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Coincidence point and common fixed point results via scalarization function

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Abstract

The main purpose of this paper is to obtain sufficient conditions for existence of points of coincidence and common fixed points for three self mappings in b-metric spaces. Next, we obtain cone b-metric version of these results by using a scalarization function. Our results extend and generalize several well known comparable results in the existing literature.

Keywords: Cone b-metric space; scalarization function; point of coincidence; common fixed point.
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1. Introduction

Over the past two decades a considerable amount of research work for the development of fixed point theory have executed by several mathematicians. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a b-metric space initiated by Bakhtin[5] and Czerwik [8]. In [14], Huang and Zhang introduced the concept of cone metric spaces as a generalization of metric spaces and proved some important fixed point theorems in such spaces. After that a series of articles have been dedicated to the improvement of fixed point theory. In most of those articles, the authors used normality property of cones in their results. Recently, Hussain and Shah[15] introduced the concept of cone b-metric spaces and studied some topological properties. In this work, we shall establish sufficient conditions for existence of points of coincidence and common fixed points for three self mappings in b-metric spaces. Finally, we prove cone b-metric version of these results by employing a scalarization function.

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2. Preliminaries

In this section we need to recall some basic notations, definitions, and necessary results from existing literature.

Definition 2.1. [8] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

Observe that if $s = 1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true. The following example illustrates the above remarks.

Example 2.2. Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.3. [22] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

Definition 2.4. [7] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.

(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 2.5. [7] In a b-metric space (X, d), the following assertions hold:

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

The following example shows that a b-metric need not be continuous.

Example 2.6. [17] Let $X = \mathbb{N} \cup \infty$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m,n) = \begin{cases} 0, \; if \; m = n, \\ \\ \mid \frac{1}{m} - \frac{1}{n} \mid, \; if \; one \; of \; m,n \; is \; even \; and \; the \; other \; is \; even \; or \; \infty, \\ \\ 5, \; if \; one \; of \; m,n \; is \; odd \; and \; the \; other \; is \; odd \, (and \; m \neq n) \; or \; \infty, \\ \\ 2, \; otherwise. \end{cases}$$

Then considering all possible cases, it can be checked that for all $m, n, p \in X$, we have

$$d(m,p) \le \frac{5}{2}(d(m,n) + d(n,p)).$$

Then, (X, d) is a b-metric space (with $s = \frac{5}{2}$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \frac{1}{2n} \to 0 \; as \; n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \not\to 5 = d(\infty, 1)$ as $n \to \infty$.

Theorem 2.7. [2] Let (X, d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have

$$\frac{1}{s^2} d(x,y) \le \liminf_{n\to\infty} \; d(x_n, y_n) \le \limsup_{n\to\infty} \; d(x_n, y_n) \le s^2 d(x,y).$$

In particular, if $x = y$, then $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \le \liminf_{n\to\infty} \; d(x_n, z) \le \limsup_{n\to\infty} \; d(x_n, z) \le sd(x,z).$$

Definition 2.8. [4] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.9. [19] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \; whenever \; Sx = Tx.$$

Proposition 2.10. [3] Let X be a nonempty set and the mappings S, T, $f : X \to X$ have a unique point of coincidence in X. If (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point.

Definition 2.11. Let (X, d) be a b-metric space with the coefficient $s \ge 1$. A mapping $T : X \to X$ is called expansive if there exists a real constant $k > s$ such that

$$d(Tx, Ty) \ge k \, d(x,y)$$

for all $x, y \in X$.

3. Main Results

In this section, we prove some point of coincidence and common fixed point results in b-metric spaces.

Theorem 3.1. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings f, g, $T : X \to X$ satisfy*

$$d(Tx, fy) \leq \alpha\, d(gx, gy) + \beta\, d(gx, Tx) + \gamma\, d(gy, fy) \tag{3.1}$$

for all $x, y \in X$, where α, β, $\gamma \geq 0$ with $\alpha + \beta + \gamma < \frac{1}{s}$. If $T(X) \cup f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, g and T have a unique point of coincidence in X. Moreover, if the pairs (T, g) and (f, g) are weakly compatible, then f, g and T have a unique common fixed point in X.

