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# Fixed Point Results in Orthogonal Modular Metric Spaces

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**Abstract**

First we generalize the notion of $O$-sets and then we establish some fixed point theorems for Banach's contraction and Suzuki type $\Theta$-contraction in the setting of orthogonal modular metric spaces. The obtained results extend, generalize and improve many fixed point results given by some authors in the literature. Some examples are furnished to demonstrate the validity of these results.

*Keywords:* Banach's contraction, Suzuki type $\Theta$-contraction, orthogonal modular metric space

## 1. Introduction and Preliminaries

In order to generalize the well known Banach contraction principle, many authors introduced various types of contraction inequalities in various type of metric spaces (see [9, 14, 15, 18] and references therein). On the other hand modular metric spaces are a natural and interesting generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and others. The concept of Modular metric spaces were introduced in [3, 4]. Here, we look at Modular metric space as the nonlinear version of the classical one introduced by Nakano [16] on vector space and modular function space introduced by Musielak [13] and Orlicz [17]. Also to study more you can see [5–8, 11, 12]

First we generalize the notion of $O$-sets and then we establish some fixed point theorems for Banach's contraction and Suzuki type $\Theta$-contraction in the setting of orthogonal modular metric spaces. The obtained results extend, generalize and improve many fixed point results given by some authors in the literature. Some examples are furnished to demonstrate the validity of these results.

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Let $X$ be a nonempty set and $\omega : (0,+\infty) \times X \times X \to [0,+\infty]$ be a function, for semplicity, we will write

$$\omega_\lambda(x,y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

**Definition 1.1 ([3, 4]).** *A function $\omega : (0,+\infty) \times X \times X \to [0,+\infty]$ is called a modular metric on $X$ if the following axioms hold:*

*(i) $x = y$ if and only if $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$;*

*(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$ and $x, y \in X$;*

*(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.*

If in the above definition, we utilize the condition

(i') $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X$;

instead of (i) then $\omega$ is said to be a pseudomodular metric on $X$. A modular metric $\omega$ on $X$ is called regular if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x,y) = 0 \quad \text{for some} \quad \lambda > 0.$$

Again, $\omega$ is called convex if for $\lambda, \mu > 0$ and $x, y, z \in X$ holds the inequality

$$\omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu}\,\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\,\omega_\mu(z,y).$$

**Remark 1.2.** *Note that if $\omega$ is a pseudomodular metric on a set $X$, then the function $\lambda \to \omega_\lambda(x,y)$ is decreasing on $(0,+\infty)$ for all $x, y \in X$. That is, if $0 < \mu < \lambda$, then*

$$\omega_\lambda(x,y) \leq \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y).$$

**Definition 1.3 ([3, 4]).** *Suppose that $\omega$ be a pseudomodular on $X$ and $x_0 \in X$ and fixed. So the two sets*

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\}$$

*and*

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \quad \text{such that} \quad \omega_\lambda(x,x_0) < +\infty\}.$$

*$X_\omega$ and $X^*_\omega$ are called modular spaces (around $x_0$).*

It is evident that $X_\omega \subset X^*_\omega$ but this inclusion may be proper in general. Assume that $\omega$ be a modular on $X$, from [3, 4], we derive that the modular space $X_\omega$ can be equipped with a (nontrivial) metric, induced by $\omega$ and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq \lambda\} \quad \text{for all} \quad x, y \in X_\omega.$$

Note that if $\omega$ is a convex modular on $X$, then according to [3, 4] the two modular spaces coincide, i.e., $X^*_\omega = X_\omega$, and this common set can be endowed with the metric $d^*_\omega$ given by

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq 1\} \quad \text{for all} \quad x, y \in X_\omega.$$

Such distances called Luxemburg distances.

Example 2.1 presented by Abdou and Khamsi [1] is an important motivation for developing the modular metric spaces theory. Other examples may be found in [3, 4].

**Definition 1.4 ([14]).** *Assume $X_\omega$ be a modular metric space, $M$ a subset of $X_\omega$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X_\omega$. Therefore,*

*(1) $(x_n)_{n\in\mathbb{N}}$ is called convergent to $x \in X_\omega$ if and only if $\omega_\lambda(x_n, x) \to 0$, as $n \to +\infty$ for all $\lambda > 0$. $x$ will be called the limit of $(x_n)$.*

*(2) $(x_n)_{n\in N}$ is called Cauchy if $\omega_\lambda(x_m, x_n) \to 0$, as $m, n \to +\infty$ for all $\lambda > 0$.*

*(3) $M$ is called closed if the limit of a convergent sequence of $M$ always belong to $M$.*

*(4) $M$ is called complete if any Cauchy sequence in $M$ is convergent to a point of $M$.*

*(5) $M$ is called $\omega$-bounded if for all $\lambda > 0$ we have $\delta_\omega(M) = \sup\{\omega_\lambda(x, y); x, y \in M\} < +\infty$.*

Recently, Moreno et al. [15] introduced the notion Banach's contraction principle for nonlinear contraction mappings in the setting of modular metric spaces as follow.

