n) - z, x_{n+1} - z\rangle \\
&\le \|(1-\beta_n) J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n - (1-\beta_n) z\|^2 + 2\beta_n \langle f(x_n) - z, x_{n+1} - z\rangle + L\|e_n\| \\
&\le (1-\beta_n)^2 \|x_n - z\|^2 + 2\beta_n \langle f(x_n) - z, x_{n+1} - z\rangle + L\|e_n\| \\
&\le (1-\beta_n)^2 \|x_n - z\|^2 + 2\beta_n \langle f(x_n) - f(z), x_{n+1} - z\rangle + 2\beta_n \langle f(z) - z, x_{n+1} - z\rangle + L\|e_n\| \\
&\le (1-\beta_n)^2 \|x_n - z\|^2 + 2k\beta_n \|x_n - z\|\|x_{n+1} - z\| + 2\beta_n \langle f(z) - z, x_{n+1} - z\rangle + L\|e_n\| \\
&\le (1-\beta_n)^2 \|x_n - z\|^2 + k\beta_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + 2\beta_n \langle f(z) - z, x_{n+1} - z\rangle + L\|e_n\|,
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\le \frac{(1-\beta_n)^2 + k\beta_n}{1-k\beta_n}\|x_n - z\|^2 + \frac{2\beta_n}{1-k\beta_n}\langle f(z) - z, x_{n+1} - z\rangle + \frac{L}{1-k\beta_n}\|e_n\| \\
&\le \frac{1-2\beta_n + k\beta_n}{1-k\beta_n}\|x_n - z\|^2 + \frac{\beta_n^2}{1-k\beta_n}\|x_n - z\|^2 + \frac{2\beta_n}{1-k\beta_n}\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad + \frac{L}{1-k\beta_n}\|e_n\| \\
&\le \Big(1 - \frac{2(1-k)\beta_n}{1-k\beta_n}\Big)\|x_n - z\|^2 + \frac{2(1-k)\beta_n}{1-k\beta_n}\Big(\frac{\beta_n N}{2(1-k)} + \frac{1}{1-k}\langle f(z) - z, x_{n+1} - z\rangle\Big) \\
&\quad + \frac{L}{1-k\beta_n}\|e_n\| \\
&\le (1-\gamma_n)\|x_n - z\|^2 + \gamma_n \delta_n + \eta_n,
\end{aligned}
$$

where $N = \sup\{\|x_n - z\|^2 : n \in \mathbb{N}\}$, $\gamma_n = \frac{2(1-k)\beta_n}{1-k\beta_n}$, $\delta_n = \frac{\beta_n N}{2(1-k)} + \frac{1}{1-k}\langle f(z) - z, x_{n+1} - z\rangle$ and $\eta_n = \frac{L}{1-k\beta_n}\|e_n\|$. By assumption $\gamma_n \to 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and we have $\limsup_{n\to 0}\delta_n \le 0$ and $\sum_{n=1}^{\infty}\eta_n < \infty$. Hence, applying Lemma 2.6, we immediately deduce that $x_n \to z$ where $z = P_Z f(z)$. $\square$

Remark 2.8. *In general, $\bigcap_{i\in I} A_i(\{x\}) \subseteq R(\boldsymbol{A},\boldsymbol{\lambda})(\{x\})$. From Theorem 2.5 of [1] we know that, if A_i's are monotone and $\bigcap_{i\in I} A_i^{-1}(0) \neq \emptyset$, then $(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) = \bigcap_{i\in I} A_i^{-1}(0)$. On the other hand, we have*

$$
\big(R(\boldsymbol{A},\boldsymbol{\lambda})\big)^{-1} = R(\boldsymbol{A}^{-1},\boldsymbol{\lambda}) \tag{2.6}
$$

see Theorem 2.2 in [1] for more details. Therefore, one can replace $R(\boldsymbol{A},\boldsymbol{\lambda})^{-1}(0) \neq \emptyset$ by $\bigcap_{i\in I} A_i^{-1}(0) \neq \emptyset$ in Theorem 2.7. The following example shows that there is a finite family of monotone operators $\{A_i : H \multimap H\}_{i\in I}$ such that $\bigcap_{i\in I} A_i^{-1}(0) = \emptyset$, but $(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) \neq \emptyset$.

