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Study on left almost-rings by anti fuzzy bi-ideals

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Abstract

This paper is a special continuation of [6]. We discuss the anti fuzzy left (resp. right, bi-, generalized bi-, $(1,2)$-) ideals in left almost rings. Our aim is to characterizated the different classes of left almost rings (llike left regular, right regular, $(2,2)$-regular, left weakly regular, right weakly regular, intra-regular) in terms of anti fuzzy left (resp. right, bi-, generalized bi-, $(1,2)$-) ideals.

Keywords: Keywords: Anti fuzzy left ideal, Anti fuzzy right ideal, Anti fuzzy bi ideal, Anti fuzzy generalized bi ideal, Anti fuzzy $(1,2)$- ideal
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1. Introduction

In 1972, a generalization of abelian semigroups was initiated by Kazim et al [14]. In ternary commutative (abelian) law: $abc = cba$, they introduced braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is the left invertive law. A groupoid S is said to be left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. An LA-semigroup is a midway structure between an abelian semigroup and a groupoid.

A groupoid S is to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [14], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial.

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Every LA-semigroup with left identity is paramedial and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

In [19], an LA-semigroup S is to be a left almost group, if there exists left identity $e \in S$, such that $ea = a$ for all $a \in S$ and for every $a \in S$ there exists $b \in S$ such that $ba = e$.

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like.

Although semigroups, group andri concentrate on theoretical aspects, they also include applications in error-correcting codes, control engineering, formal language, computer science and information science.

Algebraic structures especially rings play a prominent role in mathematics with wide ranging applications in many disciplines such as control engineering, computer arithmetics, coding theory, sequential machines and formal languages.

In [19], An LA-ring is a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Despite the fact that the structure is non-associative and non-commutative, however it possesses properties which usually come across in associative and commutative algebraic structures.

A non-empty subset A of an LA-ring R is an LA-subring of R if $a - b$ and $ab \in A$ for all $a, b \in A$ [11]. A is a left (resp. right) ideal of R if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R.

An LA-subring A of R is a bi-ideal of R if $(AR)A \subseteq A$. A non-empty subset A of R is a generalized bi-ideal of R if $(A, +)$ is an LA-group and $(AR)A \subseteq A$. Every bi-ideal of R is a generalized bi-ideal of R.

We initiate the concept of regular (resp. left regular, right regular, $(2, 2)$-regular, left weakly regular, right weakly regular, intra-regular) LA-rings. We also define the concept of anti fuzzy left (resp. right, bi-, generalized bi-, $(1, 2)$-) ideals.

We describe a study of regular (resp. left regular, right regular, $(2, 2)$-regular, left weakly regular, right weakly regular, intra-regular) LA-rings by the properties of anti fuzzy left (right, bi-, generalized bi-) ideals. In this regard, we prove that in regular (resp. left weakly regular) LA-rings, the concept of anti fuzzy (right, two-sided) ideals coincides. We also show that in right regular (resp. $(2, 2)$-regular, right weakly regular, intra-regular) LA-rings, the concept of anti fuzzy (left, right, two-sided) ideals coincides. Also in left regular LA-rings with left identity, the concept of anti fuzzy (left, right, two-sided) ideals coincides. We also characterize left weakly regular LA-rings in terms of anti fuzzy right (two-sided, bi-, generalize bi-) ideals.

2. Preliminaries and Primary Results

First time concept of fuzzy set introduced by Zadeh in his classical paper [20]. This concept has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, management science, expert systems, finite state machines, Languages, robotics, coding theory and others.

It soon invoked a natural question concerning a possible connection between fuzzy sets and algebraic systems like (set, semigroup, group, ring, near-ring, semiring, measure) theory, groupoids, real analysis, topology, differential equations and so forth. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, groupoids, semigroup, ordered semigroup, group theory, real analysis, measure theory, topology, etc. The study of fuzzy set in semigroups was established by Kuroki [15]. He also studied the fuzzy ideals and fuzzy (interior, quasi-, bi-, generalized bi-, semiprime) ideals of semigroups.

Gupta et al [4], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [16], characterized the regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi, bi-) ideals.

In [2], Biswas introduced the concept of anti fuzzy subgroups of groups and studied the basic properties of groups in terms of such ideals. Hong and Jun [5], modified the Biswas idea and applied it into BCK-algebra. Akram and Dar defined anti fuzzy left h-ideals of a hemiring and discussed the basic properties of hemiring [1].

Shal et al [18], originated the studied of intuitionistic fuzzy normal LA-subrings over left almost-ring. Islam et al [9] initiated the intuitionistics fuzzy ideals with thresholds $(\alpha, \beta]$ in left almost ring. Javaid et al [8], also studied the left almost rings by fuzzy ideals. Waqar et al [6], studied the left almost rings by using the intuitionistic fuzzy bi-ideals. Kausar et al [7], explored the direct product of finite intuitionistic anti fuzzy normal LA-subrings over LA-rings. Waqar et al [10], investigated the direct product of finite fuzzy normal LA-subrings on Left Almost-rings. Kausar et al studied the LA-Rings through their intuitionistics fuzzy Ideals in [12], and through the fuzzy bi-ideals in [13].

By a fuzzy subset μ of an LA-ring R, we mean a function $\mu : R \to [0,1]$ and the complement of μ is denoted by μ', is also a fuzzy subset of R defined as $\mu'(x) = 1 - \mu(x)$ for all $x \in R$.

A fuzzy subset μ of an LA-ring R is an anti fuzzy LA-subring of R if

$\mu(x - y) \leq max\{\mu(x), \mu(y)\}$ and $\mu(xy) \leq max\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

Equivalent definition: A fuzzy subset μ of an LA-ring R is an anti fuzzy LA-subring of R if $\mu(x + y) \leq max\{\mu(x), \mu(y)\}$, $\mu(-x) \leq \mu(x)$ and $\mu(xy) \leq max\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

A fuzzy subset μ of an LA-ring R is an anti fuzzy left (resp. right) ideal of R if $\mu(x - y) \leq max\{\mu(x), \mu(y)\}$ and $\mu(xy) \leq \mu(y)$ (resp. $\mu(xy) \leq \mu(x)$) for all $x, y \in R$. μ is called an anti fuzzy ideal of R if it is both an anti fuzzy left and an anti fuzzy right ideal of R. Every anti fuzzy ideal (whether left, right, two-sided) of R is an anti fuzzy LA-subring of R.

An anti fuzzy LA-subring μ of an LA-ring R is an anti fuzzy bi-ideal of R if $\mu((xa)y) \leq max\{\mu(x), \mu(y)\}$ for all $x, y, a \in R$.