**Definition 1.5.** *Let $X_\omega$ be a modular metric space. A map $T : X_\omega \to X_\omega$ is a contraction if there exits $k \in (0, 1)$ such that for every $\lambda > 0$ and $x, y \in X_\omega$ we have,*

$$\omega_{\lambda k}(Tx, Ty) \leq \omega_\lambda(x, y) \tag{1.1}$$

Eshaghi et al. [9] introduced the notion of orthogonal set and gave a real generalization of Banach' fixed point theorem in orthogonal metric spaces (Also see [2]). We extend and simplify the definitions of [9] to modular metric spaces by the following methods. Throughout this paper, we utilize the following version of $O$-set.

**Definition 1.6.** *Let $X \neq \varnothing$ and $\perp \in X \times X$ be an binary relation. Assume that there exists $x_0 \in X$ such that $x_0 \perp x$ for all $x \in X$. Then we say that $X$ is an orthogonal set(briefly $O$-set). We denote orthogonal set by $(X, \mathcal{A}, \perp)$. Also, suppose that $(X, \perp)$ be an $O$-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called $O$-sequence if $x_n \perp x_{n+1}$ for all $n \in \mathbb{N}$.*

**Definition 1.7.** *We say a modular metric space $X_\omega$ is an orthogonal modular metric space if $(X_\omega, \perp)$ is an $O$-set (briefly, $O$-modular metric spaces). Also, $T : X_\omega \to X_\omega$ is $\perp -$continuous in $x \in X_\omega$ if for each $O$-sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X_\omega$ if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$, then, $\lim_{n\to\infty} \omega_\lambda(Tx_n, Tx) = 0$ for all $\lambda > 0$. Furthermore, $T$ is $\perp$-continuous if $T$ is $\perp$-continuous in each $x \in X_\omega$. Also, we say $T$ is $\perp$-preserving if $Tx \perp Ty$ whence $x \perp y$. Finally, $X_\omega$ is $\omega$-orthogonally complete (in brief $\omega$-$O$-complete) if every $\omega$-Cauchy $O$-sequence is convergent.*

Now we generalize the notions of $O-$sets and $O$-modular metric spaces by the following ways.

**Definition 1.8.** *Let $X \neq \varnothing$, $\perp \in X \times X$ be an binary relation and $\varnothing \neq \mathcal{A} \subseteq X$. Assume that there exists $x_0 \in X$ such that $x_0 \perp x$ for all $x \in \mathcal{A}$. Then we say that $X$ is an orthogonal set with respect to $\mathcal{A}$ (briefly $\mathcal{A} - O$-set). We denote orthogonal set with respect to $\mathcal{A}$ by $(X, \mathcal{A}, \perp)$. Also, suppose that $(X, \mathcal{A}, \perp)$ be an $\mathcal{A} - O$-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called $\mathcal{A} - O$-sequence if $x_n \perp x_{n+1}$ for all $n \in \mathbb{N}$.*

**Definition 1.9.** *We say a modular metric space $X_\omega$ is an orthogonal modular metric space withe respect to $\mathcal{A}$ if $(X_\omega, \mathcal{A}, \perp)$ is an $\mathcal{A}-O$-set (briefly, $\mathcal{A}-O$-modular metric spaces). Also, $T : X_\omega \to X_\omega$ is $\mathcal{A}- \perp -$continuous in $x \in X_\omega$ if for each $\mathcal{A} - O$-sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X_\omega$ if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$, then, $\lim_{n\to\infty} \omega_\lambda(Tx_n, Tx) = 0$ for all $\lambda > 0$. Furthermore, $T$ is $\mathcal{A}- \perp$-continuous if $T$ is $\mathcal{A}- \perp$-continuous in each $x \in X_\omega$. Also, we say $T$ is $\mathcal{A}- \perp$-preserving if $Tx \perp Ty$ whence $x \perp y$. Finally, $X_\omega$ is $\omega - \mathcal{A}$-orthogonally complete (in brief $\omega$-$\mathcal{A} - O$-complete) if every $\omega$-Cauchy $\mathcal{A} - O$-sequence is convergent.*

Note that every $O$-set $(X, \perp)$ is an orthogonal set with respect to $\mathcal{A} = X$, but the converse is not true. The following example shows this fact.

**Example 1.10.** *Let*

$$X = \{(0,5),(1,2),(2,3),(3,4),(1,1),(2,2),(3,3),(4,4)\}$$

*and*

$$\mathcal{A} = \{(2,3),(3,4),(2,2),(3,3),(4,4)\}.$$

*For $(x,y),(u,v) \in X$, assume, $(x,y) \perp (u,v)$ if $x \leq u$ and $y \leq v$ if and only if $(x,y) \leq (u,v)$. Then by putting $x_0 = (1,2)$, $X$ is an orthogonal set with respect to $\mathcal{A}$. That is, $(1,2) \perp (x,y)$ for all $(x,y) \in \mathcal{A}$. But, $(0,5) \nleq (x,y)$ and $(x,y) \nleq (0,5)$ for all $(x,y) \in X$. That is, $(X, \perp)$ is not a $O$-set.*

**Remark 1.11.** *Let $X$ be a nonempty set. Assume, $x \perp y$ if $(x,y) \in X \times X$. Then clearly $(X, \perp)$ is an $O$-set (or $O$-set with respect to $\mathcal{A} = X$). This example say us every set is an $O$-set (by using this $\perp$). Also assume $T : X \to X$ be a given mapping. If $x \perp y$, then $(x,y) \in X \times X$. Hence, $(Tx, Ty) \in X \times X$. i.e., $Tx \perp Ty$. This say us every mapping is an $\perp$-preserving mapping. Similarly, every sequence is an $O$-sequence.*

## 2. Banach's Contraction Principle in Orthogonal Modular Metric Spaces

Motivated by works of Eshaghi et al. [9] and Moreno et al. [15] we introduce the notion Banach's contraction principle for nonlinear contraction mappings in the setting of orthogonal modular metric spaces as follows.

