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Existence of a Positive Solution for a Boundary Value Problem of some Nonlinear Fractional Differential Equation

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Abstract

In this paper we will study the existence, uniqueness and positivity of solution of a boundary value problem for nonlinear fractional differential equation, by using Leray-Schauder nonlinear alternative, Banach contraction and Guo-Krasnosel'skii fixed point theorems.

Keywords: Fractional differential equations, Fixed point theorem, Guo-Krasnosel'skii theorem, Leray-Schauder nonlinear alternative, Banach contraction theorem, Existence, uniqueness, Positive solution.
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1. Introduction

Boundary value problems for nonlinear fractional differential equations belong to the important issues for the theory of fractional differential equations and a lot of papers and books on fractional calculs are devoted to the solvability of initial fractional differential equations, see $[1-3, 8, 10, 14, 16, ...]$.

However, there are few papers that deal with the existence, uniqueness and positivity of solution to nonlinear initial fractional differential equations by the use of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory, etc...), see $[1, 13, 17, 20, ...]$.

In this paper, motivated by $[5-7, 11, 12, 15, 18...]$ we are concerned with the existence, uniqueness and positivity of solution of the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f\left(t, u(t), D_{0^+}^{\sigma} u(t)\right) = 0, \quad t \in (0,1). \\ u(0) = u'(0) = 0, \quad u(1) = \beta u(\eta), \end{cases} \tag{1.1}$$

where: (i) $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\beta > 0$, $0 < \eta < 1$ and $0 < \sigma < 1$.
(ii) $D_{0^+}^{\alpha}$ is the Riemann-Liouville differential operator, of order $2 < \alpha \leq 3$.

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El-shahed [3] , considered the following nonlinear boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u(t) + \lambda a(t) f(u(t)) = 0, \quad t \in (0,1), \ 2 < \alpha \leq 3. \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville differential derivative. He used the Guo-Krasnosel'skii fixed point theorem on cone expansion and compression to show the existence and non-existence of positive solutions for the above fractional boundary value problem.

Li, Sun, Y. Li and P. Zhao, [12] , considered the fractional differential equation of the type

$$D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \ 1 < \alpha \leq 2,$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville differential order derivative, subject to the boundary conditions

$$u(0) = 0, \quad D_{0^+}^{\alpha} u(1) = a D_{0^+}^{\beta} u(\xi), \quad 0 \leq \beta \leq 1.$$

They obtained the existence and uniqueness of solution by using Leray-Schauder nonlinear alternative and Banach contraction mapping principle.

The organization of the paper is as follows. In section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In section 3, we establish the existence and uniqueness of the solution, by using the Leray-Schauder nonlinear alternative and Banach contraction theorem. In section 4, using the Guo-Krasnosel'skii fixed point theorem, we discuss the positivity of solution. In section 5, examples are presented to illustrate the main results.

2. Preliminaries

In this section, we present the necesary definition and several important preliminary lemmas to prove our results.

Denote by $L^1([0,1], \mathbb{R})$ the Banach space of Lebesgue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\|_{L^1} = \int_0^1 |u(t)|\, dt$. Let E be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ such that $D_{0^+}^{\sigma} u(t) \in C([0,1], \mathbb{R})$, $0 < \sigma < 1$, endowed with the norm $\|u\|_E = \max_{t \in [0,1]} |u(t)| + \max_{t \in [0,1]} \left| D_{0^+}^{\sigma} u(t) \right|$.

Now we provide some background definitions.

Definition 2.1. *Let K be a set in a real or complex vector space. K is said to be convex if, for all x and y in K and all t in the interval $]0,1[$, the point $(1-t)x + ty$ is in K. In other words, every point on the line segment connecting x and y is in K.*

Definition 2.2. *Let E be a Banach space. A nonempty closed convex subset $K \subset E$ is called a cone if it satisfies the following two conditions*

(i) $x \in K$ and $\lambda \geq 0$ implies $\lambda x \in K$.

(ii) $x \in K$ and $-x \in K$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E which is given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.3. *The fractional integral*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where $\alpha > 0$, is called Riemann-Liouville fractional integral of order α of a function $f : (0, +\infty) \to \mathbb{R}$ and $\Gamma(.)$ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-s} ds.$$

Definition 2.4. *The Riemann-Liouville fractional derivative of order* $\alpha > 0$, *of a continuous function* $f : (0, +\infty) \to \mathbb{R}$ *is given by*

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds.$$

$\Gamma(.)$ *is the gamma function, provided that the right side is point-wise defined on* $(0, +\infty)$ *and* $n = [\alpha] + 1$, $[\alpha]$ *stands for the greatest integer less than* α.

Lemma 2.5. [10] *Let* $\alpha, \beta \geq 0$, $f \in L^1(0,1)$, *then*

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t), \quad I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t).$$

The following two lemmas can be found in [10, 16].

Lemma 2.6. *Let* $\alpha > 0$ *and* $u \in C(0,1) \cap L^1(0,1)$, *then fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0,$$

has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha}$$

-*n*, $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$; $n = [\alpha] + 1$,*as solution.*

Lemma 2.7. *Assume that* $u \in C(0,1) \cap L^1(0,1)$ *with a frational derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L^1(0,1)$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$; $n = [\alpha] + 1$.

