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Some common fixed point theorems for four (ψ, φ)-weakly contractive mappings satisfying rational expressions in ordered partial metric spaces

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Abstract

The aim of this paper is to prove some common fixed point theorems for four mappings satisfying (ψ, φ)-weak contractions involving rational expressions in ordered partial metric spaces. Our results extend, generalize and improve some well-known results in the literature. Also, we give two examples to illustrate our results.

Keywords: Common fixed point; rational contractions; ordered partial metric spaces; dominating and dominated mappings.
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1. Introduction and preliminaries

The existence and uniqueness of fixed points of operators has been a subject of great interest since the work of Banach [1] in 1922. There exist vast literature concerning its various generalizations and extensions. Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [2], and further studied by Nieto and Lopez [3]. Subsequently, several interesting and valuable results have appeared in this direction see for examples [4]-[12].

The concept of a partial metric space was introduced by Matthews [13] in 1994. After that, fixed point results in partial metric spaces have been studied, see for example [14]-[25].

First, we present some necessary definitions and results which will be needed in the sequel.

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Definition 1.1. [13] Let X be a nonempty set. A mapping $p : X \times X \to [0, \infty)$ is said to be a partial metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space.

If $p(x, y) = 0$, then (p_1) and (p_2) imply that $x = y$. But converse dose not hold always.

Example 1.2. [13]

1. The function $p(x, y) = \max\{x, y\}$ for all $x, y \in R^+$ defines a partial metric p on R^+.
2. If $X = \{[a, b] : a, b \in R,\ a \leq b\}$ then $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to R^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

is a metric on X.

Definition 1.3. [13] Let (X, p) be a partial metric space. Then,

(i) a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim\limits_{n\to\infty} p(x, x_n)$,

(ii) a sequence $\{x_n\}$ in a partial metric space (X, p) is said to be a Cauchy sequence if $\lim\limits_{n,m\to\infty} p(x_n, x_m)$ exists and is finite,

(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim\limits_{n,m\to\infty} p(x_n, x_m)$.

Remark 1.4. A limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot, \cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$. For example, if $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}$, $p(x_n, x) = x = p(x, x)$ for each $x \geq 1$ and so, for example, $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$.

It is easy to see that every τ_p-closed subset of a complete partial metric space is complete.

Lemma 1.5. [13] Let (X, p) be a partial metric space. Then

(i) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(ii) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n,x)=0$, if and only if

$$p(x,x)=\lim_{n\to\infty} p(x_n,x)=\lim_{n,m\to\infty} p(x_n,x_m).$$

Definition 1.6. [15] Let (X,p) be a partial metric space, $F:X\to X$ be a given mapping. We say that F is continuous at $x_0\in X$, if for every $\varepsilon>0$, there exists $\eta>0$ such that $F(B_p(x_0,\eta))\subseteq B_p(F(x_0,\varepsilon))$.

Lemma 1.7. [24] Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial metric space (X,p) such that

$$\lim_{n\to\infty} p(x_n,x)=\lim_{n\to\infty} p(x_n,x_n)=p(x,x),$$

and

$$\lim_{n\to\infty} p(y_n,y)=\lim_{n\to\infty} p(y_n,y_n)=p(y,y),$$

then $\lim_{n\to\infty} p(x_n,y_n)=p(x,y)$. In particular, $\lim_{n\to\infty} p(x_n,z)=p(x,z)$ for every $z\in X$.

Definition 1.8. Let X be a nonempty set. Then $(X,\preceq,p)$ is called an ordered partial metric space if and only if:

(i) (X,p) is a partial metric space,

(ii) $(X,\preceq)$ is a partially ordered set.

Definition 1.9. Let $(X,\preceq)$ be a partially ordered set. $x,y\in X$ are called comparable if $x\preceq y$ or $y\preceq x$ holds.

Definition 1.10. Let $(X,\preceq)$ be a partially ordered set. A mapping f on X is said to be monotone nondecreasing if for all $x,y\in X$, $x\preceq y$ implies $fx\preceq fy$.

Definition 1.11. [4], [5] Let $(X,\preceq)$ be a partially ordered set. A mapping f on X is said to be

(i) dominating if $x\preceq fx$ for all $x\in X$,

(ii) dominated if $fx\preceq x$ for all $x\in X$.

For examples illustrating the above definitions were given in [4].

Definition 1.12. [26] A function $\psi:[0,\infty)\to[0,\infty)$ is called altering distance function if

(i) ψ is increasing and continuous,

(ii) $\psi(t)=0$ if and only if $t=0$.

Now, we recall the following definition of partial-compatibility.

Definition 1.13. [23] Let (X,p) be a partial metric space and $T,g : X \to X$ be given mappings. We say that the pair (T,g) is partial-compatible if the following conditions hold:

(i) $p(x,x) = 0$ implies that $p(gx,gx) = 0$.

(ii) $\lim_{n\to\infty} p(Tgx_n, gTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to t$ and $gx_n \to t$ for some $t \in X$.

Note that Definition 1.13 extends and generalizes the notion of compatibility introduced by Jungck [27] in the setting of metric spaces.

Definition 1.14. Let (X,d) be a metric space. A mapping $f : X \to X$ is said to be weakly contraction if

$$d(fx,fy) \leq d(x,y) - \varphi(d(x,y)).$$

for all $x, y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function with $\varphi(t) = 0$ if and only if $t = 0$.

In 1997, Alber and Guerre-Delabriere [28] proved that weakly contractive mapping defined on a Hilbert space is a Picard operator. Afterwards, Rhoades [29] proved that the corresponding result is also valid when Hilbert space is replaced by a complete metric space. Dutta et al. [30] generalized the weak contractive condition and proved a fixed point theorem for a selfmap, which in turn generalizes Theorem 1 in [29] and the corresponding result in [28].

In [31], Dass and Gupta proved the following fixed point theorem.

Theorem 1.15. [31] Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that there exist α, $\beta \geq 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)[1 + d(x,Tx)]}{[1 + d(x,y)]} + \beta d(x,y), \quad \text{for all} x, y \in X. \tag{1.1}$$

Then T has a unique fixed point.

In [7], Cabrera et al. proved the above theorem in the framwark of partially ordered metric spaces. Recently, Karapinar et al. [20] obtained the following result in partial metric spaces.

Theorem 1.16. [20] Let (X,p) be a complete partial metric space and $T : X \to X$ be a mapping satisfying

$$\psi(p(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)), \quad \forall x, y \in X,$$

where

$$M(x,y) = \max\left\{ \frac{p(y,Ty)[1 + p(x,Tx)]}{1 + p(x,y)}, p(x,y) \right\},$$

and $\psi : [0,\infty) \to [0,\infty)$ is a continuous and monotone non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$ and $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

The purpose of this paper is to prove some common fixed point theorems for four mappings f, g, S and T satisfying a generalized contraction of rational type in ordered partial metric spaces, where the mappings f, g are dominated and S, T are dominating maps. Two illustrative examples are given.

2. The main results

In this section we prove some common fixed point theorems which give conditions for existence and uniqueness of a common fixed point for a generalized contraction of rational type in ordered partial metric spaces.