A fuzzy subset μ of an LA-ring R is an anti fuzzy generalized bi-ideal of R if $\mu(x - y) \leq max\{\mu(x), \mu(y)\}$ and $\mu((xa)y) \leq max\{\mu(x), \mu(y)\}$ for all $x, y, a \in R$. Every anti fuzzy bi-ideal of R is an anti fuzzy generalized bi-ideal of R. An anti fuzzy LA-subring μ of an LA-ring R is an anti fuzzy $(1, 2)$-ideal of R if $\mu((xa)(yz)) \leq max\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z, a \in R$.

Let A be a non-empty subset of an LA-ring R. Then the anti characteristic function of A is denoted by χ_A^C and defined as

$$\chi_A^C(x) = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \notin A \end{cases}$$

The product of two fuzzy subsets μ and ν is denoted by $\mu \circ \nu$ and defined by:

$$(\mu \circ \nu)(x) = \begin{cases} \wedge_{x=\sum\limits_{i=1}^{n} a_i b_i}\{\vee_{i=1}^{n}\{\mu(a_i) \vee \nu(b_i)\}\} \text{ if } x = \sum\limits_{i=1}^{n} a_i b_i,\ a_i, b_i \in R \\ 1 \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \neq \sum\limits_{i=1}^{n} a_i b_i \end{cases}$$

Now we are giving the some basic properties of an LA-ring R, which will be very helpful for the next section.

Theorem 2.1. *Let A and B be two non-empty subsets of an LA-ring R. Then the following conditions hold.*

(1) *If* $A \subseteq B$ *then* $\chi_A^C \supseteq \chi_B^C$.
(2) $\chi_A^C \circ \chi_B^C = \chi_{AB}^C$.
(3) $\chi_A^C \cup \chi_B^C = \chi_{A\cup B}^C$.
(4) $\chi_A^C \cap \chi_B^C = \chi_{A\cap B}^C$.

Proof . Straight forward. □

Example 2.2. *Let* $R = \{a, b, c, d\}$. *Define* $+$ *and* $\cdot$ *in* R *as follows* :

$+$	a	b	c	d
a	a	b	c	d
b	d	a	b	c
c	c	d	a	b
d	b	c	d	a

and

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Then R *is an LA-ring and* μ *be a fuzzy subset of* R. *We define* $\mu(a) = \mu(c) = 0,\ \mu(b) = \mu(d) = 0.7$. *Then* μ *is an anti fuzzy ideal of* R.

Lemma 2.3. *Every anti fuzzy left (resp. right, two-sided) ideal of an LA-ring R is an anti fuzzy bi-ideal of R.*

Proof . Straight forward. □

Lemma 2.4. *Every anti fuzzy bi-ideal of an LA-ring R is an anti fuzzy* $(1, 2)$*-ideal of R.*

Proof . Straight forward. □

Remark 2.5. *Every anti fuzzy left (resp. right, two-sided) ideal of an LA-ring R is an anti fuzzy* $(1, 2)$*-ideal of R.*

Proposition 2.6. *Let R be an LA-ring having the property* $a = a^2$ *for every* $a \in R$. *Then every anti fuzzy* $(1, 2)$*-ideal of R is an anti fuzzy bi-ideal of R.*

Proof . Suppose that μ is an anti fuzzy $(1, 2)$-ideal of R and $a, x, y \in R$. Thus $\mu((xa)y) = \mu((xa)(yy)) \leq max\{\mu(x), \mu(y), \mu(y)\} = max\{\mu(x), \mu(y)\}$. Therefore μ is an anti fuzzy bi-ideal of R. □

Lemma 2.7. *Let R be an LA-ring and* $\emptyset \neq A \subseteq R$. *Then A is an LA-subring of R if and only if the anti characteristic function* χ_A^C *of A is an anti fuzzy LA-subring of R.*

Proof . Straight forward. □

Lemma 2.8. *Let R be an LA-ring and* $\emptyset \neq A \subseteq R$. *Then A is a left (resp. right) ideal of R if and only if the anti characteristic function* χ_A^C *of A is an anti fuzzy left (resp. right) ideal of R.*

Proof . Straight forward. □

Proposition 2.9. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a $(1,2)$-ideal of R if and only if the anti characteristic function χ_A^C of A is an anti fuzzy $(1,2)$-ideal of R.*

Proof . Let A be a $(1,2)$-ideal of R, this implies that A is an LA-subring of R. Then χ_A^C is an anti fuzzy LA-subring of R, by the Lemma 2.7. Let $a, x, y, z \in R$. If $x, y, z \in A$, then by definition $\chi_A^C(x) = 0 = \chi_A^C(y) = \chi_A^C(z)$. Since $(xa)(yz) \in A$, A being a $(1,2)$-ideal, so $\chi_A^C((xa)(yz)) = 0$. Thus $\chi_A^C((xa)(yz)) \leq max\{\chi_A^C(x), \chi_A^C(y), \chi_A^C(z)\}$. Similarly, we have $\chi_A^C((xa)(yz)) \leq max\{\chi_A^C(x), \chi_A^C(y), \chi_A^C(z)\}$, wh $x, y, z \notin A$. Hence the anti characteristic function χ_A^C of A is an anti fuzzy $(1,2)$-ideal of R.

Conversely, assume that the anti characteristic function χ_A^C of A is an anti fuzzy $(1,2)$-ideal of R, this implies that χ_A^C is an anti fuzzy LA-subring of R. Then A is an LA-subring of R by the Lemma 2.7. Let $t \in (AR)A^2$, so $t = (xa)(yz)$, where $x, y, z \in A$, $a \in R$. So by definition $\chi_A^C(x) = 0 = \chi_A^C(y) = \chi_A^C(z)$. Since $\chi_A((xa)(yz)) \leq \chi_A(x) \vee \chi_A(y) \vee \chi_A(z) = 0$, χ_A^C being an anti fuzzy $(1,2)$-ideal of R. Thus $\chi_A^C((xa)(yz)) = 0$, i.e., $(xa)(yz) \in A$. Hence A is a $(1,2)$-ideal of R. $\Box$

Remark 2.10. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a bi-ideal of R if and only if the anti characteristic function χ_A^C of A is an anti fuzzy bi-ideal of R.*

Zadeh [20], introduced the concept of level set. Das [3], studied the fuzzy groups, level subgroups and gave the proper definition of a level set such that: let μ be a fuzzy subset of a non-empty set S, for $t \in [0,1]$, the set $\mu_t = \{x \in S \mid \mu(x) \geq t\}$, is called a level subset of the fuzzy subset μ.