Lemma 2.8. *For Riemann-Liouville fractional derivatives, we have*

$$D_{0^+}^{\beta} \int_0^t (t-s)^{\alpha-1} f(s)\, ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f(s)\, ds,$$

where $f \in C[0,1]$, α, β *are two constants with* $\alpha > \beta \geq 0$.

Proof . From

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t), \qquad I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t),$$

we get

$$\begin{aligned} D_{0^+}^{\beta} \int_0^t (t-s)^{\alpha-1} f(s)\, ds &= D_{0^+}^{\beta} \Gamma(\alpha) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \\ &= D_{0^+}^{\beta} \Gamma(\alpha) I_{0^+}^{\alpha} f(t) = \Gamma(\alpha) D_{0^+}^{\beta} I_{0^+}^{\alpha} f(t), \\ &= \Gamma(\alpha) D_{0^+}^{\beta} I_{0^+}^{\beta} I_{0^+}^{\alpha-\beta} f(t) = \Gamma(\alpha) I_{0^+}^{\alpha-\beta} f(t) \\ &= \Gamma(\alpha) \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f(s)\, ds. \end{aligned}$$

Then we obtain the result. □

Lemma 2.9. *Let* $2<\alpha\leq 3$, $\beta>0$, $0<\eta<1$, $\beta\eta^{\alpha-1}\neq 1$ *and* $y\in L^{1}[0,1]$, *then the problem*

$$D_{0+}^{\alpha}u(t)+y(t)=0,\quad 0<t<1, \tag{2.1}$$

$$u(0)=u'(0)=0,\quad u(1)=\beta u(\eta), \tag{2.2}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)\,y(s)\,ds+\frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}}\int_0^1 G(\eta,s)\,y(s)\,ds, \tag{2.3}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}, & 0\leq s\leq t\leq 1,\\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0\leq t\leq s\leq 1.\end{cases} \tag{2.4}$$

Proof . Integrating the equation (2.1) over the interval $[0,t]$ for $t\in[0,1]$, we have

$$u(t)=-I_{0+}^{\alpha}y(t)+C_1t^{\alpha}$$

-1+C$_2t^{\alpha-2}+C_3t^{\alpha-3}$.From $u(0)=u'(0)=0$ we get $C_3=C_2=0$. And, from $u(1)=\beta u(\eta)$, we deduce that

$$C_1=\frac{1}{1-\beta\eta^{\alpha-1}}\left[I_{0+}^{\alpha}y(1)-\beta I_{0+}^{\alpha}y(\eta)\right].$$

Then

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\left[-(t-s)^{\alpha-1}+t^{\alpha-1}(t-s)^{\alpha-1}\right]y(s)\,ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1(1-s)^{\alpha-1}y(s)\,ds$$

$$+\frac{t^{\alpha-1}\beta}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^{\eta}\left[\eta^{\alpha-1}(\right.$$

1-s$^{\alpha-1}-(\eta-s)^{\alpha-1}y(s)\,ds$

$$+\frac{t^{\alpha-1}\beta}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_{\eta}^1\eta^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds.$$

And, that is equivalente to

$$u(t)=\int_0^1 G(t,s)\,y(s)\,ds+\frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}}\int_0^1 G(\eta,s)\,y(s)\,ds,\quad 0\leq t\leq 1,$$

which implies the *Lemma*. □

We need some properties of functions $G(t,s)$and $D_{0+}^{\sigma}G(t,s)$.

Lemma 2.10. *The function* $G(t,s)$ *defined by* (2.4) *satisfies the following properties*

(*i*) $G(t,s)\geq 0$ *and* $G(t,s)\in C([0,1]\times[0,1],\mathbb{R}_+)$.

(*ii*) *If* $t,s\in[\tau,1]$, $\tau>0$, *then*

$$\tau^{\alpha-1}G_1(s)\leq G(t,s)\leq\frac{1}{\tau}G_1(s),$$

where $G_1(s)=\frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-1}$.

Proof . (i) The continuity of G is easily checked. For $0 \leq t \leq s \leq 1$, it is obvious that

$$G(t,s) = \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} \geq 0.$$

In the case, $0 \leq s \leq t \leq 1$, we have

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \left[(1-s)^{\alpha-1} t^{\alpha-1} - (t-s)^{\alpha-1} \right] = \frac{(t-ts)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \geq 0.$$

(ii)

If $0 \leq t \leq s \leq 1$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1} t^{\alpha-1} \leq G_1(s).$$

If $0 \leq s \leq t \leq 1$, we have

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \left[(1-s)^{\alpha-1} t^{\alpha-1} - (t-s)^{\alpha-1} \right],$$

then

$$G(t,s) \leq \frac{1}{s} G_1(s), \quad \forall s,t \in [0,1].$$

Consequently

$$G(t,s) \leq \frac{1}{\tau} G_1(s), \quad \forall s \in [\tau,1], \; t \in [0,1].$$

Now we look for lower bounds of $G(t,s)$. If $0 \leq t \leq s \leq 1$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} (1-s)^{\alpha-1} \geq \frac{1}{\Gamma(\alpha)} t^{\alpha-1} s (1-s)^{\alpha-1},$$

then

$$G(t,s) \geq t^{\alpha-1} G_1(s), \quad \forall s,t \in [0,1].$$

If $0 \leq s \leq t \leq 1$, we have

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \left[(1-s)^{\alpha-1} t^{\alpha-1} - (t-s)^{\alpha-1} \right]$$