Example 2.9. *Let $\boldsymbol{A} = (A_1, A_2)$, $\boldsymbol{A}^{-1} = (A_1^{-1}, A_2^{-1})$ and $\lambda_i = \frac{1}{2}$ for each $i = 1, 2,$. Let for each $i = 1, 2$, $A_i : H \multimap H$ be defined by*

$$
A_i(x) = \begin{cases} H & x = a_i, \\ \emptyset & x \neq a_i, \end{cases}
$$

where $a_1, a_2 \in H$ with $a_1 \neq a_2$. We have $A_i^{-1} : H \multimap H$, $A_i^{-1}(x) = \{a_i\}$ for each $i = 1, 2$. Clearly, $\bigcap_{i=1}^{2} A_i^{-1}(0) = \emptyset$.

On the other hand, for each $i = 1, 2$, we have

$$
(A_i^{-1} + \mathrm{Id})^{-1}(x) = \{x - a_i\}. \tag{2.7}
$$

By using (2.6) and (2.7), we get

$$\begin{aligned}(R(\boldsymbol{A},\lambda))^{-1}(0) &= R(\boldsymbol{A}^{-1},\boldsymbol{\lambda})(0)\\ &= \Big(\Big(\frac{1}{2}(A_1^{-1}+\mathrm{Id})^{-1}+\frac{1}{2}(A_2^{-1}+\mathrm{Id})^{-1}\Big)^{-1}-\mathrm{Id}\Big)(0)\\ &= \Big\{x\in H: 0\in \frac{1}{2}\{x-a_1\}+\frac{1}{2}\{x-a_2\}\Big\}\\ &= \Big\{\frac{a_1+a_2}{2}\Big\}.\end{aligned}$$

Therefore, $(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0)\neq\emptyset$.

Theorem 2.10. *Let* $\{A_i : H \multimap H\}_{i\in I}$ *be a finite family of maximal monotone operators with* $Z = R(\boldsymbol{A},\boldsymbol{\lambda})^{-1}(0)\neq\emptyset$, *where* $\lambda_i>0$ *and* $\sum_{i\in I}\lambda_i = 1$. *Assume that* f *is* k*-contraction of* H *into itself. Let* $\{\beta_n\}$ *be a real sequence in* $(0,1)$ *and* $\{e_n\}$ *be a sequence of errors. Let* $\{x_n\}$ *be the sequence generated by* (2.4). *Assume that the following conditions are satisfied:*

(i) $e_n\in K$, *satisfies* $\sum_{n\in\mathbb{N}}\|e_n\|<\infty$,

(ii) $\lim\limits_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n=\infty$,

then the sequence $\{x_n\}$ *converges strongly to* $z\in Z$, *where* $z = P_Z f(z)$.

Proof . Since A_i's are maximal monotone, then A_i's are monotone and satisfy the following condition:

$$\overline{\mathrm{Dom}R(\boldsymbol{A},\boldsymbol{\lambda}))}\subset K\subset \mathrm{ran}(\mathrm{Id}+R(\boldsymbol{A},\boldsymbol{\lambda})).$$

Putting $K = H$, the desired result holds. □

Theorem 2.11. *For every* $n\in\mathbb{N}$ *and* $i\in I$, *let* $A_i : H\multimap H$ *be a finite family of maximal monotone operators with* $Z = R(\boldsymbol{A},\boldsymbol{\lambda})^{-1}(0)\neq\emptyset$, *where* $\lambda_i>0$ *and* $\sum_{i\in I}\lambda_i=1$. *Let* $\{\beta_n\}$ *be a real sequence in* $(0,1)$ *and* $\{e_n\}$ *be a sequence of errors. Let* $\{x_n\}$ *be the sequence generated by* $u, x_1\in H$ *and*

$$x_{n+1} = J_{R(\boldsymbol{A},\boldsymbol{\lambda})}(\beta_n u+(1-\beta_n)x_n+e_n),\ \ n\in\mathbb{N}. \tag{2.8}$$

Assume that the following conditions are satisfied:

(i) $e_n\in K$, *satisfies* $\sum_{n\in\mathbb{N}}\|e_n\|<\infty$,

(ii) $\lim\limits_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n=\infty$,

then the sequence $\{x_n\}$ *converges strongly to* $z\in Z$, *where* $z = P_Z u$.