**Definition 2.1.** *Let $X_\omega$ be an $\mathcal{A}-O$-modular metric space. A map $T : X_\omega \to X_\omega$ is an $\mathcal{A}- \perp -\omega$-contraction if there exits $k \in (0,1)$ such that for every $\lambda > 0$ and $x, y \in X_\omega$ with $x \perp y$ we have,*

$$\omega_{\lambda k}(Tx, Ty) \leq \omega_\lambda(x,y). \tag{2.1}$$

The following example shows that contraction is $\mathcal{A}- \perp -\omega$-contraction but the converse is not true.

**Example 2.2.** *Let $X = \mathbb{N} \cup \{0\}$ endowed with the modular metric $\omega_\lambda : X \times X \times [0, +\infty)$ given by*

$$\omega_\lambda(m,n) = \frac{1}{\lambda}|m-n|.$$

*Define $T : X_\omega \to X_\omega$ by*

$$T(n) = \begin{cases} \frac{1}{2}, & n = 1 \\ 1, & n = 2 \\ 0 & n \in \{0,3,4,...\}, \end{cases}$$

*also define $m \perp n$ by $mn \leq \max\{m,n\}$ (with $\mathcal{A} = X$). Then the following cases are hold.*

- *if $m = 0$ and $n = 1$ ($m \perp n$ and $n \perp m$). Then, $Tm = 0$ and $Tn = \frac{1}{2}$, and so,*

$$\omega_{0.6\lambda}(Tm, Tn) = \frac{1}{0.6\lambda} \cdot \frac{1}{2} = \frac{1}{1.2\lambda} \leq \frac{1}{\lambda} = \omega_\lambda(m,n)$$

- *if $m = 0$ and $n = 2$ ($m \perp n$ and $n \perp m$). Then, $Tm = 0$ and $Tn = 1$, and so,*

$$\omega_{0.6\lambda}(Tm, Tn) = \frac{1}{0.6\lambda} \leq \frac{2}{\lambda} = \omega_\lambda(m,n)$$

- *if* $m = 0$ *and* $n \in \{0, 3, 4, ...\}$ *($m \perp n$ and $n \perp m$). Then,* $Tm = Tn = 0$*, and so,*

$$\omega_{0.6\lambda}(Tm, Tn) = 0 \leq \omega_\lambda(m, n)$$

- *if* $m = 1$ *and* $n = 2$ *($m \perp n$ and $n \perp m$). Then,* $Tm = \frac{1}{2}$ *and* $Tn = 1$*, and so,*

$$\omega_{0.6\lambda}(Tm, Tn) = \frac{1}{1.2\lambda} \leq \frac{1}{\lambda} = \omega_\lambda(m, n)$$

- *if* $m = 1$ *and* $n \in \{0, 3, 4, ...\}$ *($m \perp n$ and $n \perp m$). Then,* $Tm = \frac{1}{2}$ *and* $Tn = 0$*, and so,*

$$\omega_{0.6\lambda}(Tm, Tn) = \frac{1}{1.2\lambda} \leq \frac{1}{\lambda} \leq \omega_\lambda(m, n)$$

*and hence,*

$$\omega_{0.6\lambda}(Tm, Tn) \leq \omega_\lambda(m, n)$$

*holds for all* $m, n \in X_\omega$ *and* $\lambda > 0$*. That is,* $T$ *is* $\mathcal{A}- \perp -\omega-$*contraction. But,*

$$\omega_{k\lambda}(T2, T3) = \frac{1}{k\lambda} > \frac{1}{\lambda} = \omega_\lambda(2, 3)$$

*for all* $k \in (0, 1)$*. That is,* $T$ *is not Banach's contraction.*

Now we have the following results.

**Theorem 2.3.** *Let* $(X_\omega, \mathcal{A}, \perp)$ *be an* $\omega$*-*$\mathcal{A} - O$*-complete modular metric space. Let* $T : X_\omega \to X_\omega$ *be* $\mathcal{A}- \perp -$*continuous,* $\mathcal{A}- \perp -\omega-$*contraction(with Lipschitz constant* $k \in (0, 1)$*),* $\perp -$*preserving and* $TX_\omega \subseteq \mathcal{A}$*. Then* $T$ *has a unique fixed point* $x^* \in X_\omega$*. Furthermore,* $\lim_{n\to\infty} T^n x = x^*$ *for all* $x \in X_\omega$ *(i.e.,* $T$ *is a Picard operator).*

**Proof .** The proof is straightforward and is omitted here. See [3, 9]. $\square$

If in Theorem 2.3 we take $\mathcal{A} = X_\omega$ then we have the following Corollary in the setting of $O-$modular metric spaces.