$\geq$ 0,and

$$(1-s)^{\alpha-1} t^{\alpha-1} (1-s) - ($$

t-s$^{\alpha-1}$ $\geq$ 0,

$$G(t,s) \geq t^{\alpha-1} G_1(s), \quad \forall s,t \in [0,1].$$

Consequently

$$G(t,s) \geq \tau^{\alpha-1} G_1(s), \text{ for } t,s \in [\tau,1].$$

The proof is complete. □

Lemma 2.11. *The function $D_{0^+}^{\sigma} G(t,s)$, $0 \leq t \leq 1$ prossesses the following properties:*

(1) $D_{0^+}^{\sigma} G(t,s) \in C([0,1] \times [0,1])$ *and* $D_{0^+}^{\sigma} G(t,s) \geq 0$ *for* $t,s \in]0,1[$.

(2)

$$D_{0^+}^{\sigma} G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1} t^{\alpha-\sigma-1} - (t-s)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-1} t^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.5}$$

(3) *For,* $t,s \in [\tau,1]$, $\tau > 0$, *we have*

$$\tau^{\alpha-\sigma-1} G_2(s) \leq D_{0^+}^{\sigma} G(t,s) \leq \frac{1}{\tau^{\alpha-\sigma-1}} G_2($$

s,where $G_2(s) = \frac{1}{\Gamma(\alpha-\sigma)} (1-s)^{\alpha-1} s^{\alpha-\sigma-1}$.

Proof . (1) The continuity and positivity of $D_{0+}^{\sigma}G(t,s)$ is easily checked.

(2) Applying the relation $D_{0+}^{\sigma}t^{\alpha-1}=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\sigma}$
$t^{\alpha-\sigma-1}$, we get

$$\int_0^1 G(t,s)\,y(s)\,ds=\frac{1}{\Gamma(\alpha)}\int_0^t\left[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\right]y(s)\,ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}($$

1-s$^{\alpha-1}y(s)\,ds=\int_0^1\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds$.Then

$$D_{0+}^{\sigma}\int_0^1 G(t,s)\,y(s)\,ds=D_{0+}^{\sigma}\left[t^{\alpha-1}I_{0+}^{\alpha}y(1)-I_{0+}^{\alpha}y(t)\right],$$

$$=I_{0+}^{\alpha}y(1)\,D_{0+}^{\sigma}t^{\alpha-1}-D_{0+}^{\sigma}I_{0+}^{\alpha}y(t)=I_{0+}^{\alpha}y(1)\,D_{0+}^{\sigma}t^{\alpha-1}-D_{0+}^{\sigma}I_{0+}^{\alpha}y(t),$$

$$=\int_0^1\frac{(1-s)^{\alpha-1}t^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)}y(s)\,ds-\int_0^t\frac{(t-s)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)}y(s)\,ds,$$

$$=\int_0^1 D_{0+}^{\sigma}G(t,s)\,y(s)\,ds,$$

which implies that propertie (2) holds.

(3)

If $0\le t\le s\le 1$,

$$D_{0+}^{\sigma}G(t,s)=\frac{1}{\Gamma(\alpha-\sigma}$$

$(1-s)^{\alpha-1}t^{\alpha-\sigma-1}\le$
G$_2(s)$.

If $0\le s\le t\le 1$, we have

$$D_{0+}^{\sigma}G(t,s)=\frac{1}{\Gamma(\alpha-\sigma}$$

$\Big[(1-s)^{\alpha-1}t^{\alpha-\sigma}$
-1-$(t-s)^{\alpha-\sigma-1}$,

$$\le\frac{1}{\Gamma(\alpha-\sigma)}(1-s)^{\alpha-1}t^{\alpha-\sigma-1}\le\frac{1}{s^{\alpha-\sigma-1}}G_2(s).$$

Consequently

$$D_{0+}^{\sigma}G(t,s)\le\frac{1}{\tau^{\alpha-\sigma-1}}G_2(s),\quad\forall s,t\in[\tau,1].$$

Now we look for lower bounds of $G(t,s)$. If $0\le t\le s\le 1$,

$$D_{0+}^{\sigma}G(t,s)=\frac{1}{\Gamma(\alpha-\sigma}$$

$t^{\alpha-\sigma-1}(1-s)^{\alpha-1}\ge t^{\alpha-\sigma-1}G_2(s)$.If $0\le s\le t\le 1$, we have

$$D_{0+}^{\sigma}G(t,s)=\frac{1}{\Gamma(\alpha-\sigma}$$

$\Big[(1-s)^{\alpha-1}t^{\alpha-\sigma-1}-($

t-s$^{\alpha-\sigma-1} \geq 0$,

$$\geq \frac{1}{\Gamma(\alpha-\sigma)}\left[(1-s)^{\alpha-1} t^{\alpha-\sigma-1}(1-s)-(t-s)^{\alpha-\sigma-1}\right] \geq 0,$$

then

$$D_{0+}^{\sigma} G(t, s) \geq \frac{1}{\Gamma(\alpha-\sigma)}\left[s^{\alpha-\sigma-1}(1-s)^{\alpha-1} t^{\alpha-\sigma-1}\right],$$

$$D_{0+}^{\sigma} G(t, s) \geq t^{\alpha-\sigma-1} G_2 ($$

s, $\forall s, t \in [0,1]$ .Consequently,

$$D_{0+}^{\sigma} G(t, s) \geq \tau^{\alpha-\sigma}$$

-1G$_2(s), \quad \forall t \in [\tau, 1]$,
s$\in [0,1]$.
This completes the proof of the Lemma. □