Let Φ denote the set of all functions $\varphi : [0,\infty) \to [0,\infty)$ such that

(i) φ is a lower semi-continuous function,

(ii) $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 2.1. *Let $(X, \preceq, p)$ be an ordered complete partial metric space. Let $f, g, S, T : X \to X$ be four mappings such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, f, g are dominated mappings and S, T are dominating mappings. Suppose that for all comparable elements $x, y \in X$, we have*

$$\psi(p(fx, gy)) \leq \psi(M(x,y)) - \varphi(M(x,y)), \tag{2.1}$$

where

$$M(x,y) = \max\left\{\frac{p(Ty, gy)[1 + p(Sx, fx)]}{1 + p(Sx, Ty)}, p(Sx, Ty)\right\},$$

and ψ is an altering distance function and $\varphi \in \Phi$. If for a non-increasing sequence $\{x_n\}$ in X with $y_n \preceq x_n$ for all n and $\lim_{n\to\infty} p^s(y_n, z) = 0$, it follows $z \preceq x_n$ for all $n \in \mathbf{N}$, and either

(i) (f, S) is partial-compatible, f or S is continuous on (X, p^s) or

(ii) (g, T) is partial-compatible, g or T is continuous on (X, p^s),

then f, g, S and T have a common fixed point.

Proof . Let x_0 be an arbitrary point in X. Since $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, we can choose $x_1, x_2 \in X$ such that $y_0 = fx_0 = Tx_1$, and $y_1 = gx_1 = Sx_2$. Continuing this process, we define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n} = fx_{2n} = Tx_{2n+1}, \quad y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, \text{ for all } n \geq 0.$$

By the given assumptions we obtain

$$x_{2n+2} \preceq Sx_{2n+2} = gx_{2n+1} \preceq x_{2n+1} \preceq Tx_{2n+1} = fx_{2n} \preceq x_{2n}.$$

Thus, for all $n \in \mathbf{N}$ we have $x_{n+1} \preceq x_n$. Suppose that $p(y_{2n-1}, y_{2n}) > 0$ for all n. If not then $p(y_{2n-1}, y_{2n}) = 0$ for some n and so $y_{2n-1} = y_{2n}$. Further, since x_{2n} and x_{2n+1} are comparable, so from (2.1), we get

$$\begin{aligned}\psi(p(y_{2n}, y_{2n+1})) &= \psi(p(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}M(x_{2n}, x_{2n+1}) &= \max\left\{\frac{p(Tx_{2n+1}, gx_{2n+1})[1 + p(Sx_{2n}, fx_{2n})]}{1 + p(Sx_{2n}, Tx_{2n+1})}, p(Sx_{2n}, Tx_{2n+1})\right\} \\ &= \max\left\{\frac{p(y_{2n}, y_{2n+1})[1 + p(y_{2n-1}, y_{2n})]}{1 + p(y_{2n-1}, y_{2n})}, p(y_{2n-1}, y_{2n})\right\} \\ &= p(y_{2n}, y_{2n+1}).\end{aligned}$$

Hence from (2.2) we get

$$\psi(p(y_{2n},y_{2n+1})) \le \psi(p(y_{2n},y_{2n+1})) - \varphi(p(y_{2n},y_{2n+1})).$$

So $\varphi(p(y_{2n},y_{2n+1})) = 0$, and $y_{2n} = y_{2n+1}$. Similarly, we obtain $y_{2n+1} = y_{2n+2}$ and so on. Therefore $\{y_n\}$ becomes a constant sequence and y_{2n} is the common fixed point of f, g, S and T.

Now, we suppose that $p(y_{2n-1},y_{2n}) > 0$ for all $n \in \mathbf{N}$. Since x_{2n} and x_{2n+1} are comparable, from (2.1) we have

$$\begin{aligned}\psi(p(y_{2n},y_{2n+1})) &= \psi(p(fx_{2n},gx_{2n+1}))\\ &\le \psi(M(x_{2n},x_{2n+1})) - \varphi(M(x_{2n},x_{2n+1})),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}M(x_{2n},x_{2n+1}) &= \max\left\{\frac{p(Tx_{2n+1},gx_{2n+1})[1+p(Sx_{2n},fx_{2n})]}{1+p(Sx_{2n},Tx_{2n+1})}, p(Sx_{2n},Tx_{2n+1})\right\}\\ &= \max\left\{\frac{p(y_{2n},y_{2n+1})[1+p(y_{2n-1},y_{2n})]}{1+p(y_{2n-1},y_{2n})}, p(y_{2n-1},y_{2n})\right\}\\ &= \max\{p(y_{2n},y_{2n+1}), p(y_{2n-1},y_{2n})\}.\end{aligned}$$

If $M(x_{2n},x_{2n+1}) = p(y_{2n},y_{2n+1})$, then from (2.3) we obtain

$$\psi(p(y_{2n},y_{2n+1})) \le \psi(p(y_{2n},y_{2n+1})) - \varphi(p(y_{2n},y_{2n+1})),$$

Hence $\varphi(p(y_{2n},y_{2n+1})) = 0$, and so $p(y_{2n},y_{2n+1}) = 0$, gives a contradiction. Thus $M(x_{2n},x_{2n+1}) = p(y_{2n-1},y_{2n})$, and from (2.3) we obtain

$$\psi(p(y_{2n},y_{2n+1})) \le \psi(p(y_{2n-1},y_{2n})) - \varphi(p(y_{2n-1},y_{2n})) \le \psi(p(y_{2n-1},y_{2n})).$$

Since ψ is increasing, we get

$$p(y_{2n},y_{2n+1}) \le p(y_{2n-1},y_{2n}) = M(x_{2n},x_{2n+1}) \qquad \forall n \ge 0. \tag{2.4}$$

By similar arguments we can show that

$$p(y_{2n+1},y_{2n+2}) \le p(y_{2n},y_{2n+1}) = M(x_{2n+1},x_{2n+2}) \qquad \forall n \ge 0. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$p(y_n,y_{n+1}) \le p(y_{n-1},y_n) = M(x_n,x_{n+1}) \qquad \forall n \ge 0.$$

Thus, the sequence $\{p(y_n,y_{n+1})\}$ is non-increasing and so there exists $\delta \ge 0$ such that

$$\lim_{n\to\infty} p(y_n,y_{n+1}) = \lim_{n\to\infty} M(x_n,x_{n+1}) = \delta.$$

Suppose that $\delta > 0$. Then taking the upper limit as $n \to \infty$, in (2.3) and by the lower semi-continuity of φ we get

$$\limsup_{n\to\infty} \psi(p(y_{2n},y_{2n+1})) \le \limsup_{n\to\infty} \psi(M(x_{2n},x_{2n+1})) - \liminf_{n\to\infty} \varphi(M(x_{2n},x_{2n+1})).$$

Using the properties of the functions ψ and φ, we have $\psi(\delta) \leq \psi(\delta) - \varphi(\delta)$, so $\varphi(\delta) = 0$, hence $\delta = 0$, which is a contradiction. We conclude that

$$\lim_{n\to\infty} p(y_{2n}, y_{2n+1}) = \lim_{n\to\infty} M(x_{2n}, x_{2n+1}) = 0. \tag{2.6}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in the partial metric space (X,p). For this, it is sufficient to prove that $\{y_{2n}\}$ is a Cauchy sequence in (X,p). Suppose that $\{y_{2n}\}$ is not a Cauchy sequence in (X,p). Then, there is $\varepsilon > 0$ such that for an integer k there exist integers $2n(k)$, $2m(k)$ with $2m(k) > 2n(k) > k$ such that

$$p(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon, \tag{2.7}$$

for every integer k, let $m(k)$ be the least positive integer with $2m(k) > 2n(k)$, satisfying (2.7) and such that

$$p(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon. \tag{2.8}$$

Now, using (2.7) and the triangular inequality one gets

$$\begin{aligned}\varepsilon \leq p(y_{2n(k)}, y_{2m(k)}) &\leq p(y_{2n(k)}, y_{2m(k)-2}) + p(y_{2m(k)-2}, y_{2m(k)-1}) + p(y_{2m(k)-1}, y_{2m(k)})\\ &\quad - p(y_{2m(k)-2}, y_{2m(k)-2}) - p(y_{2m(k)-1}, y_{2m(k)-1}).\end{aligned}$$