**Corollary 2.4.** *Let* $(X_\omega, \perp)$ *be an* $\omega$*-*$O$*-complete modular metric space. Let* $T : X_\omega \to X_\omega$ *be* $\perp -$*continuous and* $\perp -$*preserving. If there exits* $k \in (0, 1)$ *such that for every* $\lambda > 0$ *and* $x, y \in X_\omega$ *with* $x \perp y$ *we have,*

$$\omega_{\lambda k}(Tx, Ty) \leq \omega_\lambda(x, y),$$

*then* $T$ *has a unique fixed point* $x^* \in X_\omega$*. Furthermore,* $\lim_{n\to\infty} T^n x = x^*$ *for all* $x \in X_\omega$ *(i.e.,* $T$ *is a Picard operator).*

For $\mathcal{A}- \perp -\omega-$contraction that is not $\mathcal{A}- \perp -$continuous we have the following theorem.

**Theorem 2.5.** *Let* $(X_\omega, \mathcal{A}, \perp)$ *be an* $\omega$*-*$\mathcal{A} - O$*-complete modular metric space. Let* $T : X_\omega \to X_\omega$ *be* $\mathcal{A}- \perp -\omega-$*contraction(with Lipschitz constant* $k \in (0, 1)$*),* $\mathcal{A}- \perp -$*preserving and* $TX_\omega \subseteq \mathcal{A}$*. Also, if* $\{x_n\}_{n\in\mathbb{N}}$ *be a* $\mathcal{A} - O$*-sequence with* $x_n \to x \in X_\omega$*, then* $x \perp x_n$ *for all* $n \in \mathbb{N}$*. Therefore,* $T$ *has a unique fixed point* $x^* \in X_\omega$*. Furthermore,* $\lim_{n\to\infty} T^n x = x^*$ *for all* $x \in X_\omega$ *(i.e.,* $T$ *is a Picard operator).*

**Proof .** The proof is straightforward and is omitted here. See [3, 9]. □

If in Theorem 2.5 we take $\mathcal{A} = X_\omega$ then we have the following Corollary in the setting of $O-$modular metric spaces.

**Corollary 2.6.** *Let $(X_\omega, \perp)$ be an $\omega$-$O$-complete modular metric space. Let $T : X_\omega \to X_\omega$ be a $\perp-$preserving mapping. If $\{x_n\}_{n\in\mathbb{N}}$ be a $O$-sequence with $x_n \to x \in X_\omega$, then $x \perp x_n$ for all $n \in \mathbb{N}$. Now, if there exits $k \in (0, 1)$ such that for every $\lambda > 0$ and $x, y \in X_\omega$ with $x \perp y$ we have,*

$$\omega_{\lambda k}(Tx, Ty) \leq \omega_\lambda(x, y),$$

*therefore, $T$ has a unique fixed point $x^* \in X_\omega$. Furthermore, $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X_\omega$ (i.e., $T$ is a Picard operator).*

If in Corollary 2.6 we take $x \perp y$, if $(x, y) \in X \times X$ and use Remark 1.11, then we have the following Corollary in the setting of modular metric spaces.

**Corollary 2.7.** *Let $X_\omega$ be an $\omega$-complete modular metric space. Let $T : X_\omega \to X_\omega$ be a mapping. Now, if there exits $k \in (0, 1)$ such that for every $\lambda > 0$ and $x, y \in X_\omega$ we have,*

$$\omega_{\lambda k}(Tx, Ty) \leq \omega_\lambda(x, y),$$

*therefore, $T$ has a unique fixed point $x^* \in X_\omega$. Furthermore, $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X_\omega$ (i.e., $T$ is a Picard operator).*

## 3. Suzuki Type Fixed Point Results

In this section we establish some Suzuki type results for $\Theta-$contraction in the setting of orthogonal modular metric spaces.

Consistent with Jleli and Samet [10], we denote by $\Delta_\Theta$ the set of all functions $\Theta : (0, +\infty) \to (1, +\infty)$ satisfying the following conditions:

($\Theta_1$) $\Theta$ is increasing;

($\Theta_2$) for all sequence $\{\alpha_n\} \subseteq (0, +\infty)$, $\lim\limits_{n\to+\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to+\infty} \Theta(\alpha_n) = 1$;

($\Theta_3$) there exist $0 < r < 1$ and $\ell \in (0, +\infty]$ such that $\lim\limits_{t\to 0^+} \frac{\Theta(t)-1}{t^r} = \ell$.

For Suzuki type $\Theta-$contraction mapping that is $\perp-$continuous we have the following theorem.

**Theorem 3.1.** *Let $(X_\omega, \mathcal{A}, \perp)$ be an $\omega$-$\mathcal{A} - O$-complete modular metric space with $\omega$ regular and let $T : X_\omega \to X_\omega$ be a $\mathcal{A}-\perp-$continuous, $\mathcal{A}-\perp-$preserving self-mapping and $TX_\omega \subseteq \mathcal{A}$. Assume that there exist a real number $r \in [0, 1)$ and a function $\Theta \in \Delta_\Theta$ such that for all $\lambda > 0$ and $x, y \in X_\omega$ with $x \perp y$, $\omega_\lambda(x, Tx) \leq \omega_\lambda(x, y)$ and $\omega_\lambda(Tx, Ty) > 0$, we have*

$$\Theta\big(\omega_\lambda(Tx, Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x, y)\big)\Big]^r. \tag{3.1}$$