Proof . Let $x_0 \in X$ be arbitrary and choose a point $x_1 \in X$ such that $gx_1 = Tx_0$. This is possible since $T(X) \subseteq g(X)$. Similarly, choose a point $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process, we can construct a sequence (x_n) in X such that $gx_{2k+1} = Tx_{2k},\ gx_{2k+2} = fx_{2k+1}$ for $k \geq 0$.
By (3.1), we have

$$\begin{aligned} d(gx_{2k+1}, gx_{2k+2}) &= d(Tx_{2k}, fx_{2k+1}) \\ &\leq \alpha d(gx_{2k}, gx_{2k+1}) + \beta d(gx_{2k}, Tx_{2k}) + \gamma d(gx_{2k+1}, fx_{2k+1}) \\ &= \alpha d(gx_{2k}, gx_{2k+1}) + \beta d(gx_{2k}, gx_{2k+1}) + \gamma d(gx_{2k+1}, gx_{2k+2}) \end{aligned}$$

which gives that,

$$d(gx_{2k+1}, gx_{2k+2}) \leq \frac{\alpha + \beta}{1 - \gamma} d(gx_{2k}, gx_{2k+1}). \tag{3.2}$$

Again,

$$\begin{aligned} d(gx_{2k+2}, gx_{2k+3}) &= d(fx_{2k+1}, Tx_{2k+2}) = d(Tx_{2k+2}, fx_{2k+1}) \\ &\leq \alpha d(gx_{2k+2}, gx_{2k+1}) + \beta d(gx_{2k+2}, Tx_{2k+2}) + \gamma d(gx_{2k+1}, fx_{2k+1}) \\ &= \alpha d(gx_{2k+2}, gx_{2k+1}) + \beta d(gx_{2k+2}, gx_{2k+3}) + \gamma d(gx_{2k+1}, gx_{2k+2}) \end{aligned}$$

which gives that,

$$d(gx_{2k+2}, gx_{2k+3}) \leq \frac{\alpha + \gamma}{1 - \beta} d(gx_{2k+1}, gx_{2k+2}). \tag{3.3}$$

Let $\lambda = max\left(\frac{\alpha+\beta}{1-\gamma}, \frac{\alpha+\gamma}{1-\beta}\right)$. It is easy to see that $\lambda \in [0, \frac{1}{s})$.
Combining (3.2) and (3.3), we get

$$d(gx_n, gx_{n+1}) \leq \lambda\, d(gx_{n-1}, gx_n)\ for\ all\ n \geq 1. \tag{3.4}$$

By repeated application of (3.4), we obtain

$$d(gx_n, gx_{n+1}) \leq \lambda^n\, d(gx_0, gx_1). \tag{3.5}$$

For $m, n \in \mathbb{N}$ with $m > n$, we have by repeated use of (3.5)

$$\begin{aligned}
d(gx_n, gx_m) &\leq s\left[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)\right] \\
&\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\
&\quad + s^{m-n-1}\left[d(gx_{m-2}, gx_{m-1}) + d(gx_{m-1}, gx_m)\right] \\
&\leq \left[s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n-1}\lambda^{m-1}\right] d(gx_0, gx_1) \\
&\leq \left[s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}\right] d(gx_0, gx_1) \\
&= s\lambda^n\left[1 + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-2} + (s\lambda)^{m-n-1}\right] d(gx_0, gx_1) \\
&\leq \frac{s\lambda^n}{1 - s\lambda} d(gx_0, gx_1).
\end{aligned}$$

So (gx_n) is a Cauchy sequence in $g(X)$. Since $g(X)$ is a complete subspace of X, then there exists $y \in g(X)$ such that $gx_n \to y$ as $n \longrightarrow \infty$. Consequently, there is an $u \in X$ such that $gu = y$.

Now,

$$\begin{aligned}
d(gu, fu) &\leq s[d(gu, gx_{2n+1}) + d(gx_{2n+1}, fu)] \\
&= s[d(gu, gx_{2n+1}) + d(Tx_{2n}, fu)] \\
&\leq s[d(gu, gx_{2n+1}) + \alpha d(gx_{2n}, gu) + \beta d(gx_{2n}, Tx_{2n}) + \gamma d(gu, fu)] \\
&\leq s[d(gu, gx_{2n+1}) + \alpha d(gx_{2n}, gu) + s\beta d(gx_{2n}, gu) \\
&\quad + s\beta d(gu, Tx_{2n}) + \gamma d(gu, fu)] \\
&= s[d(gu, gx_{2n+1}) + \alpha d(gx_{2n}, gu) + s\beta d(gx_{2n}, gu) \\
&\quad + s\beta d(gu, gx_{2n+1}) + \gamma d(gu, fu)]
\end{aligned}$$

which gives that

$$d(gu, fu) \leq \frac{s + \beta s^2}{1 - \gamma s} d(gu, gx_{2n+1}) + \frac{\alpha s + \beta s^2}{1 - \gamma s} d(gx_{2n}, gu).$$

