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# On the $\Phi$-reflexive property of $(X, \Upsilon)$-structures

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**Abstract**

We use $\Phi$-reflexive property on some geometrical structures(Frölicher spaces, Sikorski spaces and diffeological spaces) to prove that some results on $(X, \Upsilon)$-structures. Finally, we introduce $\mathcal{P}$-tangent bundles, $\mathcal{F}$-tangent bundles and obtain a relation between these bundles and $\Phi$-reflexive property.

*Keywords:* $\Phi$-reflexive property, differential space, diffeology, $\mathcal{P}$-tangent bundle, $\mathcal{F}$-tangent bundle.
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## 1. Introduction

There are many structures which include manifolds as special cases in the literature. Differential spaces are introduced by Roman Sikorski in 1971 [11]. In 1980, Jean-Marie Souriau presented the diffeological spaces which are developed by Patrick Iglesias and Paul Donato [8]. Alfred Frölicher offered the Frölcher spaces in 1982 [7]. Batubenge and others (2013) offered reflexive concept. They used this concept for comparing the subcategories of Frölicher, Sikorski and diffeological structures. They showed that the categories of reflexive diffeological spaces, Frölicher spaces and reflexive differential spaces are isomorphic. In the 2015s, Dehghan Nezhad and Shahriyari introduced $(X, \Upsilon)$-structures which are a generalization of manifolds, Frölicher spaces, Sikorski spaces and diffeological spaces [3]. The $(X, \Upsilon)$-structures include all above structures. We focus on diffeology, differential structures (in the sense of Sikorski), Frölcher structures and $(X, \Upsilon)$-structures. They can be assumed as special cases of $(X, \Upsilon)$-structures [3]. In this paper, we generalize the reflexive concept proposed. This generalization concept is called $\Phi$-reflexive property. We use this new concept to obtain some interesting results on some above structures. We introduce $\mathcal{P}$-tangent bundle and $\mathcal{F}$-tangent bundle. Finally, we obtain a relation between these bundles and $\Phi$-reflexive property.

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## 2. Preliminaries and notations

In this section, we summarize the general preliminary definitions of $(X, \Upsilon)$-structures and we exclude special cases. We introduce the basic concepts using the $(X, \Upsilon)$-structure. We repeat the relevant material from [3], [4] and [5], for the convenience of the reader. Throughout this paper, suppose that $M$ is a non-empty set.

**Definition 2.1. (Pseudomonoid)**[3] *Assume that $X$ is a topological space and assume that $\Upsilon$ is a collection of continuous maps on open subsets of $X$ into $X$. The collection $\Upsilon$ is said a* **pseudomonoid on** $X$ *if $\Upsilon$ satisfies the following conditions:*

- $id_X \in \Upsilon$,
- *If $f, g$ are two elements of $\Upsilon$ such that the image of $g$ is a subset of the domain of $f$, then $f \circ g \in \Upsilon$,*
- *Let $f \in \Upsilon$ and let the subset $V \subseteq dom(f)$ is open. Then the restriction map $f|_V$ is a member of $\Upsilon$.*

*The pair $(X, \Upsilon)$ is said a pseudomonoid.*

**Remark 2.2.** *Clearly, pseudomonoids are a natural generalization of pseudogroups.*

Let $M$ be a non-empty set and let $X$ be a topological space. Then a parametrization from $X$ into $M$ is a map $\phi : U \subseteq X \to M$ where $U \subseteq X$ is an open subset. A *cover* for $M$ is a collection $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in I}$ of parametrizations of $M$ such that $M = \cup_{\alpha \in I} \phi_\alpha(U_\alpha)$.

**Definition 2.3.** *Let $\Upsilon$ be a pseudomonoid on $X$. An $(X, \Upsilon)$-***atlas** *on $M$ is a cover $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ for $M$ such that if $\phi_\alpha \in \mathscr{A}$, $f \in \Upsilon$ and $\phi_\alpha \circ f$ is defined, then $\phi_\alpha \circ f \in \mathscr{A}$, A set endowed with an $(X, \Upsilon)$-atlas is called an $(X, \Upsilon)$-***structure** *and denoted by $(M, \mathscr{A})$ where, $\mathscr{A}$ is an $(X, \Upsilon)$-atlas on $M$.*

Under the above assumptions, any pseudomonoid $(X, \Upsilon)$ is an $(X, \Upsilon)$-structure too.