*Then $T$ has a fixed point.*

**Proof .** Since, $(X_\omega, \mathcal{A}, \perp)$ is an $\mathcal{A}-O-$set, then there exists $x_0 \in X_\omega$ such that,

$$x_0 \perp y \text{ for all } y \in \mathcal{A}. \tag{3.2}$$

Since $TX_\omega \subseteq \mathcal{A}$, then $Tx_0 \in \mathcal{A}$. This implies, $x_0 \perp Tx_0$. Assume,

$$x_1 := Tx_0,\ x_2 = T^2x_1,\ \ldots,\ x_n = T^n x_0 = Tx_{n-1}$$

for all $n \in \mathbb{N}$. Since $T$ is $\mathcal{A}- \perp -$preserving, then $\{x_n\}_{n\geq 0}$ is an $\mathcal{A}-O$-sequence. If there exists $n_0 \in \mathbb{N}\cup\{0\}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then $x_{n_0}$ is a fixed point of $T$ and we have nothing to prove. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. Now, assume that there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(Tx_{n_0-1}, Tx_{n_0}) = 0$ for some $\lambda > 0$. Then, $\omega$ regularity implies, $x_{n_0} = Tx_{n_0-1} = Tx_{n_0} = x_{n_0+1}$, which is a contradiction. Hence, $\omega_\lambda(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$ and $\lambda > 0$. Also, evidently,

$$\frac{1}{2}\omega_\lambda(x_{n-1}, Tx_{n-1}) \leq \omega_\lambda(x_{n-1}, x_n) \leq \omega_{\frac{\lambda}{2}}(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$ and $\lambda > 0$. So from (3.1) we can derive

$$\Theta\big(\omega_\lambda(Tx_{n-1}, Tx_n)\big) \leq \Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)^k$$

which implies that

$$\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) \leq \Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)^k. \tag{3.3}$$

Therefore,

$$\begin{aligned} 1 < \Theta\big(\omega_\lambda(x_n, x_{n+1})\big) &\leq \Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)^k \\ &\leq \Theta\big(\omega_\lambda(x_{n-2}, x_{n-1})\big)^{k^2} \leq \cdots \leq \Theta(\omega_\lambda(x_0, x_1))^{k^n}. \end{aligned} \tag{3.4}$$

Taking the limit as $n \to +\infty$ in (3.4), we get

$$\lim_{n\to+\infty} \Theta\big(\omega_\lambda(x_n, x_{n+1})\big) = 1$$

and since $\Theta \in \Delta_\Theta$, we obtain

$$\lim_{n\to+\infty} \omega_\lambda(x_n, x_{n+1}) = 0. \tag{3.5}$$

Thus there exist $0 < r < 1$ and $0 < \ell \leq +\infty$ such that

$$\lim_{n\to+\infty} \frac{\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) - 1}{[\omega_\lambda(x_n, x_{n+1})]^r} = \ell. \tag{3.6}$$

Now, let $B^{-1} \in (0, \ell)$. From the definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) - 1}{[\omega_\lambda(x_n, x_{n+1})]^r} \geq B^{-1} \quad \text{for all} \quad n \geq n_0$$

and so

$$n[\omega_\lambda(x_n, x_{n+1})]^r \leq nB[\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) - 1] \quad \text{for all} \quad n \geq n_0.$$

From Theorem 2.3, we deduce

$$n[\omega_\lambda(x_n, x_{n+1})]^r \leq nB[\Theta(\omega_\lambda(x_0, x_1))^{k^n} - 1] \quad \text{for all} \quad n \geq n_0.$$

Taking the limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n\to+\infty} n[\omega_\lambda(x_n, x_{n+1})]^r = 0. \tag{3.7}$$

From (3.7), it follows that there exists $N_0 \in \mathbb{N}$ such that

$$n[\omega_\lambda(x_n, x_{n+1})]^r \leq 1 \quad \text{for all} \quad n \geq N_0.$$

Thus

$$\omega_\lambda(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}} \quad \text{for all} \quad n \geq N_0. \tag{3.8}$$

Now, for $n \geq N_0$ and a arbitrary positive integer $p$, by (3.8), we get

$$\omega_\lambda(x_n, x_{n+p}) = \omega_{p\frac{\lambda}{p}}(x_n, x_{n+p}) \leq \sum_{i=n}^{n+p-1} \omega_{\frac{\lambda}{p}}(x_i, x_{i+1}) \leq \sum_{i=n}^{n+p-1} \frac{1}{i^{1/r}}.$$

Since $0 < r < 1$, then

$$\lim_{n\to+\infty} \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} = 0$$

and hence $\{x_n\}$ is a $\omega$-Cauchy $\mathcal{A} - O-$sequence. The hypothesis of $\omega$-$\mathcal{A} - O-$completeness of $X_\omega$ ensures that there exists $x^* \in X_\omega$ such that $\omega_\lambda(x_n, x^*) \to 0$ as $n \to +\infty$. Now, since $T$ is an $\mathcal{A}-\perp$-continuous mapping, then $\omega_\lambda(x_{n+1}, Tx^*) = \omega_\lambda(Tx_n, Tx^*) \to 0$ as $n \to +\infty$. From

$$\omega_\lambda(x^*, Tx^*) \leq \omega_{\frac{\lambda}{2}}(x^*, x_{n+1}) + \omega_{\frac{\lambda}{2}}(x_{n+1}, Tx^*),$$

taking limit as $n \to +\infty$, we get $\omega_2(x^*, Tx^*) = 0$ and hence $x^* = Tx^*$. □