Let μ be a fuzzy subset of an LA-ring R, then for all $t \in (0,1]$, we define a set $L(\mu; t) = \{x \in R \mid \mu(x) \leq t\}$, which is called a lower t-level set of μ.

Lemma 2.11. *Let μ be a fuzzy subset of an LA-ring R. Then μ is an anti fuzzy LA-subring of R if and only if the lower t-level set $L(\mu; t)$ of μ is an LA-subring of R for all $t \in (0,1]$.*

Proof . Straight forward. $\Box$

Lemma 2.12. *Let μ be a fuzzy subset of an LA-ring R. Then μ is an anti fuzzy left (resp. right) ideal of R if and only if the lower t-level set $L(\mu; t)$ of μ is a left (resp. right) ideal of R for all $t \in (0,1]$.*

Proof . Straight forward. $\Box$

Proposition 2.13. *Let μ be a fuzzy subset of an LA-ring R. Then μ is an anti fuzzy $(1,2)$-ideal of R if and only if the lower t-level set $L(\mu; t)$ of μ is a $(1,2)$-ideal of R for all $t \in (0,1]$.*

Proof . Suppose that μ is an anti fuzzy $(1,2)$-ideal of R, this implies that μ is an anti fuzzy LA-subring of R. Then $L(\mu; t)$ is an LA-subring of R by the Lemma 2.11. Let $x, y, z \in L(\mu; t)$ and $a \in R$, then by definition $\mu(x), \mu(y), \mu(z) \leq t$. Now $\mu((xa)(yz)) \leq \mu(x) \vee \mu(y) \vee \mu(z) \leq t$, μ being an anti fuzzy $(1,2)$-ideal of R, i.e., $(xa)(yz) \in L(\mu; t)$. Therefore $L(\mu; t)$ is a $(1,2)$-ideal of R.

Conversely, assume that $L(\mu; t)$ is a $(1,2)$-ideal of R, this means that $L(\mu; t)$ is an LA-subring of R. Then μ is an anti fuzzy LA-subring of R by the Lemma 2.11. We have to show that $\mu((xa)(yz)) \leq \mu(x) \vee \mu(y) \vee \mu(z)$. We suppose a contradiction $\mu((xa)(yz)) > \mu(x) \wedge \mu(y) \wedge \mu(z)$. Let $\mu(x) = t = \mu(y) = \mu(z)$, so $\mu(x), \mu(y), \mu(z) \leq t$, i.e., $x, y, z \in U(\mu; t)$. But $\mu((xa)(yz)) > t$, i.e., $(xa)(yz) \notin U(\mu; t)$, which is a contradiction. Therefore $\mu((xa)(yz)) \leq \mu(x) \vee \mu(y) \vee \mu(z)$. $\Box$

Remark 2.14. *Let μ be a fuzzy subset of an LA-ring R. Then μ is an anti fuzzy bi-ideal of R if and only if the lower t-level set $L(\mu; t)$ of μ is a bi-ideal of R for all $t \in (0,1]$.*

3. Study of anti fuzzy bi-ideals in left almost-rings

In this section, we characterize different classes of LA-ring in terms of anti fuzzy left (right, bi-, generalized bi-) ideals. An LA-ring R is called a regular if for every element $x \in R$, there exists an element $a \in R$ such that $x = (xa)x$. An LA-ring R is called an intra-regular if for every element $x \in R$, there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$.

An LA-ring R is called a left (resp. right) regular if for every element $x \in R$, there exists an element $a \in R$ such that $x = ax^2$ (resp. x^2a). An LA-ring R is called completely regular if it is regular, left regular and right regular. An LA-ring R is called a $(2,2)$-regular if for every element $x \in R$, there exists an element $a \in R$ such that $x = (x^2a)x^2$. An LA-ring R is called a locally associative LA-ring if $(a.a).a = a.(a.a)$ for all $a \in R$.

A ring R is called a left (resp. right) weakly regular if $I^2 = I$ for every left (resp. right) ideal I of R, equivalently $x \in RxRx(x \in xRxR)$ for every $x \in R$. R is called a weakly regular if it is both a left weakly regular and a right weakly regular [17]. Now we define this notion in a class of non-associative and non-commutative rings (LA-ring).

An LA-ring R is called a left (resp. right) weakly regular if for every element $x \in R$, there exist elements $a, b \in R$ such that $x = (ax)(bx)$ (resp. $x = (xa)(xb)$). An LA-ring R is called a weakly regular if it is both a left weakly regular and a right weakly regular.

Lemma 3.1. *Every anti fuzzy right ideal of an LA-ring R with left identity e, is an anti fuzzy ideal of R.*

Proof . Let μ be an anti fuzzy right ideal of R and $x, y \in R$. Thus $\mu(xy) = \mu((ex)y) = \mu((yx)e) \leq \mu(yx) \leq \mu(y)$. Hence μ is an anti fuzzy ideal of R. □

Lemma 3.2. *Every anti fuzzy right ideal of a regular LA-ring R, is an anti fuzzy ideal of R.*

Proof . Suppose that μ is an anti fuzzy right ideal of R. Let $x, y \in R$, this implies that there exists $a \in R$, such that $x = (xa)x$. Thus $\mu(xy) = \mu(((xa)x)y) = \mu((yx)(xa)) \leq \mu(yx) \leq \mu(y)$. Therefore μ is an anti fuzzy ideal of R. □

Proposition 3.3. *Let R be a regular LA-ring having the property $a = a^2$ for every $a \in R$, with left identity e. Then every anti fuzzy generalized bi-ideal of R is an anti fuzzy bi-ideal of R.*

Proof . Let μ be an anti fuzzy generalized bi-ideal of R and $x, y \in R$, this means that there exists $a \in R$ such that $x = (xa)x$. We have to show that μ is an anti fuzzy LA-subring of R. Thus

$$\begin{aligned}\mu(xy) &= \mu(((xa)x)y) = \mu(((xa)x^2)y) = \mu(((xa)(xx))y) \\ &= \mu((x((xa)x))y) \leq max\{\mu(x), \mu(y)\}.\end{aligned}$$

Hence μ is an anti fuzzy LA-subring of R. □

Lemma 3.4. *Let R be an LA-ring with left identity e. Then Ra is the smallest left ideal of R containing a.*