Taking limit as $n \to \infty$, we have $d(gu, fu) = 0$, i.e., $gu = fu$.
Again, by using (3.1)

$$\begin{aligned}
d(Tu, fu) &\leq \alpha d(gu, gu) + \beta d(gu, Tu) + \gamma d(gu, fu) \\
&= \beta d(fu, Tu).
\end{aligned}$$

This implies that $d(Tu, fu) = 0$ and so, $fu = Tu$. Therefore, $fu = gu = Tu = y$ and hence y is a common point of coincidence of f, g and T in X.

For uniqueness, assume that there exists another point of coincidence v in X such that $gx = fx = Tx = v$ for some $x \in X$. Then,

$$\begin{aligned}
d(v, y) = d(Tx, fu) &\leq \alpha\, d(gx, gu) + \beta\, d(gx, Tx) + \gamma\, d(gu, fu) \\
&= \alpha\, d(v, y) + \beta\, d(v, v) + \gamma\, d(y, y) \\
&= \alpha\, d(v, y).
\end{aligned}$$

This gives that $d(v, y) = 0$ i.e., $v = y$.
Therefore, f, g and T have a unique point of coincidence in X.

If the pairs $(f,\, g)$ and $(T,\, g)$ are weakly compatible, then by Proposition 2.10 , f, g and T have a unique common fixed point in X. $\square$

Corollary 3.2. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f,\, g : X \to X$ satisfy*

$$d(fx, fy) \leq \alpha\, d(gx, gy) + \beta\, d(gx, fx) + \gamma\, d(gy, fy)$$

for all $x, y \in X$, where $\alpha,\, \beta,\, \gamma \geq 0$ with $\alpha + \beta + \gamma < \frac{1}{s}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof . The proof follows from Theorem 3.1 by taking $T = f$.
□

Corollary 3.3. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ satisfies*

$$d(fx, fy) \leq \alpha\, d(x, y) + \beta\, d(x, fx) + \gamma\, d(y, fy)$$

for all $x, y \in X$, where $\alpha,\, \beta,\, \gamma \geq 0$ with $\alpha + \beta + \gamma < \frac{1}{s}$. Then f has a unique fixed point in X.

Proof . Taking $T = f$ and $g = I$, the identity map on X in Theorem 3.1, we obtain the desired result.
□

Corollary 3.4. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose $g : X \to X$ is onto and satisfies*

$$d(gx, gy) \geq k\, d(x, y))$$

for all $x, y \in X$, where $k > s$ is a constant. Then g has a unique fixed point in X.

Proof . Taking $T = f = I$ and $\beta = \gamma = 0$ in Theorem 3.1, we obtain the desired result.
□

Remark 3.5. Corollary 3.4 gives a sufficient condition for the existence of unique fixed point of an expansive mapping in b-metric spaces.

Theorem 3.6. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f,\, g,\, T : X \to X$ satisfy*

$$d(Tx, fy) \leq \beta\, d(Tx, gy) + \gamma\, d(fy, gx) \tag{3.6}$$

for all $x, y \in X$, where $\beta,\, \gamma \geq 0$ with max $\{\beta,\, \gamma\} < \frac{1}{s(1+s)}$. If $T(X) \cup f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f, g and T have a unique point of coincidence in X. Moreover, if the pairs $(T,\, g)$ and $(f,\, g)$ are weakly compatible, then f, g and T have a unique common fixed point in X.