**Corollary 3.2.** *Let $X_\omega$ be a complete modular metric space with $\omega$ regular and let $T : X_\omega \to X_\omega$ be a continuous and self-mapping. Assume that there exist a real number $r \in [0, 1)$ and a function $\Theta \in \Delta_\Theta$ such that for all $\lambda > 0$ and $x, y \in X_\omega$ with $\omega_\lambda(x, Tx) \leq \omega_\lambda(x, y)$ and $\omega_\lambda(Tx, Ty) > 0$, we have*

$$\Theta\big(\omega_\lambda(Tx, Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x, y)\big)\Big]^r.$$

*Then $T$ has a fixed point.*

For Suzuki type $\Theta-$contraction mapping that is not $\perp-$continuous we have the following theorem.

**Theorem 3.3.** *Let $(X_\omega, \mathcal{A}, \perp)$ be an $\omega$-$\mathcal{A} - O$-complete modular metric space with $\omega$ regular and let $T : X_\omega \to X_\omega$ be a $\mathcal{A}- \perp -$preserving self-mapping and $TX_\omega \subseteq \mathcal{A}$. Also, if $\{x_n\}_{n\in\mathbb{N}}$ be a $\mathcal{A} - O$-sequence with $x_n \to x \in X_\omega$, then $x \perp x_n$ for all $n \in \mathbb{N}$. Assume that there exist a real number $r \in [0, 1)$ and a function $\Theta \in \Delta_\Theta$ such that for all $\lambda > 0$ and $x, y \in X_\omega$ with $x \perp y$, $\frac{1}{2}\omega_\lambda(x, Tx) \leq \omega_{\frac{\lambda}{2}}(x, y)$ and $\omega_\lambda(Tx, Ty) > 0$, we have*

$$\Theta\big(\omega_\lambda(Tx, Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x, y)\big)\Big]^r. \tag{3.9}$$

*Then $T$ has a fixed point.*

**Proof .** Since, $(X_\omega, \mathcal{A}, \perp)$ is an $\mathcal{A}-O-$set, then there exists $x_0 \in X_\omega$ such that,

$$x_0 \perp y \text{ for all } y \in \mathcal{A}. \tag{3.10}$$

As in the proof of Theorem 2.3, we deduce that a Picard $\mathcal{A}-O$-sequence $\{x_n\}$ starting at $x_0$ is $\omega$-Cauchy and so converges to a point $x^* \in X_\omega$. Hence, $x^* \perp x_n$ for all $n \in \mathbb{N}$. Also from (3.3) we know that,

$$\Theta\big(\omega_\lambda(x_n, x_{n+1})\big) \le \Theta\big(\omega_\lambda(x_{n-1}, x_n)\big)^k \le \Theta\big(\omega_\lambda(x_{n-1}, x_n)\big).$$

This implies

$$\omega_\lambda(x_n, x_{n+1}) \le \omega_\lambda(x_{n-1}, x_n). \tag{3.11}$$

First assume that, for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{2}}(x_{k_n+1}, Tx^*) = 0$ and $k_n > k_{n-1}$ where $k_0 = 1$. Note that,

$$\omega_\lambda(x^*, Tx^*) \le \omega_{\frac{\lambda}{2}}(x^*, x_{k_n+1}) + \omega_{\frac{\lambda}{2}}(x_{k_n+1}, Tx^*)$$

and so we get, $\omega_1(x^*, Tx^*) = 0$. That is, $x^*$ is a fixed point of $T$. Next we assume, $\omega_\lambda(x_{n+1}, Tx^*) > 0$.

Suppose that for some $m \in \mathbb{N}$, we have

$$\frac{1}{2}\omega_\lambda(x_{m-1}, x_m) > \omega_{\frac{\lambda}{2}}(x_{m-1}, x^*) \quad \text{and} \quad \frac{1}{2}\omega_\lambda(x_m, x_{m+1}) > \omega_{\frac{\lambda}{2}}(x_m, x^*).$$

Therefore from (3.11) and the above inequalities we get,

$$\begin{aligned}
\omega_\lambda(x_{m-1}, x_m) &\le \omega_{\frac{\lambda}{2}}(x_{m-1}, x^*) + \omega_{\frac{\lambda}{2}}(x_m, x^*) \\
&< \frac{1}{2}\omega_\lambda(x_{m-1}, x_m) + \frac{1}{2}\omega_\lambda(x_m, x_{m+1}) \\
&\le \frac{1}{2}\omega_\lambda(x_{m-1}, x_m) + \frac{1}{2}\omega_\lambda(x_{m-1}, x_m) = \omega_\lambda(x_{m-1}, x_m),
\end{aligned}$$

which is a contradiction. Hence, either

$$\frac{1}{2}\omega_\lambda(x_{n-1}, x_n) \le \omega_{\frac{\lambda}{2}}(x_{n-1}, x^*) \quad \text{and} \quad \frac{1}{2}\omega_\lambda(x_n, x_{n+1}) \le \omega_{\frac{\lambda}{2}}(x_n, x^*)$$

holds for all $n \in \mathbb{N}$.