Proof . Let $x, y \in Ra$ and $r \in R$. This implies that $x = r_1a$ and $y = r_2a$, where $r_1, r_2 \in R$. Now

$$\begin{aligned}x - y &= r_1a - r_2a = (r_1 - r_2)a \in Ra \\ \text{and } rx &= r(r_1a) = (er)(r_1a) = ((r_1a)r)e = ((r_1a)(er))e \\ &= ((r_1e)(ar))e = (e(ar))(r_1e) = (ar)(r_1e) \\ &= ((r_1e)r)a \in Ra.\end{aligned}$$

Since $a = ea \in Ra$. Thus Ra is a left ideal of R containing a. Let I be another left ideal of R containing a. So $ra \in I$, where $ra \in Ra$, i.e., $Ra \subseteq I$. Hence Ra is the smallest left ideal of R containing a. □

Lemma 3.5. *Let R be an LA-ring with left identity e. Then aR is a left ideal of R.*

Proof . Straight forward. □

Proposition 3.6. *Let R be an LA-ring with left identity e. Then $aR \cup Ra$ is the smallest right ideal of R containing a.*

Proof . Let $x, y \in aR \cup Ra$, this means that $x, y \in aR$ or Ra. Since aR and Ra both are left ideals of R, so $x - y \in aR$ and Ra, i.e., $x - y \in aR \cup Ra$. We have to show that $(aR \cup Ra)R \subseteq (aR \cup Ra)$. Now

$$\begin{aligned} (aR \cup Ra)R &= (aR)R \cup (Ra)R = (RR)a \cup (Ra)(eR) \\ &\subseteq Ra \cup (Re)(aR) = Ra \cup R(aR) \\ &= Ra \cup a(RR) \subseteq Ra \cup aR = aR \cup Ra. \\ &\Rightarrow (aR \cup Ra)R \subseteq aR \cup Ra. \end{aligned}$$

As $a \in Ra$, i.e., $a \in aR \cup Ra$. Let I be another right ideal of R containing a. Since $aR \in IR \subseteq I$ and $Ra = (RR)a = (aR)R \in (IR)R \subseteq IR \subseteq I$, i.e., $aR \cup Ra \subseteq I$. Therefore $aR \cup Ra$ is the smallest right ideal of R containing a. □

Lemma 3.7. *Let R be an LA-ring. Then $\mu \circ \nu \supseteq \mu \cup \nu$, for every anti fuzzy right ideal μ and every anti fuzzy left ideal ν of R.*

Proof . Let μ be an anti fuzzy right and ν be an anti fuzzy left ideal of R and $x \in R$. If x cannot be expressible as $x = \sum_{i=1}^{n} a_i b_i$, where $a_i, b_i \in R$ and n is any positive integer, then obviously $\mu \circ \nu \supseteq \mu \cup \nu$, otherwise we have

$$\begin{aligned} (\mu \circ \nu)(x) &= \wedge_{x=\sum_{i=1}^{n} a_i b_i}\{\vee_{i=1}^{n}\{\mu(a_i) \vee \nu(b_i)\}\} \\ &\geq \wedge_{x=\sum_{i=1}^{n} a_i b_i}\{\vee_{i=1}^{n}\{\mu(a_i b_i) \vee \nu(a_i b_i)\}\} \\ &= \wedge_{x=\sum_{i=1}^{n} a_i b_i}\{\vee_{i=1}^{n}(\mu \cup \nu)(a_i b_i)\} = (\mu \cup \nu)(x). \\ &\Rightarrow \mu \circ \nu \supseteq \mu \cup \nu. \end{aligned}$$

□

Theorem 3.8. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$\mu \cup \nu = \mu \circ \nu$, for every anti fuzzy right ideal μ and every anti fuzzy left ideal ν of R.*

Proof . Suppose that (1) holds. Since $\mu \circ \nu \supseteq \mu \cup \nu$ for every anti fuzzy right ideal μ and every anti fuzzy left ideal ν of R by the Lemma 3.7. Let $x \in R$, this implies that there exists an element $a \in R$ such that $x = (xa)x$. Thus

$$\begin{aligned} (\mu \circ \nu)(x) &= \wedge_{x=\sum_{i=1}^{n} a_i b_i}\{\vee_{i=1}^{n}\{\mu(a_i) \vee \nu(b_i)\}\} \\ &\leq max\{\mu(xa), \nu(x)\} \leq max\{\mu(x), \nu(x)\} \\ &= (\mu \vee \nu)(x) = (\mu \cup \nu)(x). \\ &\Rightarrow \mu \cup \nu \supseteq \mu \circ \nu. \end{aligned}$$

Hence $\mu \cup \nu = \mu \circ \nu$, i.e., $(1) \Rightarrow (2)$. Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.4 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.6. So χ^C_{Ra} is an anti fuzzy left ideal and $\chi^C_{aR\cup Ra}$ is an anti fuzzy right ideal of R, by the Lemma 2.8. By our assumption $\chi^C_{aR\cup Ra} \cap \chi^C_{Ra} = \chi^C_{aR\cup Ra} \circ \chi^C_{Ra}$, i.e., $\chi^C_{(aR\cup Ra)\cap Ra} = \chi^C_{(aR\cup Ra)Ra}$. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. Now $(Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$. This implies that

$$(aR)(Ra) \cup (Ra)(Ra) = (aR)(Ra) \cup (aR)(Ra) = (aR)(Ra).$$

Thus $a \in (aR)(Ra)$. Then

$$\begin{aligned} a &= (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a \\ &= ((xe)(ay))a = (a((xe)y))a \in (aR)a, \text{ for any } x, y \in R. \end{aligned}$$

This means that $a \in (aR)a$, i.e., a is regular. Hence R is a regular, i.e., $(2) \Rightarrow (1)$. □

Theorem 3.9. *Let R be a regular locally associative LA-ring having the property $a = a^2$ for every $a \in R$. Then for every anti fuzzy bi-ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof . For $n = 1$. Let $a \in R$, this implies that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a = (a^2x)a^2$, because $a = a^2$. Thus

$$\begin{aligned} \mu(a) &= \mu\left((a^2x)a^2\right) \leq max\{\mu\left(a^2\right), \mu\left(a^2\right)\} = \mu\left(a^2\right) \\ &= \mu(aa) \leq max\{\mu(a), \mu(a)\} = \mu(a). \\ &\Rightarrow \mu(a) = \mu\left(a^2\right). \end{aligned}$$