Proof . Let $x_0 \in X$ be arbitrary. Following similar arguments to those given in Theorem 3.1, we can construct a sequence (x_n) in X such that $gx_{2k+1} = Tx_{2k},\; gx_{2k+2} = fx_{2k+1}$ for $k \geq 0$.
Using (3.6), we have

$$\begin{aligned} d(gx_{2k+1}, gx_{2k+2}) &= d(Tx_{2k}, fx_{2k+1}) \\ &\leq \beta d(Tx_{2k}, gx_{2k+1}) + \gamma d(fx_{2k+1}, gx_{2k}) \\ &= \beta d(gx_{2k+1}, gx_{2k+1}) + \gamma d(gx_{2k+2}, gx_{2k}) \\ &\leq \gamma\, s[d(gx_{2k+2}, gx_{2k+1}) + d(gx_{2k+1}, gx_{2k})] \end{aligned}$$

which gives that,

$$d(gx_{2k+1}, gx_{2k+2}) \leq \frac{\gamma\, s}{1-\gamma\, s} d(gx_{2k}, gx_{2k+1}). \tag{3.7}$$

Again,

$$\begin{aligned} d(gx_{2k+2}, gx_{2k+3}) &= d(fx_{2k+1}, Tx_{2k+2}) = d(Tx_{2k+2}, fx_{2k+1}) \\ &\leq \beta d(Tx_{2k+2}, gx_{2k+1}) + \gamma d(fx_{2k+1}, gx_{2k+2}) \\ &= \beta d(gx_{2k+3}, gx_{2k+1}) + \gamma d(fx_{2k+1}, fx_{2k+1}) \\ &\leq \beta\, s[d(gx_{2k+3}, gx_{2k+2}) + d(gx_{2k+2}, gx_{2k+1})] \end{aligned}$$

which gives that,

$$d(gx_{2k+2}, gx_{2k+3}) \leq \frac{\beta\, s}{1-\beta\, s} d(gx_{2k+1}, gx_{2k+2}). \tag{3.8}$$

Let $\lambda = max\left(\frac{\gamma s}{1-\gamma s}, \frac{\beta s}{1-\beta s}\right)$. It is easy to see that $\lambda \in [0, \frac{1}{s})$.
Combining (3.7) and (3.8), we get

$$d(gx_n, gx_{n+1}) \leq \lambda\, d(gx_{n-1}, gx_n) \ for\ all\ n \geq 1. \tag{3.9}$$

By repeated application of (3.9), we obtain

$$d(gx_n, gx_{n+1}) \leq \lambda^n\, d(gx_0, gx_1).$$

By an argument similar to that used in Theorem 3.1, it follows that (gx_n) is a Cauchy sequence in $g(X)$. Since $g(X)$ is a complete subspace of X, there exists $y \in g(X)$ such that $gx_n \to y$ as $n \longrightarrow \infty$. Consequently, there is an $u \in X$ such that $gu = y$.

Now,

$$\begin{aligned} d(gu, fu) &\leq s[d(gu, gx_{2n+1}) + d(gx_{2n+1}, fu)] \\ &= s[d(gu, gx_{2n+1}) + d(Tx_{2n}, fu)] \\ &\leq s[d(gu, gx_{2n+1}) + \beta d(Tx_{2n}, gu) + \gamma d(fu, gx_{2n})] \\ &\leq s[d(gu, gx_{2n+1}) + \beta d(gx_{2n+1}, gu) + s\gamma d(fu, gu) + s\gamma d(gu, gx_{2n})] \end{aligned}$$

which gives that

$$d(gu, fu) \leq \frac{s+\beta s}{1-\gamma s^2} d(gu, gx_{2n+1}) + \frac{\gamma s^2}{1-\gamma s^2} d(gu, gx_{2n}).$$

Taking limit as $n \to \infty$, we have $d(gu, fu) = 0$, i.e., $gu = fu$.
Again, by using (3.6)

$$\begin{aligned} d(Tu, fu) &\leq \beta d(Tu, gu) + \gamma d(fu, gu) \\ &= \beta d(Tu, fu). \end{aligned}$$

This implies that $d(Tu, fu) = 0$ and so, $fu = Tu$. Therefore, $fu = gu = Tu = y$ and hence y is a common point of coincidence of f, g and T in X.

For uniqueness, assume that there exists another point of coincidence v in X such that $gx = fx = Tx = v$ for some $x \in X$. Then,

$$\begin{aligned} d(v,y) = d(Tx,fu) &\leq \beta\, d(Tx,gu) + \gamma\, d(fu,gx) \\ &= \beta\, d(v,y) + \gamma\, d(y,v) \\ &= (\beta+\gamma) d(v,y). \end{aligned}$$

This gives that $d(v,y) = 0$ i.e., $v = y$.
Therefore, f, g and T have a unique point of coincidence in X.