Let, $\frac{1}{2}\omega_\lambda(x_{n-1}, x_n) \le \omega_{\frac{\lambda}{2}}(x_{n-1}, x^*)$. Than from (3.9) we get,

$$\Theta\big(\omega_\lambda(Tx_n, Tx^*)\big) \le \Theta\big(\omega_\lambda(x_n, x^*)\big)^r \le \Theta\big(\omega_\lambda(x_n, x^*)\big) \tag{3.12}$$

which implies

$$\omega_\lambda(x_{n+1}, Tx^*) \le \omega_\lambda(x_n, x^*).$$

Then

$$\lim_{n\to+\infty} \omega_\lambda(x_{n+1}, Tx^*) = 0$$

and hence

$$\omega_\lambda(x^*, Tx^*) \le \lim_{n\to+\infty} [\omega_{\frac{\lambda}{2}}(x^*, x_{n+1}) + \omega_{\frac{\lambda}{2}}(x_{n+1}, Tx^*)] = 0.$$

Thus, we get $x^* = Tx^*$. Similarly $x^*$ is fixed point of $T$ whence $\frac{1}{2}\omega_\lambda(x_n, x_{n+1}) \le \omega_{\frac{\lambda}{2}}(x_n, x^*)$. Therefore, $T$ has a fixed point. $\square$

**Corollary 3.4.** *Let $X_\omega$ be a complete modular metric space with $\omega$ regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there exist a real number $r \in [0, 1)$ and a function $\Theta \in \Delta_\Theta$ such that for all $\lambda > 0$ and $x, y \in X_\omega$ with $\frac{1}{2}\omega_\lambda(x, Tx) \leq \omega_{\frac{\lambda}{2}}(x, y)$ and $\omega_\lambda(Tx, Ty) > 0$, we have*

$$\Theta\big(\omega_\lambda(Tx, Ty)\big) \leq \Big[\Theta\big(\omega_\lambda(x, y)\big)\Big]^r.$$

*Then $T$ has a fixed point.*

**Example 3.5.** Let $X = M \cup \{(-1, 6)\}$ where $M = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$. We define a binary relation $\perp$ by

$$(x, y) \perp (u, v) \Leftrightarrow (x, y) \preceq (u, v) \Leftrightarrow x \leq u \quad \text{and} \quad y \leq v.$$

Clearly, by putting $x_0 = (0, 0)$ and $\mathcal{A} = M$, $(X, \mathcal{A}, \perp)$ is an $\mathcal{A} - O$-set (but $X$ is not a $O$-set). And define modular metric $\omega$ on $X$ by

$$\omega_\lambda((x_1, x_2), (y_1, y_2)) = \frac{1}{\lambda}\big(|x_1 - y_1| + |x_2 - y_2|\big).$$

Evidently, $X_\omega$ is an $\mathcal{A} - O-$complete modular metric space. Define $T : X_\omega \to X_\omega$ by

$$T(x_1, x_2) = \begin{cases} (x_1, 0), & \text{if} \quad (x_1, x_2) \in M \text{ with } x_1 \leq x_2 \\ (0, x_2) & \text{if} \quad (x_1, x_2) \in M \text{ with } x_1 > x_2 \\ (4, 5) & \text{if} \quad (x_1, x_2) = (-1, 6) \end{cases}$$

Also, $TX_\omega \subseteq \mathcal{A}$. Let $(x, y) \perp (u, v)$. Clearly, if $(0, 0) \perp (u, v)$, then $T(0, 0) \perp T(u, v)$. Assume that,

$$(4, 0) \perp (4, 5),\ (4, 0) \perp (5, 4),\ (0, 4) \perp (4, 5),\ (0, 4) \perp (5, 4),$$

and so

$$T(4, 0) = (0, 0) \perp (4, 0) = T(4, 5),\ T(4, 0) = (0, 0) \perp (0, 4) = T(5, 4),$$
$$T(0, 4) = (0, 0) \perp (4, 0) = T(4, 5),\ T(0, 4) = (0, 0) \perp (0, 4) = T(5, 4).$$

That is, $T$ is an $\mathcal{A}- \perp -$preserving mapping.

Let $W_n = (x_n, y_n)$ (for all $n \geq 0$) be an $O-$sequence with $W_n = (x_n, y_n) \to (x, y)$ as $n \to \infty$. Equivalently, $x_n \leq x_{n+1}$, $y_n \leq y_{n+1}$, $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then we have, $x_n \leq x$ and $y_n \leq y$ for all $n \geq 0$. That is, $W_n = (x_n, y_n) \perp (x, y)$.

Assume that, $x \perp y$, $\frac{1}{2}\omega_\lambda(x, Tx) \leq \omega_{\frac{\lambda}{2}}(x, y)$ and $\omega_\lambda(Tx, Ty) > 0$.