Now $a^2 = aa = ((a^2x)a^2)((a^2x)a^2) = (a^4x^2)a^4$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, i.e., $\mu(a^k) = \mu\left(a^{2k}\right)$. Now $a^{k+1} = a^ka = ((a^{2k}x^k)a^{2k})((a^2x)a^2) = (a^{2(k+1)}x^{k+1})a^{2(k+1)}$. Thus

$$\begin{aligned} \mu\left(a^{k+1}\right) &= \mu\left((a^{2(k+1)}x^{k+1})a^{2(k+1)}\right) \leq max\{\mu(a^{2(k+1)}), \mu\left(a^{2(k+1)}\right)\} \\ &= \mu\left(a^{2(k+1)}\right) = \mu\left(a^{k+1}a^{k+1)}\right) \\ &\leq max\{\mu\left(a^{k+1)}\right), \mu\left(a^{k+1)}\right)\} = \mu(a^{k+1)}). \\ &\Rightarrow \mu(a^{k+1}) = \mu(a^{2(k+1)}). \end{aligned}$$

Hence by induction method, the result is true for all positive integers. □

Lemma 3.10. *Let R be a $(2, 2)$-regular LA-ring. Then every anti fuzzy left (resp. right) ideal of R is an anti fuzzy ideal of R.*

Proof . Suppose that μ is an anti fuzzy right ideal of R and $x, y \in R$, this means that there exists $a \in R$ such that $x = (x^2a)x^2$. Thus $\mu(xy) = \mu(((x^2a)x^2)y) = \mu((yx^2)(x^2a)) \leq \mu(yx^2) \leq \mu(y)$. Therefore μ is an anti fuzzy ideal of R. Similarly, for left ideal. □

Remark 3.11. *The concept of anti fuzzy (left, right, two-sided) ideals coincides in $(2, 2)$-regular LA-rings.*

Proposition 3.12. *Every anti fuzzy generalized bi-ideal of $(2, 2)$-regular LA-ring R with left identity e, is an anti fuzzy bi-ideal of R.*

Proof . Assume that μ is an anti fuzzy generalized bi-ideal of R and $x, y \in R$, then there exists an element $a \in R$ such that $x = (x^2a)x^2$. We have to show that μ is an anti fuzzy LA-subring of R. Thus

$$\begin{aligned}\mu(xy) &= \mu(((x^2a)x^2)y) = \mu(((x^2a)(xx))y) \\ &= \mu((x((x^2a)x))y) \leq max\{\mu(x), \mu(y)\}.\end{aligned}$$

So μ is an anti fuzzy LA-subring of R. □

Theorem 3.13. *Let R be a $(2, 2)$-regular locally associative LA-ring. Then for every anti fuzzy bi-ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof . Same as Theorem 3.9. □

Lemma 3.14. *Let R be a right regular LA-ring. Then every anti fuzzy left (resp. right) ideal of R is an anti fuzzy ideal of R.*

Proof . Let μ be an anti fuzzy right ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = x^2a$. Thus

$$\begin{aligned}\mu(xy) &= \mu((x^2a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) \\ &= \mu((yx)(ax)) \leq \mu(yx) \leq \mu(y).\end{aligned}$$

Hence μ is an anti fuzzy ideal of R. Similarly, for left ideal. □

Remark 3.15. *The concept of anti fuzzy (left, right, two-sided) ideals coincides in right regular LA-rings.*

Proposition 3.16. *Every anti fuzzy generalized bi-ideal of a right regular LA-ring R with left identity e, is an anti fuzzy bi-ideal of R.*

Proof . Suppose that μ is an anti fuzzy generalized bi-ideal of R and $x, y \in R$, this means that there exists $a \in R$ such that $x = x^2a$. We have to show that μ is an anti fuzzy LA-subring of R. Thus

$$\begin{aligned}\mu(xy) &= \mu((x^2a)y) = \mu(((xx)(ea))y) = \mu(((ae)(xx))y) \\ &= \mu((x((ae)x))y) \leq max\{\mu(x), \mu(y)\}.\end{aligned}$$

Therefore μ is an anti fuzzy LA-subring of R. □

Theorem 3.17. *Let R be a right regular locally associative LA-ring. Then for every anti fuzzy right ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof . For $n = 1$. Let $a \in R$, then there exists an element $x \in R$ such that $a = a^2x$. Thus

$$\begin{aligned}\mu_A(a) &= \mu_A(a^2x) \leq \mu_A(a^2) = \mu_A(aa) \\ &\leq max\{\mu_A(a), \mu_A(a)\} = \mu_A(a). \\ &\Rightarrow \mu(a) = \mu\left(a^2\right).\end{aligned}$$

Now $a^2 = aa = (a^2x)(a^2x) = a^4x^2$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^k a = (a^{2k}x^k)(a^2x) = a^{2(k+1)}x^{(k+1)}$. Thus

$$\begin{aligned}
\mu(a^{k+1}) &= \mu(a^{2(k+1)}x^{(k+1)}) \leq \mu(a^{2(k+1)}) \\
&= \mu(a^{2k+2}) = \mu(a^{k+1}a^{k+1}) \\
&\leq max\{\mu\left(a^{k+1}\right), \mu\left(a^{k+1}\right)\} = \mu\left(a^{k+1}\right). \\
&\Rightarrow \mu(a^{k+1}) = \mu(a^{2(k+1)}).
\end{aligned}$$

Hence by induction method, the result is true for all positive integers. □

Lemma 3.18. *Let R be a right regular locally associative LA-ring with left identity e. Then for every anti fuzzy right ideal μ of R, $\mu(ab) = \mu(ba)$ for all $a, b \in R$.*

Proof . Let $a, b \in R$. By using Theorem 3.17 (for $n = 1$). Now

$$\begin{aligned}
\mu(ab) &= \mu((ab)^2) = \mu((ab)(ab)) \\
&= \mu((ba)(ba)) = \mu((ba)^2) = \mu(ba).
\end{aligned}$$

□

Lemma 3.19. *Let R be a left regular LA-ring with left identity e. Then every anti fuzzy left (resp. right) ideal of R is an anti fuzzy ideal of R.*

Proof . Assume that μ is an anti fuzzy right ideal of R and $x, y \in R$, then there exists an element $a \in R$ such that $x = ax^2$. Thus $\mu(xy) = \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y) = \mu(y(ax))x) \leq \mu(y(ax)) \leq \mu(y)$. So μ is an anti fuzzy ideal of R. Similarly, for left ideal. □

Remark 3.20. *The concept of anti fuzzy (left, right, two-sided) ideals coincides in left regular LA-rings with left identity.*

Proposition 3.21. *Every anti fuzzy generalized bi-ideal of a left regular LA-ring R with left identity e, is an anti fuzzy bi-ideal of R.*