Proof . First, we show that equation (2.8) is equivalent to the following equation:

$$x_{n+1} = (1-\beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}(x_n)+\beta_n u+e_n,\ \ n\in\mathbb{N}.$$

Set $y_n := \beta_n u+(1-\beta_n)x_n+e_n$. We can rewrite (2.8) as

$$y_{n+1} = (1-\beta_{n+1})J_{R(\boldsymbol{A},\boldsymbol{\lambda})}(y_n)+\beta_{n+1}u+e_{n+1},\ \ n\in\mathbb{N}, \tag{2.9}$$

Re-denoting $x_n := y_n$, $\beta_n := \beta_{n+1}$ and $e_n := e_{n+1}$ in (2.9), algorithm (2.8) reads

$$x_{n+1} = (1-\beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}(x_n) + \beta_n u + e_n, \ n \in \mathbb{N}, \tag{2.10}$$

which is exactly the algorithm that is proposed in Theorem 2.10. □

Algorithm 1 Iterative algorithms for resolvent average

Input: $x_1 \in H, \ \{\beta_n\}_{n\in\mathbb{N}} \subset (0,1), \{\lambda_i\}_{1\le i\le m} \subset (0,1), \ \{e_n\} \in H, \ \mathbf{A} = (A_1, \ldots, A_m)$
Output: x_n
  for $i = 1$ to m **do**
    $J_{A_i}(x_n) := (A_i + \mathrm{Id})^{-1}(x_n)$
  end for
  Set $J_{R(\mathbf{A},\lambda)}(x_n) = \sum_{i=1}^{m} \lambda_i J_{A_i}(x_n)$
  for $n = 1$ to ... **do**
    $x_{n+1} = \beta_n f(x_n) + (1-\beta_n)J_{R(\mathbf{A},\lambda)}(x_n) + e_n$
  end for

3. Numerical examples

In this section, we have supported our new iterative algorithm for monotone operators by numerical examples.

Example 3.1. *Let $A_1(x) = x$ and $A_2(x) = x+1$. Set $\boldsymbol{A} = (x, x+1)$, $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $f(x) = \frac{x}{2}$. Assume that $e_n = \left\{\frac{1}{n^n}\right\}$ is the sequence of errors and $\beta_n = \left\{\frac{1}{n}\right\}$ for $n \in \mathbb{N}$.*

First note that $A_1^{-1}(x) = x$ and $A_2^{-1}(x) = x-1$. So, $\boldsymbol{A}^{-1} = (x, x-1)$. Then by easy calculation, we get

$$J_{A_1^{-1}}(x_n) = (A_1^{-1} + \mathrm{Id})^{-1}(x_n) = \left\{\frac{1}{2}x_n\right\}, \tag{3.1}$$

and

$$J_{A_2^{-1}}(x_n) = \left\{\frac{1}{2}(x_n+1)\right\}. \tag{3.2}$$

By using (2.6) and (2.7), we obtain

$$\begin{aligned}
(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) &= (R(\boldsymbol{A}^{-1},\boldsymbol{\lambda}))(0)\\
&= \Big(\big(\frac{1}{2}(A_1^{-1}+\mathrm{Id})^{-1} + \frac{1}{2}(A_2^{-1}+\mathrm{Id})^{-1}\big)^{-1} - \mathrm{Id}\Big)(0)\\
&= \Big\{x \in \mathbb{R} : 0 \in \Big(\frac{1}{2}(A_1^{-1}+\mathrm{Id})^{-1}(x) + \frac{1}{2}(A_2^{-1}+\mathrm{Id})^{-1}(x)\Big)\Big\}\\
&= \Big\{x \in \mathbb{R} : 0 \in \frac{1}{2}\Big(\frac{1}{2}x\Big) + \frac{1}{2}\Big(\frac{1}{2}(x+1)\Big)\Big\}\\
&= \Big\{-\frac{1}{2}\Big\}.
\end{aligned}$$

Therefore, $Z = (R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) = \left\{-\frac{1}{2}\right\}$. Hence,

$$P_Z(f(z)) = P_{\{-\frac{1}{2}\}}\Big(f\Big(-\frac{1}{2}\Big)\Big) = P_{\{-\frac{1}{2}\}}(1) = -\frac{1}{2} \tag{3.3}$$