**Definition 2.4. (Diffeology)**[8] *A plot of $M$ is a map $p : U \subseteq \mathbb{R}^n \to M$ where the subset $U$ is open, for some $n \in \mathbb{N}$. A set $\mathcal{D}$ of plots is said a* **diffeology on** $M$ *if satisfying the following properties:*

- *The constant function $p : \mathbb{R}^n \to \{m\} \subseteq M$ is an element of $\mathcal{D}$, for every $m \in M$ and for all $n \in \mathbb{N}$.*
- *Assume that $p : U \to M$ is a plot where for any element $u \in U$ there exists an open subset $V \subseteq U$ contains $u$ such that $p\,|_V \in \mathcal{D}$. Then $p \in \mathcal{D}$.*
- *The map $p \circ F$ is an element of $\mathcal{D}$, for all $p : U \subseteq \mathbb{R}^n \to M \in \mathcal{D}$ and for all smooth map $F : V \subseteq \mathbb{R}^m \to U \subseteq \mathbb{R}^n (m, n \in \mathbb{N})$.*

*The pair $(M, \mathcal{D})$ is said a* **diffeological space**. *The strongest topology on $M$ which every plot is continuous is said* **D-topology**.

**Definition 2.5. (Differential Space)**[11] *A non-empty family $\mathcal{F}$ of real-functions on $M$, along with the weakest topology on $M$ is called a* **Sikorski structure (or differential structure)**, *on $M$ if the following conditions are being held:*

- *All elements of $\mathcal{F}$ are continuous functions,*
- *For every $f_1, ..., f_m \in \mathcal{F}$ and $F \in C^\infty(\mathbb{R}^m)$, composition $F(f_1, ..., f_m)$ is an element of $\mathcal{F}$,*
- *Assume that $f : M \to \mathbb{R}$ is a real function where for all $m \in M$, there exists an open subset $U_m \subseteq M$ contains $m$ and a function $g \in \mathcal{F}$ such that $f\mid_{U_m} = g\mid_{U_m}$ . Then $f$ is an element of $\mathcal{F}$.*

*The pair $(M, \mathcal{F})$ is said a* **Sikorski space (or differential space)**.

**Definition 2.6. (Frölicher spaces)***[7] A family of real-value functions $\mathcal{F}$ with a collection of curves $\mathcal{C}$ is said a* **Frölicher structure** *if $\mathcal{F}$ and $\mathcal{C}$ satisfy the following properties:*

- *A curve $c : \mathbb{R} \to M$ is an element of $\mathcal{C}$ if and only if $f \circ c \in C^\infty(\mathbb{R})$, for any function $f \in \mathcal{F}$.*
- *A real-value function $f : M \to \mathbb{R}$ is an element of $\mathcal{F}$ if and only if for any curve $c \in \mathcal{C}$, we have $f \circ c \in C^\infty(\mathbb{R})$.*

*The triple $(M, \mathcal{C}, \mathcal{F})$ is called a* **Frölicher space**.

**Proposition 2.7.** *[3] The $(X, \Upsilon)$-structures are generalization of manifolds, Frölicher spaces, Sikorski spaces and diffeological spaces.*

## 3. $\Phi$-reflexive property

In [2], authors offered the concept of reflexive. They used this concept to obtain isomorphisms between some subcategories of Fölicher, differential and diffeology spaces. In this section, we introduce a generalization of reflexive concept what is said $\Phi$-reflexive property.