Proof . Let μ be an anti fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = ax^2$. We have to show that μ is an anti fuzzy LA-subring of R. Thus $\mu(xy) = \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y) \leq max\{\mu(x), \mu(y)\}$. Hence μ is an anti fuzzy LA-subring of R. □

Remark 3.22. *It is easy to see that, if R is a left regular locally associative LA-ring with left identity e. Then for every anti fuzzy left ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer. And also for every anti fuzzy left ideal μ of R, $\mu(ab) = \mu(ba)$ for all $a, b \in R$.*

Theorem 3.23. *Let R be a regular and right regular locally associative LA-ring. Then for every anti fuzzy right ideal μ of R, $\mu(a^n) = \mu(a^{3n})$ for all $a \in R$, where n is any positive integer.*

Proof . For $n = 1$. Let $a \in R$, this means that there exists an element $x \in R$ such that $a = (ax)a$ and $a = a^2x$. Now $a = (ax)a = (ax)(a^2x) = a^3x^2$. Thus

$$\begin{aligned}\mu(a) &= \mu(a^3x^2) \leq \mu(a^3) = \mu(aa^2) \leq max\{\mu(a), \mu(a^2)\}\\ &\leq max\{\mu(a), \mu(a), \mu(a)\} = \mu(a).\\ &\Rightarrow \mu(a) = \mu(a^3).\end{aligned}$$

Here $a^2 = aa = (a^3x^2)(a^3x^2) = a^6x^4$, then the result is true for $n = 2$. Assume that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{3k})$. Now $a^{k+1} = a^k a = (a^{3k}x^{2k})(a^3x^2) = a^{3(k+1)}x^{2(k+1)}$. Thus

$$\begin{aligned}\mu(a^{k+1}) &= \mu(a^{3(k+1)}x^{2(k+1)}) \leq \mu(a^{3(k+1)}) = \mu(a^{3k+3})\\ &= \mu(a^{k+1}a^{2k+2}) \leq max\{\mu(a^{k+1}), \mu(a^{2k+2})\}\\ &\leq max\{\mu(a^{k+1}), \mu(a^{k+1}), \mu(a^{k+1})\} = \mu(a^{k+1}).\\ &\Rightarrow \mu(a^{k+1}) = \mu(a^{3(k+1)}).\end{aligned}$$

Hence by induction method, the result is true for all positive integers. □

Lemma 3.24. *Let R be a right weakly regular LA-ring. Then every anti fuzzy left (resp. right) ideal of R is an anti fuzzy ideal of R.*

Proof . Suppose that μ is an anti fuzzy right ideal of R and $x, y \in R$, this means that there exist $a, b \in R$ such that $x = (xa)(xb)$. Thus

$$\begin{aligned}\mu(xy) &= \mu(((xa)(xb))y) = \mu((((xb)a)x)y)\\ &= \mu((((ab)x)x)y) = \mu((yx)((ab)x))\\ &= \mu((yx)(nx)) \text{ say } ab = n\\ &\leq \mu(yx) \leq \mu(y).\end{aligned}$$

Therefore μ is an anti fuzzy ideal of R. Similarly, for left ideal. □

Remark 3.25. *The concept of anti fuzzy (left, right, two-sided) ideals coincides in right weakly regular LA-rings.*

Proposition 3.26. *Every anti fuzzy generalized bi-ideal of a right weakly regular LA-ring R with left identity e, is an anti fuzzy bi-ideal of R.*

Proof . Assume that μ is an anti fuzzy generalized bi-ideal of R and $x, y \in R$, then there exist elements $a, b \in R$ such that $x = (xa)(xb)$. We have to show that μ is an anti fuzzy LA-subring of R. Thus $\mu(xy) = \mu(((xa)(xb))y) = \mu((x((xa)b))y) \leq max\{\mu(x), \mu(y)\}$. So μ is an anti fuzzy LA-subring of R. □

Lemma 3.27. *Let R be a left weakly regular LA-ring. Then every anti fuzzy right ideal of R is an anti fuzzy ideal of R.*

Proof . Let μ be an anti fuzzy right ideal of R and $x, y \in R$, this implies that there exist $a, b \in R$ such that $x = (ax)(bx)$. Thus $\mu(xy) = \mu(((ax)(bx))y) = \mu(y(bx))(ax) \leq \mu(y(bx)) \leq \mu(y)$. Hence μ is an anti fuzzy ideal of R. □

Lemma 3.28. *Let R be a left weakly regular LA-ring with left identity e. Then every anti fuzzy left ideal of R is an anti fuzzy ideal of R.*

Proof . Suppose that μ is an anti fuzzy left ideal of R and $x, y \in R$, this means that there exist $a, b \in R$ such that $x = (ax)(bx)$. Thus

$$\begin{aligned} \mu(xy) &= \mu(((ax)(bx))y) = \mu(((ab)(xx))y) \\ &= \mu((x((ab)x))y) = \mu((y((ab)x))x) \leq \mu(x). \end{aligned}$$

Therefore μ is an anti fuzzy ideal of R. □

Proposition 3.29. *Every anti fuzzy generalized bi-ideal of a left weakly regular LA-ring R with left identity e, is an anti fuzzy bi-ideal of R.*

Proof . Assume that μ is an anti fuzzy generalized bi-ideal of R and $x, y \in R$, then there exist elements $a, b \in R$ such that $x = (ax)(bx)$. We have to show that μ is an anti fuzzy LA-subring of R. Thus

$$\begin{aligned} \mu(xy) &= \mu(((ax)(bx))y) = \mu(((ab)(xx))y) \\ &= \mu((x((ab)x))y) \leq max\{\mu(x), \mu(y)\}. \end{aligned}$$

So μ is an anti fuzzy LA-subring of R. □

Remark 3.30. *It is easy to see that, if R is a left (resp. right) weakly regular locally associative LA-ring. Then for every anti fuzzy left (resp. right) ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Theorem 3.31. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*
(1) *R is a left weakly regular.*
(2) *$\mu \cup \nu = \mu \circ \nu$ for every anti fuzzy right ideal μ and for every anti fuzzy left ideal ν of R.*

Proof . Suppose that (1) holds. Since $\mu \circ \nu \supseteq \mu \cup \nu$ for every anti fuzzy right ideal μ and every anti fuzzy left ideal ν of R by the Lemma 3.7. Let $x \in R$, this implies that there exist $a, b \in R$ such that $x = (ax)(bx) = (ab)(xx) = x((ab)x)$. Now

$$\begin{aligned} (\mu \circ \nu)(x) &= \wedge_{x=\sum_{i=1}^{n} a_i b_i} \{\vee_{i=1}^{n} \{\mu(a_i) \vee \nu(b_i)\}\} \\ &\leq \mu(x) \vee \nu((ab)x) \leq \mu(x) \vee \nu(x) = (\mu \cup \nu)(x). \\ &\Rightarrow \mu \cup \nu \supseteq \mu \circ \nu. \end{aligned}$$