Definition 2.12. *We define the operator $T: E \longrightarrow E$ by*

$$Tu(t) = \int_0^1 G(t, s) f\left(s, u(s), D_{0+}^{\sigma} u(s)\right) ds$$

$$+\frac{\beta t^{\alpha-1}}{1-\beta \eta^{\alpha-1}} \int_0^1 G($$

$\eta, sf\left(s, u(s), D_{0+}^{\sigma} u(s)\right) ds, \quad t \in [0,1]$.(2.1) *The function $u \in E$ is a solution of the BVP (1.1) if and only if $Tu = u$; (u is a fixed point of T).*

Definition 2.13. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

3. Existence and uniqueness results

Now we give some results to prove the existence and uniqueness of a solution for the fractional boundary value broblem (1.1).

Theorem 3.1. *Assume that there exists a nonnegative function $k, h \in L^1([0,1], \mathbb{R}_+)$, such that*

$$|f(t, x, y) - f(t, u, v)| \leq k(t)|x-u| + h(t)|y-v|, \tag{3.7}$$

$$\forall x, y, u, v \in \mathbb{R}, \ t \in [0,1],$$

such that

$$C = \gamma\left(1+\frac{\beta}{1-\beta \eta^{\alpha-1}}\right)$$

$\int_0^1 (G_1(s) + G_2(s))(k(s) + h(s)) ds < 1.$ *Where,* $\gamma = \max\left\{\frac{1}{\tau}, \frac{1}{\tau^{\alpha-\sigma-1}}\right\}, \ 0 < \tau < 1.$
Then the fractional boundary value broblem (1.1), *has a unique solution in E*

Proof . We shall use the Banach contraction principle to prove that the operator T defined by (2.6) has a fixed point. We shall show that T is a contraction. Let $u, v \in E$, we have

$$|Tu(t) - Tv(t)| \leq \int_0^1 G(t, s)\left|f\left(s, u(s), D_{0+}^{\sigma} u(s)\right) - f\left(s, v(s), D_{0+}^{\sigma} v(s)\right)\right| ds$$

$$+\frac{\beta}{1-\beta \eta^{\alpha-1}} \int_0^1 G(\eta, s)$$

$\left|f\left(s,u\left(s\right),D_{0+}^{\sigma}u\left(s\right)\right)-f\left(s,v\left(s\right),D_{0+}^{\sigma}v\left(s\right)\right)\right|ds$.So, we can obtain

$$\left|Tu\left(t\right)-Tv\left(t\right)\right|\leq$$

$$\frac{1}{\tau}\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$\int_0^1 G_1\left(s\right)\left[k\left(s\right)\left|u\left(s\right)-v\left(s\right)\right|+h\left(s\right)\left|D_{0+}^{\sigma}u\left(s\right)-D_{0+}^{\sigma}v\left(s\right)\right|\right]ds$,then

$$\left|Tu\left(t\right)-Tv\left(t\right)\right|\leq\frac{1}{\tau}\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\left\|u-v\right\|_E\int_0^1 G_1\left(s\right)\left[k\left(s\right)+h\left(s\right)\right]ds.$$

And

$$\begin{aligned}\left|D_{0+}^{\sigma}Tu\left(t\right)-D_{0+}^{\sigma}Tv\left(t\right)\right| &\leq \frac{1}{\tau^{\alpha-\sigma-1}}\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\\ &\quad\times\left\|u-v\right\|_E\int_0^1 G_2\left(s\right)\left[k\left(s\right)+h\left(s\right)\right]ds.\end{aligned}$$

By using

$$C=\gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$\int_0^1\left(G_1\left(s\right)+G_2\left(s\right)\right)\left(k\left(s\right)+h\left(s\right)\right)ds<1$,where $\gamma=\max\left\{\frac{1}{\tau},\frac{1}{\tau^{\alpha-\sigma-1}}\right\}$, $0<\tau<1$. Obviously, we have

$$\left\|Tu-Tv\right\|_E\leq C\left\|u-v\right\|_E.$$

Then T is a contraction, so it has a unique fixed point which is the unique solution of the fractional boundary value broblem (1.1). □

We will employ the following Leray-Schauder nonlinear alternative [17].