If the pairs $(f,\, g)$ and $(T,\, g)$ are weakly compatible, then by Proposition 2.10 , f, g and T have a unique common fixed point in X. □

Corollary 3.7. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f,\, g : X \to X$ satisfy*

$$d(fx,fy) \leq \beta\, d(fx,gy) + \gamma\, d(fy,gx)$$

for all $x,y \in X$, where $\beta,\, \gamma \geq 0$ with max $\{\beta,\, \gamma\} < \frac{1}{s(1+s)}$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof . Proof follows from Theorem 3.6 by taking $T = f$.
□

Corollary 3.8. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose $f : X \to X$ satisfies*

$$d(fx,fy) \leq \beta\, [d(x,fy) + d(y,fx)]$$

for all $x,y \in X$, where $0 \leq \beta < \frac{1}{s(1+s)}$. Then f has a unique fixed point in X.

Proof . Proof follows from Theorem 3.6 by taking $T = f$, $g = I$ and $\gamma = \beta$.
□

Theorem 3.9. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping such that for each positive integer n,*

$$d(T^n x, T^n y) \leq a_n\, d(x,y) \tag{3.10}$$

for all $x,y \in X$, where $a_n > 0$ is independent of x, y. If the series $\displaystyle\sum_{n=1}^{\infty} s^n a_n$ is convergent, then T has a unique fixed point in X.

Proof . Let $x_0 \in X$ be arbitrary. We can construct a sequence (x_n) in X such that $x_n = Tx_{n-1} = T^n x_0$ for $n = 1,\, 2,\, 3,\, \cdots$.
Then by using (3.10), we get

$$d(x_n, x_{n+1}) = d(T^n x_0, T^n x_1) \leq a_n\, d(x_0, x_1). \tag{3.11}$$

For all m, $n \in \mathbb{N}$ with $m > n$, by repeated application of (3.11)

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + s^{m-n-1} d(x_{m-2}, x_{m-1}) + s^{m-n-1} d(x_{m-1}, x_m) \\ &\leq [sa_n + s^2 a_{n+1} + \cdots + s^{m-n-1} a_{m-2} + s^{m-n} a_{m-1}] d(x_0, x_1) \\ &\leq [s^n a_n + s^{n+1} a_{n+1} + \cdots + s^{m-2} a_{m-2} + s^{m-1} a_{m-1}] d(x_0, x_1) \\ &= \left(\sum_{r=n}^{m-1} s^r a_r \right) d(x_0, x_1). \end{aligned} \tag{3.12}$$

If $x_1 = x_0$, then a fixed point of T is obtained. So, we assume that $x_1 \neq x_0$. Let k be a positive integer with $k > d(x_0, x_1)$. Since the series $\sum_{n=1}^{\infty} s^n a_n$ is convergent, for $\epsilon > 0$, there exists a positive integer n_0 such that

$$\sum_{r=n}^{m-1} s^r a_r < \frac{\epsilon}{k} \ if \ m > n > n_0.$$

It follows from (3.12) that for $m > n > n_0$,

$$d(x_n, x_m) < \frac{\epsilon}{k} d(x_0, x_1) < \epsilon.$$

So, (x_n) is a Cauchy sequence in X. Since X is complete there exists $u \in X$ such that $x_n \to u$. Now,

$$\begin{aligned} d(u, Tu) &\leq s[d(u, x_{n+1}) + d(x_{n+1}, Tu)] \\ &= s[d(u, x_{n+1}) + d(Tx_n, Tu)] \\ &\leq s[d(u, x_{n+1}) + a_1 d(x_n, u)] \\ &\longrightarrow 0 \quad as \ n \to \infty, \end{aligned}$$

which gives that, $Tu = u$ and so, u becomes a fixed point of T.
For uniqueness, assume that there exists another fixed point v in X such that $Tv = v$. Then,

$$d(u, v) = d(T^n u, T^n v) \leq a_n \, d(u, v).$$

If $d(u, v) > 0$, then $a_n \geq 1$ for all n. Since $s \geq 1$, it follows that $s^n a_n \not\longrightarrow 0$ as $n \to \infty$. which contradicts the fact that the series $\sum_{n=1}^{\infty} s^n a_n$ is convergent. Therefore, $d(u, v) = 0$ and so, $u = v$. □

As an application of Theorem 3.9, we have the following result.