Hence $\mu \cup \nu = \mu \circ \nu$, i.e., $(1) \Rightarrow (2)$. Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 3.4 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.6. So χ^C_{Ra} is an anti fuzzy left ideal and $\chi^C_{aR\cup Ra}$ is an anti fuzzy right ideal of R, by the Lemma 2.8. Then by our assumption $\chi^C_{aR\cup Ra} \cap \chi^C_{Ra} = \chi^C_{aR\cup Ra} \circ \chi^C_{Ra}$, i.e., $\chi^C_{(aR\cup Ra)\cap Ra} = \chi^C_{(aR\cup Ra)Ra}$ by the Theorem 2.1. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. This implies that $a \in (aR)(Ra)$ or $a \in (Ra)(Ra)$. If $a \in (Ra)(Ra)$, then R is a left weakly regular. If $a \in (aR)(Ra)$, then

$$\begin{aligned} (aR)(Ra) &= ((ea)(RR))(Ra) = ((RR)(ae))(Ra) \\ &= (((ae)R)R)(Ra) = ((aR)R)(Ra) \\ &= ((RR)a)(Ra) = (Ra)(Ra). \end{aligned}$$

Hence R is a left weakly regular, i.e., $(2) \Rightarrow (1)$. □

Theorem 3.32. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a left weakly regular.*

(2) *$\mu \cup \gamma \supseteq \mu \circ \gamma$ for every anti fuzzy bi-ideal μ and for every anti fuzzy ideal γ of R.*

(3) *$\nu \cup \gamma \supseteq \nu \circ \gamma$ for every anti fuzzy generalized bi-ideal ν and for every anti fuzzy ideal γ of R.*

Proof . Assume that (1) holds. Let ν be an anti fuzzy generalized bi-ideal and γ be an anti fuzzy ideal of R. Let $x \in R$, this means that there exist $a, b \in R$ such that $x = (ax)(bx) = (ab)(xx) = x((ab)x)$. Thus

$$\begin{aligned}
(\mu \circ \gamma)(x) &= \wedge_{x=\sum_{i=1}^{n} a_i b_i}\{\vee_{i=1}^{n}\{\mu(a_i) \vee \gamma(b_i)\}\} \\
&\leq \mu(x) \vee \gamma((ab)x) \leq \mu(x) \vee \gamma(x) \\
&= (\mu \vee \gamma)(x) = (\mu \cup \gamma)(x). \\
&\Rightarrow \nu \cup \gamma \supseteq \nu \circ \gamma.
\end{aligned}$$

Therefore $(1) \Rightarrow (3)$. It is clear that $(3) \Rightarrow (2)$. Suppose that (2) holds. Then $\mu \cup \gamma \supseteq \mu \circ \gamma$, where μ is an anti fuzzy right ideal of R. Since $\mu \circ \gamma \supseteq \mu \cup \gamma$, so $\mu \circ \gamma = \mu \cup \gamma$. Therefore R is a left weakly regular by the Theorem 3.31, i.e., $(2) \Rightarrow (1)$. $\square$

Theorem 3.33. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a left weakly regular.*

(2) *$\mu \cup \gamma \cup \delta \supseteq (\mu \circ \gamma) \circ \delta$ for every anti fuzzy bi-ideal μ, for every anti fuzzy ideal γ and for every anti fuzzy right ideal δ of R.*

(3) *$\nu \cup \gamma \cup \delta \supseteq (\nu \circ \gamma) \circ \delta$ for every anti fuzzy generalized bi-ideal ν, for every anti fuzzy ideal γ and for every anti fuzzy right ideal δ of R.*

Proof . Suppose that (1) holds. Let ν be an anti fuzzy generalized bi-ideal, γ be an anti fuzzy ideal and δ be an anti fuzzy right ideal of R. Let $x \in R$, then there exist elements $a, b \in R$ such that $x = (ax)(bx)$. Now

$$\begin{aligned}
x &= (ax)(bx) = (xb)(xa) \\
xb &= ((ax)(bx))b = ((xx)(ba))b \\
&= (b(ba))(xx) = c(xx) = x(cx) \quad \text{say } c = b(ba)
\end{aligned}$$

Thus

$$\begin{aligned}
((\nu \circ \gamma) \circ \delta)(x) &= \wedge_{x=\sum_{i=1}^{n} a_i b_i}\{\vee_{i=1}^{n}\{(\nu \circ \gamma)(a_i) \vee \delta(b_i)\}\} \\
&\leq (\nu \circ \gamma)(xb) \vee \delta(xa) \\
&\leq (\nu \circ \gamma)(xb) \vee \delta(x) \\
&= \wedge_{xb=\sum_{i=1}^{n} p_i q_i}\{\vee_{i=1}^{n}\{\nu(p_i) \vee \gamma(q_i)\}\} \vee \delta(x) \\
&\leq \nu(x) \vee \gamma(cx) \vee \delta(x) \\
&\leq \nu(x) \vee \gamma(x) \vee \delta(x) = (\mu \cup \gamma \cup \delta)(x). \\
&\Rightarrow \mu \cup \gamma \cup \delta \supseteq (\nu \circ \gamma) \circ \delta.
\end{aligned}$$

Hence $(1) \Rightarrow (3)$. Since $(3) \Rightarrow (2)$, every anti fuzzy bi-ideal of R is an anti fuzzy generalized bi-ideal of R. Assume that (2) is true. Then $\mu \cup \gamma \cup R \supseteq (\mu \circ \gamma) \circ R$, where μ is an anti fuzzy right ideal of R, i.e., $\mu \cup \gamma \supseteq \mu \circ \gamma$. Since $\mu \circ \gamma \supseteq \mu \cup \gamma$, so $\mu \circ \gamma = \mu \cup \gamma$. Hence R is a left weakly regular by the Theorem 3.31, i.e., $(2) \Rightarrow (1)$. $\square$

Lemma 3.34. *Let R be an intra-regular LA-ring. Then every anti fuzzy left (resp. right) ideal of R is an anti fuzzy ideal of R.*