Let $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n) J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n + e_n, \;\; n \in \mathbb{N}, \tag{3.4}$$

with starting point $x_1 \in \mathbb{R}$. Clearly,

$$\begin{aligned} J_{A_1}(x_n) &= (A_1 + \mathrm{Id})^{-1}(x_n) \\ &= \{y \in \mathbb{R} : x_n \in (A_1 + \mathrm{Id})(y)\} \\ &= \Big\{\frac{1}{2} x_n\Big\}. \end{aligned} \tag{3.5}$$

and similarly,

$$J_{A_2}(x_n) = \Big\{\frac{1}{2}(x_n - 1)\Big\}. \tag{3.6}$$

Substituting (3.5) *and* (3.6) *into* (3.4)*, we obtain*

$$\begin{aligned} x_{n+1} &= \beta_n f(x_n) + (1-\beta_n) J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n + e_n \\ &= \beta_n f(x_n) + (1-\beta_n) \sum_{i=1}^{2} \lambda_i J_{A_i} x_n + e_n \\ &= \frac{1}{2n} x_n + \frac{1}{2}\big(1 - \frac{1}{n}\big)\Big(\frac{1}{2} x_n + \frac{1}{2}(x_n - 1)\Big) + \frac{1}{n^n}, \;\; n \in \mathbb{N}. \end{aligned}$$

It follows from Theorem 2.7 that $\{x_n\}$ converges, say to x. Since $\{x_n\}$ is bounded, by letting $n \to \infty$ in the above equality we obtain

$$x = 0 + \frac{1}{2}\Big(\frac{1}{2} x + \frac{1}{2}(x-1)\Big).$$

Therefore, $x = -\frac{1}{2}$. The numerical results with starting point $x_1 = 0$, which are shown in Table 1, shows that $x_n \to -\frac{1}{2}$.

Table 1: Results for given starting point $x_1 = 0$ in Example 3.1

n	1	10	100	1000	2000	3000	4000	5000	10000	...
x_n	0	-0.42656	-0.49489	-0.49949	-0.49975	-0.49983	-0.49987	-0.4999	-0.4999	...

Example 3.2. *Let $A_1(x) = 2x - 1$, $A_2(x) = x$, $A_3(x) = x + 1$ and $A_4(x) = 2x + 3$. Set $\boldsymbol{A} = (2x-1, x, x+1, 2x+3)$, $f(x) = \frac{2x}{3}$ and $\lambda_i = \frac{1}{4}$ for each $1 \le i \le 4$. Assume that $e_n = \big\{\frac{1}{n^n}\big\}$ is the sequence of errors and $\beta_n = \big\{\frac{1}{n}\big\}$ for $n \in \mathbb{N}$. We have $\boldsymbol{A}^{-1} = \big(\frac{1+x}{2}, x, x-1, \frac{x-3}{2}\big)$. Then by easy calculation, we get*

$$\begin{aligned} &J_{A_1^{-1}}(x_n) = \Big\{\frac{1}{3}\big(2x_n - 1\big)\Big\}, \;\; J_{A_2^{-1}}(x_n) = \Big\{\frac{1}{2} x_n\Big\}, \\ &J_{A_3^{-1}}(x_n) = \Big\{\frac{1}{2}(x_n + 1)\Big\}, \;\; J_{A_4^{-1}}(x_n) = \Big\{\frac{1}{3}(2x_n + 3)\Big\}. \end{aligned} \tag{3.7}$$

By using (2.6) *and* (3.7)*, we obtain*

$$(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) = \Big\{-\frac{1}{2}\Big\}.$$

Therefore, $Z = (R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(0) = \{\frac{-1}{2}\}$*. Hence*

$$P_Z(f(z)) = P_{\{-\frac{1}{2}\}}\Big(f\big(-\frac{1}{2}\big)\Big) = P_{\{-\frac{1}{2}\}}\big(-\frac{1}{4}\big) = -\frac{1}{2} \tag{3.8}$$

Let $\{x_n\}$ *be the sequence generated by*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n + e_n,\ n \in \mathbb{N}, \tag{3.9}$$

with starting point $x_1 \in \mathbb{R}$*. We have*

$$\begin{aligned} J_{A_1}(x_n) &= \Big\{\frac{1}{3}(x_n+1)\Big\},\ J_{A_2}(x_n) = \Big\{\frac{1}{2}x_n\Big\},\\ J_{A_3}(x_n) &= \Big\{\frac{1}{2}(x_n-1)\Big\},\ J_{A_4}(x_n) = \Big\{\frac{1}{3}(x_n-3)\Big\}. \end{aligned} \tag{3.10}$$