Theorem 3.10. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq k \, d(x, y) \tag{3.13}$$

for all $x, y \in X$, where $k \in [0, \frac{1}{s})$ is a constant. Then T has a unique fixed point in X.

Proof . For $x, y \in X$, we obtain by using (3.13) that

$$d(T^2 x, T^2 y) \leq k \, d(Tx, Ty) \leq k^2 d(x, y).$$

By induction, we have

$$d(T^n x, T^n y) \leq k^n d(x,y)$$

for all $x, y \in X$. Since $k \in [0, \frac{1}{s})$, it follows that the series $\sum_{n=1}^{\infty} s^n k^n$ is convergent. So, Theorem 3.9 applies to obtain a unique fixed point of T. □

We conclude with an example.

Example 3.11. Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}^+$ be such that

$$d(x,y) = \mid x - y \mid^p$$

for any $x,\, y \in X$, where $p > 1$ is a constant. Then (X, d) is a b-metric space with $s = 2^{p-1}$. Let us define $T, f, g : X \to X$ as

$$\begin{aligned} fx &= \frac{x}{16},\ for\ all\ x \in \left[0, \frac{1}{2}\right) \\ &= \frac{x}{12},\ for\ all\ x \in \left[\frac{1}{2}, 1\right]; \end{aligned}$$

$$\begin{aligned} Tx &= \frac{x}{12},\ for\ all\ x \in \left[0, \frac{1}{2}\right) \\ &= \frac{x}{16},\ for\ all\ x \in \left[\frac{1}{2}, 1\right] \end{aligned}$$

and

$$gx = \frac{x}{2},\ for\ all\ x \in X.$$

Now we verify that for every $x, y \in X$ one has

$$d(Tx, fy) \leq \alpha\, d(gx, gy) + \beta\, d(gx, Tx) + \gamma\, d(gy, fy)$$

where $\alpha,\, \beta,\, \gamma \geq 0$ with $\alpha + \beta + \gamma < \frac{1}{s}$.

Case-I If $x, y \in [0, \frac{1}{2})$, then

$$\begin{aligned} d(Tx, fy) &= \mid Tx - fy \mid^p = \mid \frac{x}{12} - \frac{y}{16} \mid^p \leq 2^{p-1} \left((\frac{x}{12})^p + (\frac{y}{16})^p \right) \\ &= \frac{2^{p-1}}{5^p} \left((\frac{5x}{12})^p + (\frac{5y}{16})^p \right) \leq \frac{2^{p-1}}{5^p} \left((\frac{5x}{12})^p + (\frac{7y}{16})^p \right). \end{aligned}$$

Also,

$$\begin{aligned} d(Tx, gx) + d(fy, gy) &= \mid Tx - gx \mid^p + \mid fy - gy \mid^p \\ &= \mid \frac{x}{12} - \frac{x}{2} \mid^p + \mid \frac{y}{16} - \frac{y}{2} \mid^p \\ &= (\frac{5x}{12})^p + (\frac{7y}{16})^p. \end{aligned}$$

Therefore,

$$d(Tx, fy) \leq \frac{2^{p-1}}{5^p}\,[d(Tx, gx) + d(fy, gy)].$$

Case-II If $x, y \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} d(Tx, fy) &= \mid Tx - fy \mid^p = \mid \frac{x}{16} - \frac{y}{12} \mid^p \leq 2^{p-1} \left((\frac{x}{16})^p + (\frac{y}{12})^p \right) \\ &= \frac{2^{p-1}}{5^p} \left((\frac{5x}{16})^p + (\frac{5y}{12})^p \right) < \frac{2^{p-1}}{5^p} \left((\frac{7x}{16})^p + (\frac{5y}{12})^p \right). \end{aligned}$$

Also,

$$\begin{aligned} d(Tx, gx) + d(fy, gy) &= \mid Tx - gx \mid^p + \mid fy - gy \mid^p \\ &= \mid \frac{x}{16} - \frac{x}{2} \mid^p + \mid \frac{y}{12} - \frac{y}{2} \mid^p \\ &= (\frac{7x}{16})^p + (\frac{5y}{12})^p. \end{aligned}$$

Therefore,

$$d(Tx, fy) < \frac{2^{p-1}}{5^p}\,[d(Tx, gx) + d(fy, gy)].$$

Case-III If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} d(Tx, fy) &= \mid Tx - fy \mid^p = \mid \frac{x}{12} - \frac{y}{12} \mid^p \leq 2^{p-1} \left((\frac{x}{12})^p + (\frac{y}{12})^p \right) \\ &= \frac{2^{p-1}}{12^p} (x^p + y^p) = \frac{2^{p-1}}{5^p} . \frac{5^p}{12^p} (x^p + y^p). \end{aligned}$$