Lemma 3.2. *Let F be Banach space and Ω be a bounded open subset of F, $0\in\Omega$. $T:\overline{\Omega}\to F$ be a completely continuous operator. Then, either there exists $x\in\partial\Omega$, $\lambda>1$ such that $T\left(x\right)=\lambda x$, or there exists a fixed point $x^*\in\overline{\Omega}$*

Theorem 3.3. *We assume that $f\left(t,0,0\right)\neq0$, there exist nonnegative functions $k,l,h\in L^1\left(\left[0,1\right],\mathbb{R}^+\right)$ and ϕ_1, $\phi_2\in C(\mathbb{R}^+,\mathbb{R}^+)$ nondecreasing, such that*

$$\left|f\left(t,u,v\right)\right|\leq k\left(t\right)\phi_1\left(\left|u\right|\right)+h\left(t\right)\phi_2\left(\left|v\right|\right)+l\left(t\right),\ \forall u,v\in\mathbb{R},\ t\in\left[0,1\right],\tag{3.8}$$

and there exists $m>0$ such that

$$M_1\max\left\{\phi_1\left(\left\|u\right\|_E\right),\phi_2\left(\left\|u\right\|_E\right)\right\}+M_2<m.$$

Where,

$$M_1=\gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$\int_0^1\left(G_1\left(s\right)+G_2\left(s\right)\right)\left(k\left(s\right)+h\left(s\right)\right)ds,(3.1)$

$$M_2=\gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$\int_0^1\left(G_1\left(s\right)+G_2\left(s\right)\right)l\left(s\right)ds.(3.2)$ *Then the fractional boundary value problem (1.1) has at least one non-trivial solution $u^*\in E$.*

Proof . To prove this *Theorem*, we apply *Lemma* 3.2. First, we need to prove that T is completely continuous

1) T is continuous.

From the continuity of f and G, we conclude that T is continous operator

2) Let $B_r = \{u \in E : \|u\|_E \leq r\}$ a bounded subset in E. We will prove that $T(\Omega \cap B_r)$ is relatively compact:

(i) $T(\Omega \cap B_r)$ is uniformly bounded. For some $u \in \Omega \cap B_r$, we have:

$$\begin{aligned} |Tu(t)| &\leq \gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \\ &\times \int_0^1 G_1(s)\left[k(s)\phi_1(u(s))+h(s)\phi_2\left(D_{0^+}^{\sigma}u(s)\right)+l(s)\right]ds. \end{aligned}$$

And

$$\begin{aligned} \left|D_{0^+}^{\sigma}Tu(t)\right| &\leq \gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \\ &\times \int_0^1 G_2(s)\left[k(s)\phi_1(u(s))+h(s)\phi_2\left(D_{0^+}^{\sigma}u(s)\right)+l(s)\right]ds. \end{aligned}$$

Then,

$$\|Tu\|_E \leq \gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\left[M_1 \max\left\{\phi_1\left(\|u\|_E\right), \phi_2\left(\|u\|_E\right)\right\}+M_2\right],$$

then, $T(\Omega \cap B_r)$ is uniformly bounded.

(ii) $T(\Omega \cap B_r)$ is equicontinuous.

Let $u \in \Omega \cap B_r$, t_1, $t_2 \in [0,1]$; $t_1 < t_2$, we have:

$$|Tu(t_2)-Tu(t_1)| \leq \int_0^1 \left|\left[G(t_2,s)-G(t_1,s)\right]f\left(s,u(s),D_{0^+}^{\sigma}u(s)\right)\right|ds$$

$$+\frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)\beta}{1-\beta\eta^{\alpha-1}}\int_0^1 G(\eta,s)\left|f\left(s,u(s),D_{0^+}^{\sigma}u(s)\right)\right|ds.$$

$$|Tu(t_2)-Tu(t_1)| \leq \frac{L\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\Gamma(\alpha)}$$

$$\times\left[\int_0^1 (1-s)^{\alpha-1}ds+\frac{\beta}{1-\beta\eta^{\alpha-1}}\int_0^1 G(\eta,s)\,ds\right].$$

$$\text{where,}\quad L = \max_{\substack{0<s<1 \\ \|u\|_E \leq r}} \left|f\left(s,u(s),D_{0^+}^{\sigma}u(s)\right)\right|,$$

and

$$\left|D_{0^+}^{\sigma}Tu(t_2)-D_{0^+}^{\sigma}\right.$$

$\text{Tu}(t_1) \leq \frac{L\left(t_2^{\alpha-\sigma-1}-t_1^{\alpha-\sigma-1}\right)}{\Gamma(\alpha-\sigma}$

$\times\left[\int_0^1 (1-s)^{\alpha-1}ds+\frac{\beta}{1-\beta\eta^{\alpha-1}}\int_0^1 G(\eta,s)\,ds\right]$,when $t_1 \to t_1 : |Tu(t_2)-Tu(t_1)|$ and $\left|D_{0^+}^{\sigma}Tu(t_2)-D_{0^+}^{\sigma}Tu(t_1)\right|$ tend to 0.

Consequently, $T(\Omega \cap B_r)$ is equicontinuous. From Arzela-Ascoli theorem, we deduce that T is a completely continuous operator.

Let $\Omega=\{u\in E:\|u\|_E<m\}$. We assume that $u\in\partial\Omega$, $\lambda>1$ such that $Tu=\lambda u$, then

$$\lambda m=\lambda\|u\|_E=\|Tu\|_E=\|Tu\|_\infty+\|D^\sigma_{0+}Tu\|_\infty,$$

since $\|Tu\|_\infty=\max\limits_{t\in[0,1]}|Tu(t)|$, we have

$$\|Tu\|_\infty\leq\frac{1}{\tau}\int_0^1 G_1(s)\left[k(s)\phi_1(\|u\|_\infty)+h(s)\phi_2\left(\|D^\sigma_{0+}u\|_\infty\right)+l(s)\right]ds$$

$$+\frac{\beta}{1-\beta\eta^{\alpha-1}}\int_0^1\frac{1}{\tau}G_1(s)\left[k(s)\phi_1(\|u\|_\infty)+h(s)\phi_2\left(\|D^\sigma_{0+}u\|_\infty\right)+l(s)\right]ds.$$

$$\|Tu\|_\infty \leq \gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\times$$
$$\int_0^1 G_1(s)\left[k(s)\phi_1(\|u\|_\infty)+h(s)\phi_2\left(\|D^\sigma_{0+}u\|_\infty\right)+l(\right.$$

sds.