Letting $k \to \infty$, in the above inequality and from (2.6), (2.8) it follows that

$$\lim_{k\to\infty} p(y_{2n(k)}, y_{2m(k)}) = \varepsilon. \tag{2.9}$$

Also, by the triangular inequality, we have

$$p(y_{2n(k)}, y_{2m(k)-1}) \leq p(y_{2n(k)}, y_{2m(k)}) + p(y_{2m(k)}, y_{2m(k)-1}) - p(y_{2m(k)}, y_{2m(k)}),$$

and

$$p(y_{2n(k)}, y_{2m(k)}) \leq p(y_{2n(k)}, y_{2m(k)-1}) + p(y_{2m(k)-1}, y_{2m(k)}) - p(y_{2m(k)-1}, y_{2m(k)-1}).$$

Letting $k \to \infty$, in the two above inequalities and using (2.6) and (2.9) we have

$$\lim_{k\to\infty} p(y_{2n(k)}, y_{2m(k)-1}) = \varepsilon. \tag{2.10}$$

Similarly,

$$\begin{aligned}p(y_{2n(k)-1}, y_{2m(k)-2}) &\leq p(y_{2n(k)-1}, y_{2n(k)}) + p(y_{2n(k)}, y_{2m(k)-1}) + p(y_{2m(k)-1}, y_{2m(k)-2})\\ &\quad - p(y_{2n(k)}, y_{2n(k)}) - p(y_{2m(k)-1}, y_{2m(k)-1}),\end{aligned}$$

and

$$\begin{aligned}p(y_{2n(k)}, y_{2m(k)-1}) &\leq p(y_{2n(k)}, y_{2n(k)-1}) + p(y_{2n(k)-1}, y_{2m(k)-2}) + p(y_{2m(k)-2}, y_{2m(k)-1})\\ &\quad - p(y_{2n(k)-1}, y_{2n(k)-1}) - p(y_{2m(k)-2}, y_{2m(k)-2}).\end{aligned}$$

Letting $k \to \infty$, in the two above inequalities and using (2.6) and (2.10) we have

$$\lim_{k\to\infty} p(y_{2n(k)-1}, y_{2m(k)-2}) = \varepsilon. \tag{2.11}$$

Since $x_{2n(k)}$, $x_{2m(k)-1}$ are comparable, then from (2.1), we obtain

$$\begin{aligned}\psi(p(y_{2n(k)}, y_{2m(k)-1})) &= \psi(p(fx_{2n(k)}, gx_{2m(k)-1}))\\ &\leq \psi(M(x_{2n(k)}, x_{2m(k)-1})) - \varphi(M(x_{2n(k)}, x_{2m(k)-1})),\end{aligned} \tag{2.12}$$

where

$$M(x_{2n(k)}, x_{2m(k)-1}) = \max\left\{\frac{p(Tx_{2m(k)-1}, gx_{2m(k)-1})[1+p(Sx_{2n(k)}, fx_{2n(k)})]}{1+p(Sx_{2n(k)}, Tx_{2m(k)-1})}, p(Sx_{2n(k)}, Tx_{2m(k)-1})\right\}$$
$$= \max\left\{\frac{p(y_{2m(k)-2}, y_{2m(k)-1})[1+p(y_{2n(k)-1}, y_{2n(k)})]}{1+p(y_{2n(k)-1}, y_{2m(k)-2})}, p(y_{2n(k)-1}, y_{2m(k)-2})\right\}.$$

Letting $k \to \infty$ in (2.12) and from(2.6), (2.10),(2.11), we get

$$\psi(\varepsilon) \leq \psi(\max\{0, \varepsilon\}) - \varphi(\max\{0, \varepsilon\}) = \psi(\varepsilon) - \varphi(\varepsilon).$$

Hence $\varphi(\varepsilon) = 0$, i.e. $\varepsilon = 0$, which is a contradiction. Thus we proved that $\{y_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is complete then from Lemma 1.5 (X, p^s) is a complete metric space. Therefore there exists $z \in X$, such that $\lim_{n\to\infty} p^s(y_n, z) = 0$. Also, from Lemma 1.5 we obtain

$$p(z, z) = \lim_{n\to\infty} p(y_n, z) = \lim_{m,n\to\infty} p(y_n, y_m). \tag{2.13}$$

Moreover, since $\{y_n\}$ is a Cauchy sequence in the metric space (X, p^s), then $\lim_{m,n\to\infty} p^s(y_n, y_m) = 0$. On the other hand, by (p_2) and (2.6), we have $p(y_n, y_n) \leq p(y_n, y_{n+1}) \to 0$, as $n \to \infty$ and hence we get

$$\lim_{n\to\infty} p(y_n, y_n) = 0. \tag{2.14}$$

Therefore from the definition of p^s and (2.14), we have $\lim_{m,n\to\infty} p(y_n, y_m) = 0$. Hence, from (2.13), we have

$$p(z, z) = \lim_{n\to\infty} p(y_n, z) = \lim_{m,n\to\infty} p(y_n, y_m) = 0. \tag{2.15}$$

Then we conclude that

$$\lim_{n\to\infty} p(y_{2n}, z) = \lim_{n\to\infty} p(fx_{2n}, z) = \lim_{n\to\infty} p(Tx_{2n+1}, z) = 0,$$

$$\lim_{n\to\infty} p(y_{2n+1}, z) = \lim_{n\to\infty} p(gx_{2n+1}, z) = \lim_{n\to\infty} p(Sx_{2n+2}, z) = 0.$$

Assume that S is continuous on (X, p^s). Then

$$\lim_{n\to\infty} p^s(SSx_{2n+2}Sfx_{2n+2}) = 0.$$

Also, since the (f, S) is partial-compatible, we have $\lim_{n\to\infty} p(fSx_{2n+2}, Sfx_{2n+2}) = 0$. Further, since $p(z, z) = 0$, then again the partial-compatibility of the pair (f, S) gives that $p(Sz, Sz) = 0$.