Also,

$$\begin{aligned} d(Tx, gx) + d(fy, gy) &= \mid Tx - gx \mid^p + \mid fy - gy \mid^p \\ &= \mid \frac{x}{12} - \frac{x}{2} \mid^p + \mid \frac{y}{12} - \frac{y}{2} \mid^p \\ &= (\frac{5x}{12})^p + (\frac{5y}{12})^p \\ &= \frac{5^p}{12^p} (x^p + y^p). \end{aligned}$$

Therefore,

$$d(Tx, fy) \leq \frac{2^{p-1}}{5^p}\,[d(Tx, gx) + d(fy, gy)].$$

Case-IV If $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2})$, then

$$\begin{aligned} d(Tx, fy) &= \mid Tx - fy \mid^p = \mid \frac{x}{16} - \frac{y}{16} \mid^p \leq 2^{p-1} \left((\frac{x}{16})^p + (\frac{y}{16})^p \right) \\ &= \frac{2^{p-1}}{16^p} (x^p + y^p) = \frac{2^{p-1}}{7^p} . \frac{7^p}{16^p} (x^p + y^p). \end{aligned}$$

Also,

$$\begin{aligned} d(Tx, gx) + d(fy, gy) &= \mid Tx - gx \mid^p + \mid fy - gy \mid^p \\ &= \mid \frac{x}{16} - \frac{x}{2} \mid^p + \mid \frac{y}{16} - \frac{y}{2} \mid^p \\ &= (\frac{7x}{16})^p + (\frac{7y}{16})^p \\ &= \frac{7^p}{16^p} (x^p + y^p). \end{aligned}$$

Therefore,

$$d(Tx, fy) \leq \frac{2^{p-1}}{7^p}\,[d(Tx, gx) + d(fy, gy)] < \frac{2^{p-1}}{5^p}\,[d(Tx, gx) + d(fy, gy)].$$

Thus, we have

$$d(Tx, fy) \leq \alpha\, d(gx, gy) + \beta\, d(gx, Tx) + \gamma\, d(gy, fy)$$

for all $x, y \in X$, where $\alpha = \beta = \gamma = \frac{2^{p-1}}{5^p}$ with $s(\alpha + \beta + \gamma) = 3s.\frac{2^{p-1}}{5^p} < 4s.\frac{2^{p-1}}{5^p} = (\frac{4}{5})^p < 1$ since $s = 2^{p-1}$.
We see that $T(X) \cup f(X) \subseteq g(X)$, $g(X)$ is complete, (T, g) and (f, g) are weakly compatible. Therefore, all the conditions of Theorem 3.1 are satisfied and $0 \in X$ is the unique common fixed point of f, g and T.

4. Fixed points via scalarization functions

Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

(i) *P is closed, nonempty and $P \neq \{\theta\}$;*

(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \ \Rightarrow\ ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

For any cone $P \subseteq E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$(equivalently, $y \succeq x$) if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \textit{ implies } \|x\| \leq k\,\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. Throughout this section, we suppose that E is a real Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\preceq$ is a partial ordering on E with respect to P.

Definition 4.1. [14] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) *$\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;*

(ii) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*

(iii) *$d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 4.2. [15] Let X be a nonempty set and E a real Banach space with cone P. A vector valued function $p : X \times X \to E$ is said to be a cone b-metric function on X with the constant $s \geq 1$ if the following conditions are satisfied:

(i) *$\theta \preceq p(x, y)$ for all $x, y \in X$ and $p(x, y) = \theta$ if and only if $x = y$;*

(ii) *$p(x, y) = p(y, x)$ for all $x, y \in X$;*

$p(x, y) \preceq s\,(p(x, z) + p(z, y))$ for all $x, y, z \in X$.

The pair (X, p) is called a cone b-metric space.