$$\|Tu\|_\infty\leq\gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\left[\phi_1(\|u\|_E)\int_0^1 G_1(s)k(s)ds\right.$$
$$\left.+\phi_2(\|u\|_E)\int_0^1 G_1(s)h(s)ds+\int_0^1 G_1(s)l(s)ds\right],$$

and

$$\|D^\sigma_{0+}Tu\|_\infty\leq\gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\left[\phi_1(\|u\|_E)\int_0^1 G_2(s)k(s)ds\right.$$
$$\left.+\phi_2(\|u\|_E)\int_0^1 G_2(s)h(s)ds+\int_0^1 G_2(s)l(s)ds\right].$$

Then, we get

$$\|Tu\|_E\leq M_1\max\{\phi_1(\|u\|_E),\phi_2(\|u\|_E)\}+M_2,$$

and we have

$$\lambda m=\lambda\|u\|_E=\|Tu\|_E\leq M_1\max\{\phi_1(\|u\|_E),\phi(\|u\|_E)\}+M_2\leq m.$$

Consequently $\lambda<1$. This contradicts $\lambda>1$. By applying *Lemma* 3.2, T has a fixed point $u^*\in\overline{\Omega}$ and then the fractional boundary value broblem (1.1), has a nontrivial solution $u^*\in E$. The proof is complete. □

4. Positivity of the solution

In this section, we discuss the existence of positive solution for *fractional boundary value problem* (1.1). We make the following additional assumptions.

(Q1) $f(t,u,v)=a(t)f_1(u,v)$ where $a\in C((0,1),\mathbb{R}_+)$ and $f_1\in C(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$.

(Q2) $0<\int_0^1[G_1(s)+G_2($
$sa(s)ds<\infty$.

Definition 4.1. *A function $u(t)$ is called positive solution for the fractional boundary value problem* (1.1) *if $u(t)\geq 0$, $\forall t\in[0,1]$ and satisfies the B.V.P..*(1.1)

Lemma 4.2. *Let $u \in E$, the solution of the fractional boundary value problem* (1.1) *is nonnegative and satisfies*

$$\min_{t\in[0,1]} \left(u(t) + D_{0+}^{\sigma} u(t)\right) \geq \frac{\mu}{\gamma} \|u\|_E ,$$

where γ is defined in theorem 3.1 and $\mu = \min\left\{\tau^{\alpha-1}, \tau^{\alpha-\sigma-1}\right\}$.

Proof . Let $u \in E$, it is obvious that $u(t)$ is nonnegative, $t \in [0,1]$. From *Lemma* 2.10 and 2.11 , we have

$$u(t) \leq \frac{1}{\tau}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \int_0^1 G_1(s)\, a(s)\, f_1\left(u(s), D_{0+}^{\sigma} u(s)\right) ds,$$

and

$$D_{0+}^{\sigma} u(t) \leq \frac{1}{\tau^{\alpha-\sigma-1}}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$\int_0^1 G_2(s)\, a(s)\, f_1\left(u(s), D_{0+}^{\sigma} u(s)\right) ds$.Then

$$\|u\|_E \leq \gamma\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \int_0^1 (G_1(s) + G_2(s)$$

s$a(s)\, f_1\left(u(s), D_{0+}^{\sigma} u(s)\right) ds$.Hence

$$\gamma^{-1}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$^{-1}\|u\|_E \leq \int_0^1 (G_1(s) + G_2(s))\, a(s)\, f_1\left(u(s), D_{0+}^{\sigma} u(s)\right) ds$.On the other hand, for all $t \in [\tau, 1]$, we obtain

$$u(t) \geq \tau^{\alpha-1}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \int_0^1 G_1(s)\, a(s)\, f_1\left(u(s), D_{0+}^{\sigma} u(s)\right) ds,$$

and

$$D_{0+}^{\sigma} u(t) \geq \tau^{\alpha-\sigma-1}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \int_0^1 G_2(s)\, a(s)\, f_1\left(u(s), D_{0+}^{\sigma}\right.$$

u$(s)\, ds$.Therefore, we have

$$\begin{aligned} \min_{t\in[\tau,1]} \left(u(t) + D_{0+}^{\sigma} u(t)\right) \geq\ & \mu\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \times \\ & \int_0^1 (G_1(s) + G_2(s))\, a(s)\, f_1\left(u(s), D_{0+}^{\sigma} u(s)\right) ds. \end{aligned}$$

$$\min_{t\in[\tau,1]} \left(u(t) + D_{0+}^{\sigma} u(t)\right) \geq \mu\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \gamma^{-1}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)^{-1} \|u\|_E .$$

$$\min_{t\in[0,1]} \left(u(t) + D_{0+}^{\sigma} u(t)\right) \geq \frac{\mu}{\gamma} \|u\|_E .$$

Therefore, The proof is complete. □

Definition 4.3. *We define the cone K by*

$$K = \left\{u \in E,\ u(t) \geq 0,\ \min_{t\in[\tau,1]} \left(u(t) + D_{0+}^{\sigma}\right.\right.$$

$u(t) \geq \frac{\mu}{\gamma} \|u\|_E$.