We need to show that $\lim_{n\to\infty} p(fSx_{2n+2}, gx_{2n+1}) = p(Sz, z)$, $\lim_{n\to\infty} p(SSx_{2n+2}, fSx_{2n+2}) = 0$ and $\lim_{n\to\infty} p(SSx_{2n+2}, Tx_{2n+1}) = p(Sz, z)$. So, since

$$p^s(fSx_{2n+2}, gx_{2n+1}) \leq p^s(fSx_{2n+2}, Sfx_{2n+2}) + p^s(Sfx_{2n+2}, gx_{2n+1}),$$

and

$$p^s(Sfx_{2n+2}, gx_{2n+1}) \leq p^s(Sfx_{2n+2}, fSx_{2n+2}) + p^s(fSx_{2n+2}, gx_{2n+1}),$$

letting $n \to \infty$, in the two above inequalities and using the continuity of S and the partial-compatibility of the pair (f, S) we have

$$\lim_{n\to\infty} p^s(fSx_{2n+2}, gx_{2n+1}) = p^s(Sz, z).$$

On the other hand

$$p^s(fSx_{2n+2}, gx_{2n+1}) = 2p(fSx_{2n+2}, gx_{2n+1}) - p(fSx_{2n+2}, fSx_{2n+2}) - p(gx_{2n+1}, gx_{2n+1}),$$

that is

$$2p(fSx_{2n+2}, gx_{2n+1}) = p^s(fSx_{2n+2}, gx_{2n+1}) + p(fSx_{2n+2}, fSx_{2n+2}) + p(gx_{2n+1}, gx_{2n+1}).$$

Taking limit as $n \to \infty$ we conclude that

$$2 \lim_{n\to\infty} p(fSx_{2n+2}, gx_{2n+1}) = p^s(Sz, z) = 2p(Sz, z).$$

Hence $\lim_{n\to\infty} p(fSx_{2n+2}, gx_{2n+1}) = p(Sz, z)$.

Since S is continuous, and $\{y_n\}$ converges to z in (X, p), hence

$$\lim_{n\to\infty} p(SSx_{2n+2}, Sz) = \lim_{n\to\infty} p(Sy_{2n+1}, Sz) = p(Sz, Sz) = 0.$$

Thus,

$$\lim_{n\to\infty} p(Sfx_{2n+2}, Sz) = \lim_{n\to\infty} p(Sy_{2n+2}, Sz) = p(Sz, Sz) = 0.$$

Then by triangular inequality we obtain

$$\begin{aligned} p(SSx_{2n+2}, fSx_{2n+2}) \leq\ & p(SSx_{2n+2}, Sz) + p(Sz, Sfx_{2n+2}) \\ & + p(Sfx_{2n+2}, fSx_{2n+2}) - p(Sfx_{2n+2}, Sfx_{2n+2}). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} p(SSx_{2n+2}, fSx_{2n+2}) = 0.$$

From Lemma 1.7 we obtain

$$\lim_{n\to\infty} p(SSx_{2n+2}, Tx_{2n+1}) = p(Sz, z).$$

Now, since, $Sx_{2n+2} = gx_{2n+1} \preceq x_{2n+1}$, so from (2.1), we obtain

$$\psi(p(fSx_{2n+2}, gx_{2n+1})) \leq \psi(M(Sx_{2n+2}, x_{2n+1})) - \varphi(M(Sx_{2n+2}, x_{2n+1})), \tag{2.16}$$

where

$$M(Sx_{2n+2}, x_{2n+1}) = \max\left\{\frac{p(Tx_{2n+1}, gx_{2n+1})[1 + p(SSx_{2n+2}, fSx_{2n+2})]}{1 + p(SSx_{2n+2}, Tx_{2n+1})}, p(SSx_{2n+2}, Tx_{2n+1})\right\}.$$

From (2.16), taking the upper limit as $n \to \infty$, we have $\psi(p(Sz, z)) \leq \psi(p(Sz, z)) - \varphi(p(Sz, z))$, and so $\varphi(p(Sz, z)) = 0$. Hence $Sz = z$.

On other hand, since $gx_{2n+1} \preceq x_{2n+1}$ and $\lim_{n\to\infty} gx_{2n+1} = z$, it follows that $z \preceq x_{2n+1}$. Thus from (2.1), we obtain

$$\psi(p(fz, gx_{2n+1})) \leq \psi(M(z, x_{2n+1})) - \varphi(M(z, x_{2n+1})), \tag{2.17}$$

where

$$\begin{aligned} M(z, x_{2n+1}) &= \max\left\{\frac{p(Tx_{2n+1}, gx_{2n+1})[1 + p(Sz, fz)]}{1 + p(Sz, Tx_{2n+1})}, p(Sz, Tx_{2n+1})\right\} \\ &= \max\left\{\frac{p(y_{2n}, y_{2n+1})[1 + p(z, fz)]}{1 + p(z, y_{2n})}, p(z, y_{2n})\right\}. \end{aligned}$$

On taking the upper limit in (2.17) as $n \to \infty$, we get $\psi(p(fz, z)) \leq \psi(p(z, z)) - \varphi(p(z, z)))$, so $\psi(p(fz, z)) \leq 0$, and $fz = z = Sz$.

Since $f(X) \subseteq T(X)$, there exists a point $w \in X$ such that $fz = Tw$. Suppose that $gw \neq Tw$. Since $w \preceq Tw = fz \preceq z$ implies $w \preceq z$. From (2.1), we obtain

$$\psi(p(Tw, gw)) = \psi(p(fz, gw)) \leq \psi(M(z, w)) - \varphi(M(z, w)), \tag{2.18}$$

where

$$\begin{aligned} M(z, w) &= \max\left\{\frac{p(Tw, gw)[1 + p(Sz, fz)]}{1 + p(Sz, Tw)}, p(Sz, Tw)\right\} \\ &= \max\{p(Tw, gw), 0\} = p(Tw, gw). \end{aligned}$$

Hence from (2.18), we get $\psi(p(Tw, gw)) \leq \psi(p(Tw, gw)) - \varphi(p(Tw, gw))$, a contradiction. Therefore, $Tw = gw$. Since g is dominated map and T is dominating map,

$$w \preceq Tw = z \quad \text{and} \quad z = gw \preceq w \quad \Rightarrow \quad w = z.$$

Hence $Sz = fz = Tz = gz = z$. Thus f, g, S and T have a common fixed point. The proof is similar when f is continuous. Similarly, the result follows when (ii) holds. □

Corollary 2.2. *Let $(X, \preceq, p)$ be an ordered complete partial metric space. Let $f, g, S, T : X \to X$ be four mappings such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, f, g are dominated mappings and S, T are dominating mappings. Suppose that for all comparable elements $x, y \in X$, we have*

$$p(fx, gy) \leq M(x, y) - \varphi(M(x, y)),$$

where

$$M(x, y) = \max\left\{\frac{p(Ty, gy)[1 + p(Sx, fx)]}{1 + p(Sx, Ty)}, p(Sx, Ty)\right\},$$

and $\varphi \in \Phi$. If for a non-increasing sequence $\{x_n\}$ in X with $y_n \preceq x_n$ for all n and $\lim_{n\to\infty} p^s(y_n, z) = 0$, it follows $z \preceq x_n$ for all $n \in \mathbf{N}$, and either

(i) (f, S) is partial-compatible, f or S is continuous on (X, p^s) or

(ii) (g, T) is partial-compatible, g or T is continuous on (X, p^s),

then f, g, S and T have a common fixed point.