Definition 4.3. [10, 11, 12] The nonlinear scalarization function $\xi_e : E \to \mathbb{R}$, where $e \in int\,(P)$ is defined as follows:

$$\xi_e(y) = inf\{r \in \mathbb{R} : y \in re - P\} \; for \; all \; y \in E.$$

Lemma 4.4. [10, 11, 12] For each $r \in \mathbb{R}$ and $y \in E$, the following statements are satisfied:

(i) $\xi_e(y) \leq r \Longleftrightarrow y \in re - P$,

(ii) $\xi_e(y) > r \Longleftrightarrow y \notin re - P$,

(iii) $\xi_e(y) \geq r \Longleftrightarrow y \notin re - int\,(P)$,

(iv) $\xi_e(y) < r \Longleftrightarrow y \in re - int\,(P)$,

(v) $\xi_e(\cdot)$ is positively homogeneous and continuous on E,

(vi) if $y_1 \in y_2 + P\,(i.e.\, y_2 \preceq y_1)$, then $\xi_e(y_2) \leq \xi_e(y_1)$,

(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ for all $y_1,\, y_2 \in E$.

Remark 4.5. [11]

(a) Clearly $\xi_e(\theta) = 0$.

(b) It is worth mentioning that the reverse statement of (vi) in Lemma 4.4 does not hold in general.

Theorem 4.6. [11] Let (X, p) be a cone b-metric space. Then, $d_p : X \times X \to [0, \infty)$ defined by $d_p = \xi_e \circ p$ is a b-metric.

Definition 4.7. [15] Let (X, p) be a cone b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n > n_0$, $p(x_n, x) \ll c$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$;

(ii) (x_n) is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n, m > n_0$, $p(x_n, x_m) \ll c$;

(iii) (X, p) is a complete cone b-metric space if every Cauchy sequence is convergent.

Theorem 4.8. [11] Let (X, p) be a cone b-metric space, $x \in X$ and (x_n) be a sequence in X. Set $d_p = \xi_e \circ p$. Then the following statements hold:

(i) (x_n) converges to x in cone b-metric space (X, p) if and only if $d_p(x_n, x) \to 0$ as $n \to \infty$,

(ii) (x_n) is a Cauchy sequence in cone b-metric space (X, p) if and only if (x_n) is a Cauchy sequence in (X, d_p),

(iii) (X, p) is a complete cone b-metric space if and only if (X, d_p) is a complete b-metric space.

Theorem 4.9. *Let (X, p) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings f, g, $T : X \to X$ satisfy*

$$p(Tx, fy) \preceq \alpha\, p(gx, gy)) + \beta\, p(gx, Tx) + \gamma\, p(gy, fy) \tag{4.1}$$

for all $x, y \in X$, where α, β, $\gamma \geq 0$ with $\alpha + \beta + \gamma < \frac{1}{s}$. If $T(X) \cup f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of (X, p), then f, g and T have a unique point of coincidence in X. Moreover, if the pairs (T, g) and (f, g) are weakly compatible, then f, g and T have a unique common fixed point in X.

Proof . Taking $d_p = \xi_e \circ p$, it follows that d_p is a b-metric. Using Theorem 4.8, we conclude that $g(X)$ is a complete subspace of (X, d_p). By applying Lemma 4.4, we obtain from (4.1) that

$$d_p(Tx, fy) \leq \alpha\, d_p(gx, gy)) + \beta\, d_p(gx, Tx) + \gamma\, d_p(gy, fy)$$

for all $x, y \in X$, where α, β, $\gamma \geq 0$ with $\alpha + \beta + \gamma < \frac{1}{s}$. Now, Theorem 3.1 applies to obtain the desired result. □

By using the techniques above, we can derive the following theorems.

Theorem 4.10. Let (X, p) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings f, g, $T : X \to X$ satisfy

$$p(Tx, fy) \preceq \beta\, p(Tx, gy) + \gamma\, p(fy, gx)$$

for all $x, y \in X$, where β, $\gamma \geq 0$ with $max\,\{\beta, \gamma\} < \frac{1}{s(1+s)}$. If $T(X) \cup f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of (X, p), then f, g and T have a unique point of coincidence in X. Moreover, if the pairs (T, g) and (f, g) are weakly compatible, then f, g and T have a unique common fixed point in X.

Theorem 4.11. Let (X, p) be a complete cone b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a mapping such that for each positive integer n,

$$p(T^n x, T^n y) \preceq a_n\, p(x, y)$$

for all $x, y \in X$, where $a_n > 0$ is independent of x, y. If the series $\displaystyle\sum_{n=1}^{\infty} s^n a_n$ is convergent, then T has a unique fixed point in X.

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