Substituting (3.10) *into* (3.9)*, we obtain*

$$\begin{aligned} x_{n+1} &= \beta_n f(x_n) + (1-\beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n + e_n\\ &= \frac{2}{3n}x_n + \frac{1}{4}\Big(1-\frac{1}{n}\Big)\Big(\frac{1}{6}\big(10x_n-7\big)\Big) + \frac{1}{n^n},\ n \in \mathbb{N}. \end{aligned}$$

Now, Theorem 2.7 implies the convergence of $\{x_n\}$*, say to* x*. By letting* $n \to \infty$ *in the above equality we obtain*

$$x = 0 + \frac{1}{24}(10x-7);$$

i.e., $x = -\frac{1}{2}$*. The numerical results in Table 2 with starting point* $x_1 = 0$ *show that* $x_n \to -\frac{1}{2}$*.*

Table 2: Results for given starting point $x_1 = 0$ in Example 3.2

n	1	10	100	1000	2000	3000	4000	5000	10000	...
x_n	0	-0.45656	-0.49708	-0.49971	-0.49985	-0.4999	-0.49992	-0.49994	-0.49997	...

Example 3.3. *Let* $\boldsymbol{A} = (x^3-1, x-1, (x-1)^3)$*,* $f(x) = \dfrac{99x}{100}$ *and* $\lambda_i = \frac{1}{3}$ *for every* $1 \le i \le 3$*. Assume that* $e_n = \{\frac{1}{n^n}\}$ *and* $\beta_n = \{\frac{1}{n+1}\}$*. We have* $\boldsymbol{A}^{-1} = \big((1+x)^{\frac{1}{3}}, 1+x, 1+x^{\frac{1}{3}}\big)$*. Then*

$$J_{A_1^{-1}}(x_n) = \Big\{x_n + \frac{(\frac{2}{3})^{\frac{1}{3}}}{h_1(x_n)} - \frac{h_1(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}}\Big\},\ J_{A_2^{-1}}(x_n) = \Big\{\frac{1}{2}(x_n-1)\Big\},$$

$$J_{A_3^{-1}}(x_n) = \Big\{x_n - \frac{(\frac{2}{3})^{\frac{1}{3}}}{h_2(x_n)} + \frac{h_2(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} - 1\Big\}, \tag{3.11}$$

where $h_1(x_n) = \Big(9+9x_n-\sqrt{3}\sqrt{31+54x_n+27x_n^2}\Big)^{\frac{1}{3}}$ *and* $h_2(x_n) = \Big(9-9x_n+\sqrt{3}\sqrt{31-54x_n+27x_n^2}\Big)^{\frac{1}{3}}$. *Let* $\{x_n\}$ *be the sequence generated by*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n + e_n,\ n\in\mathbb{N}, \tag{3.12}$$

with starting point $x_1 \in \mathbb{R}$. *We have*

$$J_{A_1}(x_n) = \Big\{\frac{h_1(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} - \frac{(\frac{2}{3})^{\frac{1}{3}}}{h_1(x_n)}\Big\},\ J_{A_2}(x_n) = \Big\{\frac{1}{2}(1+x_n)\Big\},$$

$$J_{A_3}(x_n) = \Big\{\frac{(\frac{2}{3})^{\frac{1}{3}}}{h_2(x_n)} - \frac{h_2(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} + 1\Big\}. \tag{3.13}$$

Substituting (3.13) *into* (3.12), *we obtain*

$$\begin{aligned} x_{n+1} &= \beta_n f(x_n) + (1-\beta_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n + e_n \\ &= \frac{99}{100(n+1)}x_n + \frac{1}{3}\big(1-\frac{1}{n+1}\big)\Big(\frac{3}{2}+\frac{1}{2}x_n + \frac{(\frac{2}{3})^{\frac{1}{3}}}{h_2(x_n)} - \frac{h_2(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} \\ &\quad + \frac{h_1(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} - \frac{(\frac{2}{3})^{\frac{1}{3}}}{h_1(x_n)}\Big) + \frac{1}{n^n},\ n\in\mathbb{N}. \end{aligned}$$

The numerical results in Table 3 with starting point $x_1 = 0$ *show that* $x_n \to 1$.

Table 3: Results for given starting point $x_1 = 0$ in Example 3.3

n	1	10	100	1000	2000	5000	10000	...
x_n	0	1.01732	0.999754	0.999976	0.999988	0.999995	0.999998	...

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