Proof . In Theorem 2.1, taking $\psi(t) = t$ for all $t \in [0, \infty)$. □

Corollary 2.3. *Let $(X, \preceq, p)$ be an ordered complete partial metric space. Let $f, g, S, T : X \to X$ be four mappings such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, f, g are dominated mappings and S, T are dominating mappings. Suppose that for all comparable elements $x, y \in X$, we have*

$$p(fx, gy) \leq k \max\left\{\frac{p(Ty, gy)[1 + p(Sx, fx)]}{1 + p(Sx, Ty)}, p(Sx, Ty)\right\},$$

where $k \in (0, 1)$. If for a non-increasing sequence $\{x_n\}$ in X with $y_n \preceq x_n$ for all n and $\lim_{n\to\infty} p^s(y_n, z) =$ 0, it follows $z \preceq x_n$ for all $n \in \mathbf{N}$, and either *(i)* (f, S) *is partial-compatible,* f *or* S *is continuous on* (X, p^S) *or*

(ii) (g, T) *is partial-compatible,* g *or* T *is continuous on* (X, p^S),

then f, g, S *and* T *have a common fixed point.*

Proof . In Theorem 2.1, taking $\psi(t) = t$ and $\varphi(t) = (1-k)t$, for all $t \in [0, \infty)$. □

Corollary 2.4. *Let* $(X, \preceq, p)$ *be an ordered complete partial metric space. Let* $f, g, S, T : X \to X$ *be four mappings such that* $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, f, g *are dominated mappings and* S, T *are dominating mappings. Suppose that for all comparable elements* $x, y \in X$, *we have*

$$p(fx, gy) \leq \alpha \frac{p(Ty, gy)[1 + p(Sx, fx)]}{1 + p(Sx, Ty)} + \beta p(Sx, Ty),$$

where α, $\beta \geq 0$ *with* $\alpha + \beta < 1$. *If for a non-increasing sequence* $\{x_n\}$ *in* X *with* $y_n \preceq x_n$ *for all* n *and* $\lim_{n \to \infty} p^s(y_n, z) = 0$, *it follows* $z \preceq x_n$ *for all* $n \in \mathbf{N}$, *and either*

(i) (f, S) *is partial-compatible,* f *or* S *is continuous on* (X, p^S) *or*

(ii) (g, T) *is partial-compatible,* g *or* T *is continuous on* (X, p^S),

then f, g, S *and* T *have a common fixed point.*

Proof . In Corollary 2.3, taking $k = \alpha + \beta$, we get

$$\alpha \frac{p(Ty, gy)[1 + p(Sx, fx)]}{1 + p(Sx, Ty)} + \beta p(Sx, Ty) \leq k \max \left\{ \frac{p(Ty, gy)[1 + p(Sx, fx)]}{1 + p(Sx, Ty)}, p(Sx, Ty) \right\}.$$

Hence we apply Corollary 2.3. □

If we put $f = g$ in Theorem 2.1 we have the following corollary.

Corollary 2.5. *Let* $(X, \preceq, p)$ *be an ordered complete partial metric space. Let* $f, S, T : X \to X$ *be three mappings such that* $f(X) \subseteq T(X)$, $f(X) \subseteq S(X)$, f *is dominated mapping and* S, T *are dominating mappings. Suppose that for all comparable elements* $x, y \in X$, *we have*

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where

$$M(x, y) = \max \left\{ \frac{p(Ty, fy)[1 + p(Sx, fx)]}{1 + p(Sx, Ty)}, p(Sx, Ty) \right\},$$

and ψ *is an altering distance function and* $\varphi \in \Phi$. *If for a non-increasing sequence* $\{x_n\}$ *in* X *with* $y_n \preceq x_n$ *for all* n *and* $\lim_{n \to \infty} p^s(y_n, z) = 0$, *it follows* $z \preceq x_n$ *for all* $n \in \mathbf{N}$, *and either*

(i) (f, S) *is partial-compatible,* f *or* S *is continuous on* (X, p^s) *or*

(ii) (f, T) *is partial-compatible,* f *or* T *is continuous on* (X, p^s),

then f, S *and* T *have a common fixed point.*

If we put $S = T$ in Theorem 2.1 we have the following corollary.

Corollary 2.6. *Let $(X, \preceq, p)$ be an ordered complete partial metric space. Let $f, g, T : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq T(X)$, f, g are dominated mappings and T is dominating mapping. Suppose that for all comparable elements $x, y \in X$, we have*

$$\psi(p(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where

$$M(x, y) = \max\left\{\frac{p(Ty, gy)[1 + p(Tx, fx)]}{1 + p(Tx, Ty)}, p(Tx, Ty)\right\},$$

and ψ is an altering distance function and $\varphi \in \Phi$. If for a non-increasing sequence $\{x_n\}$ in X with $y_n \preceq x_n$ for all n and $\lim_{n\to\infty} p^s(y_n, z) = 0$, it follows $z \preceq x_n$ for all $n \in \mathbf{N}$, and either

(i) (f, T) is partial-compatible, f or T is continuous on (X, p^s) or

(ii) (g, T) is partial-compatible, g or T is continuous on (X, p^s),

then f, g and T have a common fixed point.

Further, if we put $f = g$ and $S = T$ in Theorem 2.1 we have the following corollary.

Corollary 2.7. *Let $(X, \preceq, p)$ be an ordered complete partial metric space. Let $f, T : X \to X$ be mappings such that $f(X) \subseteq T(X)$, f is dominated mapping and T is dominating mapping. Suppose that for all comparable elements $x, y \in X$, we have*

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where

$$M(x, y) = \max\left\{\frac{p(Ty, fy)[1 + p(Tx, fx)]}{1 + p(Tx, Ty)}, p(Tx, Ty)\right\},$$

and ψ is an altering distance function and $\varphi \in \Phi$. If one of the following two conditions is satisfied

(i) (f, T) is partial-compatible, f or T is continuous on (X, p^s), or

(ii) if for a non-increasing sequence $\{x_n\}$ in X with $y_n \preceq x_n$ for all n and $\lim_{n\to\infty} p^s(y_n, z) = 0$, it follows $z \preceq x_n$ for all $n \in \mathbf{N}$.

Then f and T have a common fixed point.

Putting $T = S = I$ in Theorem 2.1 we have the following corollary.

Corollary 2.8. *Let $(X, \preceq, p)$ be an ordered complete partial metric space. Let $f, g : X \to X$ be mappings such that f, g are dominated mappings. Suppose that for all comparable elements $x, y \in X$, we have*

$$\psi(p(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where

$$M(x, y) = \max\left\{\frac{p(y, gy)[1 + p(x, fx)]}{1 + p(x, y)}, p(x, y)\right\},$$

and ψ is an altering distance function and $\varphi \in \Phi$. If one of the following two conditions is satisfied:

(i) f or g is continuous on (X, p^s), or

(ii) If for a non-increasing sequence $\{x_n\}$ in X and $\lim_{n\to\infty} p^s(x_n,z)=0$, implies that $z\preceq x_n$ for all $n\in\mathbf{N}$.

Then f and g have a common fixed point.

If we take $f=g$ and $S=T=I$ in Theorem 2.1, we obtain the following corollary which improved Theorem 2 in [7].

Corollary 2.9. *Let $(X,\preceq,p)$ be an ordered complete partial metric space. Let $f:X\to X$ be mappings such that f is dominated mapping. Suppose that for all comparable elements $x,y\in X$, we have*

$$\psi(p(fx,fy))\le\psi(M(x,y))-\varphi(M(x,y)),$$

where

$$M(x,y)=\max\left\{\frac{p(y,fy)[1+p(x,fx)]}{1+p(x,y)},p(x,y)\right\},$$

and ψ is an altering distance function and $\varphi\in\Phi$. If one of the following two conditions is satisfied:

(i) f is continuous on (X,p^s), or

(ii) if for a non-increasing sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} p^s(x_n,z)=0$, implies that $z\preceq x_n$ for all $n\in\mathbf{N}$.