If $x \perp y$, then,

$$(x, y) \in \Big\{\big((0, 0), (4, 0)\big), \big((0, 0), (0, 4)\big), \big((0, 0), (4, 5)\big), \big((0, 0), (5, 4)\big)$$
$$\big((4, 0), (4, 5)\big), \big((4, 0), (5, 4)\big), \big((0, 4), (4, 5)\big), \big((0, 4), (5, 4)\big)\Big\}$$

If, $\omega_\lambda(Tx, Ty) > 0$, then elements of the above set reduce to,

$$(x, y) \in \Big\{\big((0, 0), (4, 5)\big), \big((0, 0), (5, 4)\big), \big((4, 0), (4, 5)\big), \big((4, 0), (5, 4)\big)$$
$$, \big((0, 4), (4, 5)\big), \big((0, 4), (5, 4)\big)\Big\}$$

Now since,

$$\begin{aligned}
\frac{1}{2}\omega_\lambda\big((0,0),T(0,0)\big) &= 0 \le \frac{18}{\lambda} = \omega_{\frac{\lambda}{2}}\big((0,0),(4,5)\big),\\
\frac{1}{2}\omega_\lambda\big((0,0),T(0,0)\big) &= 0 \le \frac{18}{\lambda} = \omega_{\frac{\lambda}{2}}\big((0,0),(5,4)\big),\\
\frac{1}{2}\omega_\lambda\big((4,0),T(4,0)\big) &= \frac{2}{\lambda} \le \frac{10}{\lambda} = \omega_{\frac{\lambda}{2}}\big((4,0),(4,5)\big),\\
\frac{1}{2}\omega_\lambda\big((4,0),T(4,0)\big) &= \frac{2}{\lambda} \le \frac{10}{\lambda} = \omega_{\frac{\lambda}{2}}\big((4,0),(5,4)\big),\\
\frac{1}{2}\omega_\lambda\big((0,4),T(0,4)\big) &= \frac{2}{\lambda} \le \frac{10}{\lambda} = \omega_{\frac{\lambda}{2}}\big((0,4),(4,5)\big)
\end{aligned}$$

and

$$\frac{1}{2}\omega_\lambda\big((0,4),T(0,4)\big) = \frac{2}{\lambda} \le \frac{10}{\lambda} = \omega_{\frac{\lambda}{2}}\big((0,4),(5,4)\big),$$

then we have the following cases:

- if $(x,y) = \big((0,0),(4,5)\big)$, then,
$$\omega_\lambda(T(0,0),T(4,5)) = \frac{4}{\lambda} \le \frac{7.38}{\lambda} = 0.82\omega_\lambda((0,0),(4,5))$$
- if $(x,y) = \big((0,0),(5,4)\big)$, then,
$$\omega_\lambda(T(0,0),T(5,4)) = \frac{4}{\lambda} \le \frac{7.38}{\lambda} = 0.82\omega_\lambda((0,0),(5,4))$$
- if $(x,y) = \big((4,0),(4,5)\big)$, then,
$$\omega_\lambda(T(4,0),T(4,5)) = \frac{4}{\lambda} \le \frac{4.1}{\lambda} = 0.82\omega_\lambda((4,0),(4,5))$$
- if $(x,y) = \big((4,0),(5,4)\big)$, then,
$$\omega_\lambda(T(4,0),T(5,4)) = \frac{4}{\lambda} \le \frac{4.1}{\lambda} = 0.82\omega_\lambda((4,0),(5,4))$$
- if $(x,y) = \big((0,4),(4,5)\big)$, then,
$$\omega_\lambda(T(0,4),T(4,5)) = \frac{4}{\lambda} \le \frac{4.1}{\lambda} = 0.82\omega_\lambda((0,4),(4,5))$$
- if $(x,y) = \big((0,4),(5,4)\big)$, then,
$$\omega_\lambda(T(0,4),T(5,4)) = \frac{4}{\lambda} \le \frac{4.1}{\lambda} = 0.82\omega_\lambda((0,4),(5,4))$$

and so we can write,

$$e^{\omega_\lambda(Tx,Ty)\sqrt{\omega_\lambda(Tx,Ty)}} \le e^{0.82\omega_\lambda(x,y)\sqrt{0.82\omega_\lambda(x,y)}} = \left[e^{\omega_\lambda(x,y)\sqrt{\omega_\lambda(x,y)}}\right]^{(0.82)^{1.5}}$$

Define $\Theta : (0,\infty) \to (1,\infty)$ by $\Theta(t) = e^{t\sqrt{t}}$. Clearly, $\Theta \in \Delta_\Theta$. Then from the above inequality we can write,

$$\Theta(\omega_\lambda(Tx,Ty)) \le [\Theta(\omega_\lambda(x,y))]^{(0.82)^{1.5}}.$$

Therefore all conditions of Theorem 3.3 hold and $T$ has a fixed point (here, $(0,0)$ is fixed point of $T$).

If $x=(4,5)$ and $y=(5,4)$, then,

$$\Theta(d(Tx,Ty))=\Theta(8)\geq\Theta(2)>[\Theta(d(x,y))]^r$$

for all $r\in(0,1)$ and $\Theta\in\Delta_\Theta$. Hence results of Jleli and Samet [10] can not be applied for this example.

If $x=(0,0)$ and $y=(-1,6)$ then we get,

$$\theta(r)d(x,Tx)=0\leq d(x,y) \text{ but } d(Tx,Ty)>rd(x,y)$$

where,

$$\theta(r)=\begin{cases}1 & \text{if } \ 0\leq r\leq(\sqrt{5}-1)/2\\ (1-r)r^{-2} & \text{if } \ (\sqrt{5}-1)/2<r<2^{-1/2}\\ (1+r)^{-1} & \text{if } \ 2^{-1/2}\leq r<1.\end{cases}$$

and so Theorem of Suzuki [18] can not be applied for this example.

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