Proof . Suppose that μ is an anti fuzzy right ideal of R. Let $x, y \in R$, this implies that there exist $a_i, b_i \in R$, such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Thus $\mu(xy) = \mu(((a_i x^2)b_i)y) = \mu((yb_i)(a_i x^2)) \leq \mu(yb_i) \leq \mu(y)$. Hence μ is an anti fuzzy ideal of R. $\square$

Proposition 3.35. *Every anti fuzzy generalized bi-ideal of an intra-regular LA-ring R with left identity e, is an anti fuzzy bi-ideal of R.*

Proof . Let μ be an anti fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. We have to show that μ is an anti fuzzy LA-subring of R. Now

$$\begin{aligned} x &= (a_i x^2)b_i = (a_i x^2)(eb_i) = (a_i e)(x^2 b_i) \\ &= (a_i e)((xx)b_i) = (a_i e)((b_i x)x) = (x(b_i x))(ea_i) \\ &= (x(b_i x))a_i = (a_i(b_i x))x = (a_i(b_i x))(ex) \\ &= (xe)((b_i x)a_i) = (b_i x)((xe)a_i) = (b_i x)((a_i e)x) \\ &= (x(a_i e))(xb_i) = x((x(a_i e))b_i) = xn, \quad \text{say } n = (x(a_i e))b_i \end{aligned}$$

Thus $\mu(xy) = \mu((xn)y) \leq max\{\mu(x), \mu(y)\}$. Hence μ is an anti fuzzy LA-subring of R. $\square$

Theorem 3.36. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*
(1) *R is an intra-regular.*
(2) *$\mu \cap \nu \subseteq \mu \circ \nu$ for every anti fuzzy right ideal ν and for every anti fuzzy left ideal μ of R.*

Proof . Assume that (1) holds. Let $x \in R$, then there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now

$$\begin{aligned} x &= (a_i x^2)b_i = (a_i(xx))b_i = (x(a_i x))(eb_i) \\ &= (xe)((a_i x)b_i) = (a_i x)((xe)b_i). \end{aligned}$$

Thus

$$\begin{aligned} (\mu \circ \nu)(x) &= \wedge_{x=\sum_{i=1}^{n} a_i b_i}\{\vee_{i=1}^{n}\mu(a_i) \vee \nu(b_i)\} \\ &\leq max\{\mu(a_i x), \nu((xe)b_i)\} \\ &\leq max\{\mu(x), \nu(x)\} = (\mu \vee \nu)(x). \\ &\Rightarrow \mu \cup \nu \supseteq \mu \circ \nu. \end{aligned}$$

Hence (1) $\Rightarrow$ (2). Suppose that (2) is true and $a \in R$, then Ra is a left ideal of R containing a by the Lemma 3.4 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 3.6. This means that χ_{Ra}^{C} is an anti fuzzy left ideal and $\chi_{aR\cup Ra}^{C}$ is an anti fuzzy right ideal of R, by the Lemma 2.8. By our supposition $\chi_{aR\cup Ra}^{C} \cap \chi_{Ra}^{C} \subseteq \chi_{Ra}^{C} \circ \chi_{aR\cup Ra}^{C}$, i.e., $\chi_{(aR\cup Ra)\cap Ra}^{C} \subseteq \chi_{(Ra)(aR\cup Ra)}^{C}$. Thus $(aR\cup Ra)\cap Ra \subseteq Ra(aR\cup Ra)$. Since $a \in (aR\cup Ra)\cap Ra$, i.e., $a \in Ra(aR\cup Ra) = (Ra)(aR)\cup(Ra)(Ra)$. Now

$$\begin{aligned} (Ra)(aR) &= (Ra)((ea)(RR)) = (Ra)((RR)(ae)) \\ &= (Ra)(((ae)R)R) = (Ra)((aR)R) \\ &= (Ra)((RR)a) = (Ra)(Ra). \end{aligned}$$

This implies that

$$\begin{aligned}(Ra)(aR) \cup (Ra)(Ra) &= (Ra)(Ra) \cup (Ra)(Ra) \\ &= (Ra)(Ra) = ((Ra)a)R \\ &= ((Ra)(ea))R = ((Re)(aa))R \\ &= (Ra^2)R.\end{aligned}$$

Thus $a \in (Ra^2)R$, i.e., a is an intra regular. Therefore R is an intra-regular, i.e., $(2) \Rightarrow (1)$. □

Theorem 3.37. *Let R be an intra-regular locally associative LA-ring. Then for every anti fuzzy right ideal μ of R, $\mu(a^n) = \mu(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof . For $n = 1$. Let $a \in R$, this implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Thus

$$\begin{aligned}\mu(a) &= \mu((x_i a^2)y_i) \leq \mu(x_i a^2) \leq \mu(a^2) \\ &= \mu(aa) \leq max\{\mu(a), \mu(a)\} = \mu(a). \\ &\Rightarrow \mu(a) = \mu(a^2).\end{aligned}$$

Now $a^2 = aa = ((x_i a^2)y_i)((x_i a^2)y_i) = (x_i^2 a^4)y_i^2$, then the result is true for $n = 2$. Assume that the result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^k a = ((x_i^k a^{2k})y_i^k)((x_i a^2)y_i) = (x_i^{k+1}a^{2(k+1)})y_i^{k+1}$. Thus

$$\begin{aligned}\mu(a^{k+1}) &= \mu((x_i^{k+1}a^{2(k+1)})y_i^{k+1}) \leq \mu(x_i^{k+1}a^{2(k+1)}) \\ &\leq \mu(a^{2(k+1)}) = \mu(a^{k+1}a^{k+1}) \\ &\leq max\{\mu\left(a^{(k+1)}\right), \mu\left(a^{(k+1)}\right)\} = \mu\left(a^{(k+1)}\right). \\ &\Rightarrow \mu(a^{k+1}) = \mu(a^{2(k+1)}).\end{aligned}$$

Hence by induction method, the result is true for all positive integers. □

Proposition 3.38. *Let R be an intra-regular locally associative LA-ring with left identity e. Then for every anti fuzzy right ideal μ of R, $\mu(ab) = \mu(ba)$ for all $a, b \in R$.*

Proof . Same as Lemma 3.18. □

Conclusion 1. *Our purpose is to encourage the study and maturity of non associative algebraic structure (LA-ring) The objective is to explain original methodological developments on LA-rings, which will be very helpful for upcoming theory of algebraic structure. The ideal of fuzzy set is to characterize of LA-ring are captivating a great attention of algebraist. The aim of this paper is to investigate, the study of LA-rings by using the anti fuzzy left(resp. right, bi-, generalized bi-, $(1, 2)$-ideals.*

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