Then f has a fixed point.

By removing the continuity and compatibility assumptions in Theorem 2.1, we prove the following theorem.

Theorem 2.10. *Let $(X,\preceq,p)$ be an ordered complete partial metric space. Let $f,g,S,T:X\to X$ be four mappings such that $f(X)\subseteq T(X)$, $g(X)\subseteq S(X)$, f, g are dominated mappings and S, T are dominating mappings. Suppose that the condition (2.1) holds for all comparable elements $x,y\in X$, and ψ and φ are the same as in Theorem 2.1. Let one of $f(X)$, $g(X)$, $S(X)$ or $T(X)$ be a closed subset of X. If for a non-increasing sequence $\{x_n\}$ in X with $y_n\preceq x_n$ for all n and $\lim_{n\to\infty}p^s(y_n,z)=0$, it follows $z\preceq x_n$ for all $n\in\mathbf{N}$, then f, g, S and T have a common fixed point.*

Proof . Proceeding exactly as in Theorem 2.1, we have that $\{y_n\}$ is a Cauchy sequence in (X,p). Also,

$$\lim_{n\to\infty}p(y_{2n+1},z)=\lim_{n\to\infty}p(gx_{2n+1},z)=\lim_{n\to\infty}p(Sx_{2n+2},z)=p(z,z)=0.$$

Suppose that $S(X)$ is a closed subset of X. Hence there exists $u\in X$ such that $Su=z$. We show that $p(fu,z)=0$. Since $gx_{2n+1}\preceq x_{2n+1}$ and $\lim_{n\to\infty}gx_{2n+1}=z$ it follows that $z\preceq x_{2n+1}$, and $u\preceq Su=z$. Hence $u\preceq x_{2n+1}$, so from (2.1) we obtain

$$\psi(p(fu,gx_{2n+1}))\le\psi(M(u,x_{2n+1}))-\varphi(M(u,x_{2n+1})),\tag{2.19}$$

where

$$\begin{aligned}M(u,x_{2n+1})&=\max\left\{\frac{p(Tx_{2n+1},gx_{2n+1})[1+p(Su,fu)]}{1+p(Su,Tx_{2n+1})},p(Su,Tx_{2n+1})\right\}\\&=\max\left\{\frac{p(y_{2n},y_{2n+1})[1+p(z,fu)]}{1+p(z,y_{2n})},p(z,y_{2n})\right\}.\end{aligned}$$

Letting $n \to \infty$ in (2.19) and by (2.15) we get $\psi(p(fu, z)) = 0$. Thus we conclude that $fu = z = Su$. As f is dominated and S is dominating maps. then

$$u \preceq Su = z \quad \text{and} \quad z = fu \preceq u.$$

Hence $z = u$. Thus $fz = Sz = z$. From $f(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Tv$. We show that $p(gv, z) = 0$. From (2.1) we get

$$\psi(p(z, gv)) = \psi(p(fz, gv)) \leq \psi(M(z, v)) - \varphi(M(z, v)), \tag{2.20}$$

where

$$M(z, v) = \max\left\{\frac{p(Tv, gv)[1 + p(Sz, fz)]}{1 + p(Sz, Tv)}, p(Sz, Tv)\right\} = p(z, gv).$$

Therefore from (2.20) we deduce that

$$\psi(p(z, gv)) \leq \psi(p(z, gv)) - \varphi(p(z, gv)).$$

Hence $\varphi(p(z, gv)) = 0$, so $gv = z$. Since g is dominated and T is dominating maps. Then

$$v \preceq Tv = z \quad \text{and} \quad z = gv \preceq v.$$

Hence $z = v$. Thus $fz = Sz = gz = Tz = z$. That is z is a common fixed point of f, g, S and T. The proof is similar when $f(X)$, $g(X)$ or $T(X)$ is a closed subset of X. □

Now, we shall prove the uniqueness of the common fixed point as in the following theorem.

Theorem 2.11. *In addition to the hypotheses of Theorem* 2.1 *(or Theorem* 2.10*) assume that for all* $(x, y) \in X \times X$, *there exists* $z \in X$ *such that* $z \preceq x$ *and* $z \preceq y$. *Then,* f, g, S *and* T *have a unique common fixed point.*

Proof . The set of common fixed points of f, g, S and T is not empty due to Theorem 2.1 (or Theorem 2.10). Suppose that u and v are two common fixed points of f, g, S and T, that is, $fu = gu = Su = Tu = u$ and $fv = gv = Sv = Tv = v$. Theorem 2.1 (or Theorem 2.10) gives us that $p(u, u) = p(v, v) = 0$. By assumption, there exists $z_0 \in X$ such that

$$z_0 \preceq u \quad \text{and} \quad z_0 \preceq v. \tag{2.21}$$

Now, proceeding similarly to the proof of Theorem 2.1 (or Theorem 2.10), we can define the sequences $\{z_n\}$ and $\{w_n\}$ in X as follows

$$w_{2n} = fz_{2n} = Tz_{2n+1}, \quad w_{2n+1} = gz_{2n+1} = Sz_{2n+2}, \ \text{ for all } n \geq 0.$$

Since f, g are dominated mappings and S, T are dominating mappings we have

$$z_{2n+2} \preceq Sz_{2n+2} = gz_{2n+1} \preceq z_{2n+1} \preceq Tz_{2n+1} = fz_{2n} \preceq z_{2n} \ \text{ for all } n \geq 0.$$

Thus, for all $n \geq 0$ we have $z_{n+1} \preceq z_n \preceq z_0 \preceq u$. Further, in similar way for the proof of Theorem 2.1 we can get

$$\lim_{n\to\infty} p(w_n, w_{n+1}) = 0. \tag{2.22}$$

As $z_{2n} \preceq u$, putting $x = z_{2n}$ and $y = u$ in (2.1), we obtain

$$\psi(p(w_{2n}, u)) = \psi(p(fz_{2n}, gu)) \leq \psi(M(z_{2n}, u)) - \varphi(M(z_{2n}, u)),$$

where

$$M(z_{2n}, u) = \max\left\{\frac{p(Tu, gu)[1 + p(Sz_{2n}, fz_{2n})]}{1 + p(Sz_{2n}, Tu)}, p(Sz_{2n}, Tu)\right\} = p(w_{2n-1}, u).$$

Thus

$$\psi(p(w_{2n}, u)) \leq \psi(p(w_{2n-1}, u)) - \varphi(p(w_{2n-1}, u)) \leq \psi(p(w_{2n-1}, u)).$$

Since ψ is increasing, we have

$$p(w_{2n}, u) \leq p(w_{2n-1}, u). \tag{2.23}$$

Also, since $z_{2n+1} \preceq u$, putting $x = u$ and $y = z_{2n+1}$ in (2.1), we have

$$\psi(p(u, w_{2n+1})) = \psi(p(fu, gz_{2n+1})) \leq \psi(M(u, z_{2n+1})) - \varphi(M(u, z_{2n+1})), \tag{2.24}$$

where

$$\begin{aligned} M(u, z_{2n+1}) &= \max\left\{\frac{p(Tz_{2n+1}, gz_{2n+1})[1 + p(Su, fu)]}{1 + p(Su, Tz_{2n+1})}, p(Su, Tz_{2n+1})\right\} \\ &= \max\left\{\frac{p(w_{2n}, w_{2n+1})}{1 + p(u, w_{2n})}, p(u, w_{2n})\right\}. \end{aligned}$$