K is a non-empty closed and convex subset of E.

Lemma 4.4. [7] *The operator defined in* (2.6) *is completely continuous and satisfies* $T(K) \subseteq K$.

To establish the existence of positive solutions for problem (1.1), we will employ the following Guo–Krasnosel'skii fixed point theorem [8]

Theorem 4.5. *Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$\mathcal{A} : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K,$$

be a completely continuous operator. In addition suppose either
(*i*) $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_2$; *or*
(*ii*) $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_2$,
holds. Then $\mathcal{A}$ has a fixed point in $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

The main result of this section is the following

Theorem 4.6. *Let* (Q_1) *and* (Q_2) *hold,* $0 < \beta\eta^{\alpha-1} < 1$ *and assume that*

$$f_0 = \lim_{(|u|+|v|)\to 0} \frac{f_1(u,v)}{|u|+|v|}, \quad f_\infty = \lim_{(|u|+|v|)\to\infty} \frac{f_1(u,v)}{|u|+|v|} \quad \text{exists.}$$

Then the problem (1.1) *has at least one positive solution in the case*
(*i*) $f_0 = 0$ *and* $f_\infty = \infty$ (*superlinear*) *or*
(*ii*) $f_0 = \infty$ *and* $f_\infty = 0$ (*sublinear*).

Proof . *We shall prove that the problem BVP* (1.1) *has at least one positive solution in both cases, superlinear and sublinear. For this we use Theorem 4.5. We prove the superlinear case. Since* $f_0 = 0$, *then for any* $\varepsilon > 0$, $\exists\delta_1 > 0$, *such that* $f_1(u,v) \leq \varepsilon(|u|+|v|)$, *for* $|u|+|v| < \delta_1$. *Let* Ω_1 *be an open set in* E *defined by*

$$\Omega_1 = \{y \in E \ / \ ||y||_E < \delta_1\},$$

then, for any $u \in K \cap \partial\Omega_1$, *it yields*

$$Tu(t) \leq \frac{1}{\tau}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\int_0^1 G_1(s)\, a(s)\, f_1\left(u(s), D_{0^+}^\sigma u(s)\right) ds.$$

Therefore

$$\|Tu\|_\infty \quad \leq \varepsilon\frac{1}{\tau}\|u\|_E\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\int_0^1 G_1(s)\, a(s)\, ds,$$

and

$$D_{0^+}^\sigma Tu(t) \leq \frac{1}{\tau^{\alpha-\sigma-1}}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$\int_0^1 G_2(s)\, a(s)\, f_1\left(u(s), D_{0^+}^\sigma u(s)\right) ds.$*So*

$$\|D_{0^+}^\sigma Tu\|_\infty \quad \leq \varepsilon\frac{1}{\tau^{\alpha-\sigma-1}}\|u\|_E\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$\int_0^1 G_1(s)\, a(s)\, ds$*If we choose* $\varepsilon = \left[\gamma\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\int_0^1 \left[G_1(s) + G_2(s)\right] a(s)\, ds\right]^{-1}$, *then it yields*

$$\|Tu\|_E \leq \|u\|_E, \quad \forall u \in K \cap \partial\Omega_1.$$

Now from $f_\infty = \infty$,*we conclude that for any* $M > 0$, *there exists* $H > 0$, *such that* $f_1(u,v) \geq M(|u|+|v|)$ *for* $|u|+|v| \geq H$. *Let*

$$H_1 = \max\left\{2\delta_1, \frac{\gamma}{\mu}H\right\}.$$

Denote by Ω_2 *the open set*

$$\Omega_2 = \{y \in E \ / \ \|y\|_E < H_1\}.$$

For any $u \in K \cap \partial\Omega_2$*, we have*

$$\min_{t\in[\tau,1]} \left(u(t) + D_{0+}^{\sigma} u(t)\right) \geq \frac{\mu}{\gamma} ||u||_E ,$$

$$= \frac{\mu}{\gamma} H_1 \geq H,$$

let $u \in K \cap \partial\Omega_2$ *then*

$$Tu(t) \geq \tau^{\alpha-1} \int_0^1 \left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) G_1(s)\, a(s)\, f_1\left(u(s), D_{0+}^{\sigma} u(s)\right) ds,$$

$$Tu(t) \geq \tau^{\alpha-1} \left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) M \int_0^1 G_1(s)\, a(s)\, ds\, \|u\|_E ,$$

and

$$D_{0+}^{\sigma} Tu(t) \geq \tau^{\alpha-\sigma}$$

-1$\int_0^1 G_2(s)\, a(s)\, f_1\left(u(s), D_{0+}^{\sigma} u(s)\right) ds$

$$D_{0+}^{\sigma} Tu(t) \geq M\tau^{\alpha-\sigma}$$

-1$\|u\|_E \int_0^1 G_2(s)\, a(s)\, ds$,*and choosing* $M = \left[\mu \int_0^1 \left[G_1(s) + G_2(s)\right] a(s)\, ds\right]^{-1}$, *we get*

$$\|Tu\|_E \geq \|u\|_E , \forall u \in K \cap \partial\Omega_2.$$

By the first part of Theorem 4.5, T has at least one fixed point in $K \cap \left(\bar{\Omega}_2 \diagdown \Omega_1\right)$ *, such that;* $H \leq ||y|| \leq H_1$. *This completes the superlinear case of Theorem 4.6.* ***Case II*** *Now, we assume that* $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear case). Proceding as above and by the second part of Theorem 4.5, we prove the sublinear case. This achieves the proof of Theorem 4.6.* □