(I) If $M(u, z_{2n+1}) = \frac{p(w_{2n}, w_{2n+1})}{1+p(u, w_{2n})}$, then from (2.22) we obtain $\lim_{n\to\infty} M(u, z_{2n+1}) = 0$. Therefore from (2.24) we have $\lim_{n\to\infty} \psi(p(u, w_{2n+1})) = 0$. Hence

$$\lim_{n\to\infty} p(u, w_{2n+1}) = 0. \tag{2.25}$$

(II) If $M(u, z_{2n+1}) = p(u, w_{2n})$, so from (2.24) we have

$$\psi(p(u, w_{2n+1})) \leq \psi(p(u, w_{2n})) - \varphi(p(u, w_{2n})) \leq \psi(p(u, w_{2n})), \tag{2.26}$$

Since ψ is increasing, we obtain

$$p(u, w_{2n+1}) \leq p(u, w_{2n}). \tag{2.27}$$

Combining (2.23) and (2.27) we conclude that

$$p(u, w_{n+1}) \leq p(u, w_n) \qquad \forall n \geq 0. \tag{2.28}$$

So, the sequence $\{p(u, w_n)\}$ is non-increasing and bounded below, so there exists $\gamma \geq 0$ such that

$$\lim_{n\to\infty} p(u, w_n) = \gamma. \tag{2.29}$$

Suppose that $\gamma > 0$. Then from (2.26) taking the upper limit as $n \to \infty$, and by the lower semi-continuity of φ we get

$$\limsup_{n\to\infty} \psi(p(u, w_{2n+1})) \leq \limsup_{n\to\infty} \psi(p(u, w_{2n})) - \liminf_{n\to\infty} \varphi(p(u, w_{2n})).$$

Using the properties of the functions ψ and φ, we have $\psi(\gamma) \leq \psi(\gamma) - \varphi(\gamma)$, so $\gamma = 0$, which is a contradiction. We conclude that $\lim_{n\to\infty} p(u, w_n) = 0$.

From (I) and (II) we conclude that
$$\lim_{n\to\infty} p(u, w_{2n}) = 0. \tag{2.30}$$
Similarly, using the same argument we can get
$$\lim_{n\to\infty} p(v, w_{2n}) = 0. \tag{2.31}$$
Since $p(u,v) \leq p(u,w_{2n}) + p(w_{2n},v) - p(w_{2n},w_{2n})$, and from (2.22), (2.30), (2.31), we conclude that $p(u,v) \leq 0$. Therefore $u = v$. □

To support our results, we give the following examples.

Example 2.12. Let $X = [0,1]$ endowed with usual order $\leq$ and (X,p) be a complete partial metric space, where $p : X \times X \to R^+$ is defined by $p(x,y) = max\{x,y\}$ and let $\psi, \varphi : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = bt$ and $\varphi(t) = (b-1)t$, where $1 \leq b \leq 2$. Let $f, g, S, T : X \to X$ be defined by
$$fx = \frac{x}{2}, \qquad gx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ \frac{1}{4} & \text{if } x \in (\frac{1}{2}, 1] \end{cases},$$
$$Sx = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ x & \text{if } x \in (\frac{1}{2}, 1] \end{cases}, \quad Tx = \begin{cases} \frac{3}{2}x & \text{if } x \in [0, \frac{1}{2}] \\ 1 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}.$$
Then $f(X) \subseteq T(X)$ $g(X) \subseteq S(X)$. The table shows that f, g are dominated and S, T are dominating mappings.

for each $x \in [0,1]$	$fx \leq x$	$gx \leq x$	$x \leq Sx$	$x \leq Tx$
$x \in [0, \frac{1}{2}]$	$fx = \frac{x}{2} \leq x$	$gx = 0 \leq x$	$x \leq Sx = 2x$	$x \leq Tx = \frac{3}{2}x$
$x \in (\frac{1}{2}, 1]$	$fx = \frac{x}{2} \leq x$	$gx = \frac{1}{4} \leq x$	$x \leq Sx = x$	$x \leq Tx = 1$

(f,S) is partial-compatible maps and f is a continuous map. To show that f, g, S and T satisfy condition (2.1) for all $x,y \in X$, we consider the following cases

(i) If $x, y \in [0, \frac{1}{2}]$, then
$$M(x,y) = \max\left\{\frac{p(\frac{3}{2}y, 0)[1 + p(2x, \frac{x}{2})]}{1 + p(2x, \frac{3}{2}y)}, p(2x, \frac{3}{2}y)\right\} = \max\left\{\frac{\frac{3}{2}y[1+2x]}{1+p(2x,\frac{3}{2}y)}, p(2x, \frac{3}{2}y)\right\}.$$

We have two cases:

(a) If $p(2x, \frac{3}{2}y) = 2x$ then $M(x,y) = \max\left\{\frac{3}{2}y, 2x\right\} = 2x$. Hence
$$\psi(p(fx,gy)) = \psi(p(\frac{x}{2}, 0)) = \psi(\frac{x}{2}) = \frac{bx}{2} \leq 2x = M(x,y) = \psi(M(x,y)) - \phi(M(x,y)).$$

(b) If $p(2x, \frac{3}{2}y) = \frac{3}{2}y$ then $M(x,y) = \max\left\{\frac{\frac{3}{2}y[1+2x]}{1+\frac{3}{2}y}, \frac{3}{2}y\right\}$. Hence
$$\psi(p(fx,gy)) = \psi(\frac{x}{2}) = \frac{bx}{2} \leq 2x \leq \frac{3}{2}y \leq M(x,y) = \psi(M(x,y)) - \phi(M(x,y)).$$

(ii) If $x \in [0, \frac{1}{2}]$, $y \in (\frac{1}{2}, 1]$, then

$$M(x,y) = \max\left\{\frac{p(1,\frac{1}{4})[1+p(2x,\frac{x}{2})]}{1+p(2x,1)}, p(2x,1)\right\} = \max\left\{\frac{1+2x}{2}, 1\right\} = 1.$$

Hence

$$\psi(p(fx,gy)) = \psi(p(\frac{x}{2},\frac{1}{4})) = \psi(\frac{1}{4}) = \frac{b}{4} \leq M(x,y) = \psi(M(x,y)) - \phi(M(x,y)).$$

(iii) if $x \in (\frac{1}{2}, 1]$, $y \in [0, \frac{1}{2}]$, then

$$M(x,y) = \max\left\{\frac{p(\frac{3}{2}y,0)[1+p(x,\frac{x}{2})]}{1+p(x,\frac{3}{2}y)}, p(x,\frac{3}{2}y)\right\} = \max\left\{\frac{\frac{3}{2}y[1+x]}{1+p(x,\frac{3}{2}y)}, p(x,\frac{3}{2}y)\right\}.$$

We have two cases:

(a) if $p(x,\frac{3}{2}y) = x$ then $M(x,y) = \max\left\{\frac{3}{2}y, x\right\} = x$. Hence

$$\begin{aligned}\psi(p(fx,gy)) = \psi(p(\frac{x}{2},0)) = \psi(\frac{x}{2}) = \frac{bx}{2} &\leq x = M(x,y)\\ &= \psi(M(x,y)) - \phi(M(x,y)).\end{aligned}$$