5. Examples

In order to illustrate our result, we give the following examples:

Example 5.1. *Consider the following fractional boundary value problem*

$$\begin{cases} D_{0+}^{\frac{5}{2}} u(t) + \frac{t^3}{4} u + (1-t)^2 D_{0+}^{\frac{1}{3}} u(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u(1) = \beta u(\eta), \end{cases} \tag{J1}$$

set

$$\beta = \frac{1}{3}, \ \eta = \frac{1}{4}$$

and

$$f(t, u, v) = \frac{t^3}{4} u + (1-t)^2 v.$$

One can choose

$$\begin{cases} k(t) = \frac{t^3}{4} \\ h(t) = (1-t)^2 \end{cases}, \quad t \in [0,1],$$

$k, h \in L^1([0,1], \mathbb{R}^+)$ *are nonnegative functions, where*

$$\begin{aligned} |f(t,x,y) - f(t,u,v)| &\leq \frac{t^3}{4} |x-u| + (1-t)^2 |y-v| \\ &\leq k(t)\, |x-u| + h(t)\, |y-v| . \end{aligned}$$

and,

$$C=\gamma\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)$$

$\int_0^1\left(G_1(s)+G_2(s)\right)\left(k(s)+h(s)\right)ds<1$.*Hence, by Theorem* 3.1*, the fractional boundary value problem* $(J1)$ *has a unique solution in* E.

Example 5.2. *Consider the following fractional boundary value problem*

$$\left\{\begin{array}{c} D_{0^+}^{\frac{5}{2}}u(t)+\frac{t^2}{4}u+(1+t)^2D_{0^+}^{\frac{1}{4}}u(t)+\frac{1+t^2}{2}=0, \quad 0<t<1, \\ u(0)=u'(0)=0, \quad u(1)=\beta u(\eta), \end{array}\right. \tag{J2}$$

set

$$\beta=\frac{1}{2},\ \eta=\frac{1}{5}.$$

Where, $\alpha=\frac{5}{2}$, $\sigma=\frac{1}{4}$ *and*

$$f(t,u,v)=\frac{t^2}{4}u+(1+t)^2v+\frac{1+t^2}{2},\ \forall u,v\in\mathbb{R},\ t\in[0,1].$$

One can choose

$$\left\{\begin{array}{c} k(t)=\frac{t^2}{4} \\ h(t)=(1+t)^2 \\ l(t)=\frac{1+t^2}{2} \end{array}\right., \quad t\in[0,1],$$

$k,h,l\in L^1([0,1],\mathbb{R}^+)$ *are nonnegative functions, where*

$$|f(t,u,v)|\le k(t)\phi_1(|u|)+h(t)\phi_2(|v|)+l(t),\ \forall u,v\in\mathbb{R},\ t\in[0,1].$$

By Theorem 16, we can see that, there exists $m>0$ *such that*

$$M_1\max\{\phi_1(\|u\|_E),$$

$\phi_2(\|u\|_E)+M_2<m,$

where, M_1 *and* M_2 *are given by the formulas* (3.9) *and* (3.10)*, and the fractional boundary value problem* $(J2)$ *has at least one nontrivial solution in* E.

Example 5.3. *Consider the following fractional boundary value problem*

$$\left\{\begin{array}{c} D_{0^+}^{\frac{5}{2}}u(t)+t^2u^2+\frac{t^2}{4}\left(D_{0^+}^{\frac{1}{3}}u(t)\right)^2=0, \quad 0<t<1, \\ u(0)=u'(0)=0, \quad u(1)=\beta u(\eta), \end{array}\right. \tag{J3}$$

where, $0<\beta\eta^{\alpha-1}<1$; *and*

$$f(t,u,v)=t^2\left(u^2+\frac{1}{4}v^2\right)=a(t)f_1(u,v),$$

$a(t)=t^2\in C((0,1),\mathbb{R}_+)$, $f_1(u,v)\in C(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$. *Then*

$$f_0=\lim_{(|u|+|v|)\to 0}\frac{f_1(u,v)}{|u|+|v|}=0, \quad and \quad f_\infty=\lim_{(|u|+|v|)\to\infty}\frac{f_1(u,v)}{|u|+|v|}=\infty.$$

By Theorem 4.6 (i), *the fractional boundary value problem* $(J3)$ *has at least one positive solution.*

In this paper, motivated by some recent papers, we studied the existence, uniqueness and positivity of solution for a boundary value problem of nonlinear fractional differential equations, we established the existence and uniqueness of solution by applying, Leray-Schauder nonlinear alternative and Banach contraction theorem, and we discussed the existence of positive solution by applying Guo-Krasnosel'skii theorem. In the last, as applications, examples are presented to illustrate the main results.

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