(b) If $p(x,\frac{3}{2}y) = \frac{3}{2}y$ then $M(x,y) = \max\left\{\frac{\frac{3}{2}y[1+x]}{1+\frac{3}{2}y}, \frac{3}{2}y\right\}$. Hence

$$\psi(p(fx,gy)) = \psi(\frac{x}{2}) = \frac{bx}{2} \leq x \leq \frac{3}{2}y \leq M(x,y) = \psi(M(x,y)) - \phi(M(x,y)).$$

(iv) if $x, y \in (\frac{1}{2}, 1]$, then

$$M(x,y) = \max\left\{\frac{p(1,\frac{1}{4})[1+p(x,\frac{x}{2})]}{1+p(x,1)}, p(x,1)\right\} = \max\left\{\frac{1+x}{2}, 1\right\} = 1.$$

Hence

$$\psi(p(fx,gy)) = \psi(p(\frac{x}{2},\frac{1}{4})) = \psi(\frac{x}{2}) = \frac{bx}{2} \leq x \leq M(x,y) = \psi(M(x,y)) - \phi(M(x,y)).$$

Thus, the mappings f, g, S and T satisfy the condition (2.1). Therefore all conditions given in Theorem 2.1 are satisfied. Moreover, 0 is the unique common fixed point of f, g, S and T.

Example 2.13. Let $X = [0,3]$ endowed with usual order $\leq$ and (X,p) be a complete partial metric space, where $p : X \times X \to R^+$ is defined by $p(x,y) = max\{x,y\}$ and let $\psi, \varphi : [0,\infty) \to [0,\infty)$ be defined by $\psi(t) = 3t$ and $\varphi(t) = \frac{1}{3}t$. Let $f, g, S, T : X \to X$ be defined by

$$fx = \begin{cases} \frac{x^2}{2} & \text{if } x \in [0,1) \\ \frac{1}{4} & \text{if } x \in [1,3] \end{cases}, \qquad gx = \begin{cases} 0 & \text{if } x \in [0,1) \\ \frac{1}{2} & \text{if } x \in [1,3] \end{cases},$$

$$Sx = \begin{cases} 3\sqrt{x} & \text{if } x \in [0,1) \\ x & \text{if } x \in [1,3] \end{cases}, \qquad Tx = \begin{cases} 2\sqrt{x} & \text{if } x \in [0,1) \\ 3 & \text{if } x \in [1,3] \end{cases}.$$

Then $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and $S(X)$ is a closed subset of X. The table shows that f, g are dominated and S, T are dominating mappings.

for each $x \in [0,3]$	$fx \leq x$	$gx \leq x$	$x \leq Sx$	$x \leq Tx$
$x \in [0,1)$	$fx = \frac{x^2}{2} \leq x$	$gx = 0 \leq x$	$x \leq Sx = 3\sqrt{x}$	$x \leq Tx = 2\sqrt{x}$
$x \in [1,3]$	$fx = \frac{1}{4} \leq x$	$gx = \frac{1}{2} \leq x$	$x \leq Sx = x$	$x \leq Tx = 3$

Now, we show that f, g, S and T satisfy condition (2.1) for all $x, y \in X$, we consider the following cases

(i) If $x, y \in [0,1)$, then

$$\begin{aligned} M(x,y) &= \max\left\{\frac{p(2\sqrt{y},0)[1+p(3\sqrt{x},\frac{x^2}{2})]}{1+p(3\sqrt{x},2\sqrt{y})}, p(3\sqrt{x},2\sqrt{y})\right\} \\ &= \max\left\{\frac{2\sqrt{y}[1+3\sqrt{x}]}{1+p(3\sqrt{x},2\sqrt{y})}, p(3\sqrt{x},2\sqrt{y})\right\}. \end{aligned}$$

We have two cases:

(a) If $p(3\sqrt{x},2\sqrt{y}) = 3\sqrt{x}$ then $M(x,y) = 3\sqrt{x}$. Hence

$$\psi(p(fx,gy)) = \psi(\frac{x^2}{2}) = \frac{3x^2}{2} \leq 3\sqrt{x} \leq 9\sqrt{x} - \sqrt{x} = \psi(M(x,y)) - \phi(M(x,y)).$$

(b) if $p(3\sqrt{x},2\sqrt{y}) = 2\sqrt{y}$ then $M(x,y) = \max\left\{\frac{2\sqrt{y}[1+3\sqrt{x}]}{1+2\sqrt{y}}, 2\sqrt{y}\right\}$. Hence

$$\psi(p(fx,gy)) = \psi(\frac{x^2}{2}) = \frac{3x^2}{2} \leq 3\sqrt{x} \leq 2\sqrt{y} \leq M(x,y) \leq \psi(M(x,y)) - \phi(M(x,y)).$$

(ii) If $X \in [0,1)$, $y \in [1,3]$, then

$$M(x,y) = \max\left\{\frac{p(3,\frac{1}{2})[1+p(3\sqrt{x},\frac{x^2}{2})]}{1+p(3\sqrt{x},3)}, p(3\sqrt{x},3)\right\} = 3.$$

Hence

$$\psi(p(fx,gy)) = \psi(p(\frac{x^2}{2},\frac{1}{2})) = \psi(\frac{1}{2}) = \frac{3}{2} \leq M(x,y) \leq \psi(M(x,y)) - \phi(M(x,y)).$$

(iii) If $X \in [1,3]$, $y \in [0,1)$, then

$$M(x,y) = \max\left\{\frac{p(2\sqrt{y},0)[1+p(x,\frac{1}{4})]}{1+p(x,2\sqrt{y})}, p(x,2\sqrt{y})\right\} = \max\left\{\frac{2\sqrt{y}[1+x]}{1+p(x,2\sqrt{y})}, p(x,2\sqrt{y})\right\}.$$

We have two cases:

(a) If $p(x,2\sqrt{y}) = x$ then $M(x,y) = \max\{2\sqrt{y}, x\} = x$. Hence

$$\psi(p(fx,gy)) = \psi(p(\frac{1}{4},0)) = \psi(\frac{1}{4}) = \frac{3}{4} \leq M(x,y) \leq \psi(M(x,y)) - \phi(M(x,y)).$$

(b) if $p(x, 2\sqrt{y}) = 2\sqrt{y}$ then $M(x,y) = \max\left\{\frac{2\sqrt{y}[1+x]}{1+2\sqrt{y}}, 2\sqrt{y}\right\}$. Hence

$$\psi(p(fx, gy)) = \frac{3}{4} \leq x \leq 2\sqrt{y} \leq M(x,y) \leq \psi(M(x,y)) - \phi(M(x,y)).$$

(iv) if $x, y \in [1,3]$, then

$$M(x,y) = \max\left\{\frac{p(3,\frac{1}{2})[1+p(x,\frac{1}{4})]}{1+p(x,3)}, p(x,3)\right\} = \max\left\{\frac{3[1+x]}{4}, 3\right\} = 3.$$

Hence

$$\psi(p(fx, gy)) = \psi(p(\frac{1}{4}, \frac{1}{2})) = \psi(\frac{1}{2}) = \frac{3}{2} \leq M(x,y) \leq \psi(M(x,y)) - \phi(M(x,y)).$$

Thus, the mappings f, g, S and T satisfy the condition (2.1). Therefore all conditions given in Theorem 2.10 are satisfied. Moreover, 0 is the unique common fixed point of f, g, S and T.

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