**Definition 3.1.** *[2] Let $\mathcal{D}_0$ be a collection of parametrizations $p : U \subseteq \mathbb{R}^n \to M$ where the subset $U$ is open, for some $n \in \mathbb{N}$. Suppose that $\mathcal{F}_0$ is a family of functions $f : M \to \mathbb{R}$. Consider two following sets:*

$$\Pi\mathcal{F}_0 := \{p : U \to M \mid U \subseteq \mathbb{R}^n \text{ is an open subset, } n \in \mathbb{N} \text{ where } f \circ p \in C^\infty(U), \forall f \in \mathcal{F}_0\},$$
$$\Phi\mathcal{D}_0 := \{f : M \to \mathbb{R} \mid f \circ p \in C^\infty(U), \forall (p : U \to M) \in \mathcal{D}_0\}.$$

*The set $\mathcal{D}_0$ or $\mathcal{F}_0$ is called* **reflexive** *if $\mathcal{D}_0 = \Pi\Phi\mathcal{D}_0$ or $\mathcal{F}_0 = \Phi\Pi\mathcal{F}_0$(resp.).*

**Definition 3.2. (Reflexive Diffeologies)***[2] Assume that $\mathcal{D}$ is a diffeology on $M$. Then $\mathcal{D}$ is said reflexive if and only if $\mathcal{D} = \Pi\Phi\mathcal{D}$. Similarly, Let $(M, \mathcal{F})$ be a differential space. Then $\mathcal{F}$ is said reflexive if and only if $\mathcal{F} = \Phi\Pi\mathcal{F}$.*

**Proposition 3.3.** *[2] Three categories of reflexive diffeological spaces, Frölicher spaces and reflexive differential spaces are isomorphic.*

In the following definition, we propose $\Phi$-reflexive property.

**Definition 3.4.** *Suppose that $X, X_1, X_2$ are three topological spaces. An $X$-***parametrization** *of $M$ is a map $\phi : U \subseteq X \to M$ where the subset $U$ is open. An $X$-***function** *is a map from the whole of $M$ into topological space $X$.*
*Let $\Phi$ be a continuous $X_1$-parametrizations collection of $X_2$. Consider an $X_2$-functions family $\mathcal{F}_0$ on $M$ and consider a collection of $X_1$-parametrizations $\mathcal{P}_0$ of $M$. Define two following sets:*

$$\Phi_*\mathcal{F}_0 := \{X_1 - parametrizations\ p : U \to M \mid \text{for any } f \in \mathcal{F}, f \circ p \in \Phi\},$$
$$\Phi^*\mathcal{P}_0 := \{f : M \to X_2 \mid \text{for any } p \in \mathcal{P}, f \circ p \in \Phi\}.$$

**Lemma 3.5.** *Two operators $\Phi_*$ and $\Phi^*$ are inclusion-reserving and $\Phi_*\Phi^*\mathcal{P}_0 \supseteq \mathcal{P}_0$, $\Phi^*\Phi_*\mathcal{F}_0 \supseteq \mathcal{F}_0$.*

**Definition 3.6.** *We say a collection $\mathcal{P}_0$ of $M$ has* **$\Phi$-reflexive property** *if and only if $\Phi_*\Phi^*\mathcal{P}_0 = \mathcal{P}_0$. Similarly, We say an $X$-function family $\mathcal{F}_0$ on $M$ has* **$\Phi$-reflexive property** *if and only if $\Phi^*\Phi_*\mathcal{F}_0 = \mathcal{F}_0$.*

The $\Phi$-reflexive property defined in Definition 3.6 is a generalization of reflexive property defined in Definition 3.1 (see the following example).

**Example 3.7.** *Let $\Phi$ denote all smooth real-value functions on all open subsets of $\mathbb{R}^n$'s, for all $n \in \mathbb{N}$. Suppose that $\mathcal{P}_0$ is a parametrizations collection from some open subsets $\mathbb{R}^n$'s into $M$. Assume that $\mathcal{F}_0$ is a real-value functions family on $M$. The sets $\mathcal{F}_0$ and $\mathcal{P}_0$ have $\Phi$-reflexive property if and only if they are reflexive by the Definition 3.1.*

**Proposition 3.8.** *Suppose that $\Phi$ be a continuous $X_1$-parametrizations collection of a topological space $X_2$. Assume that $\mathcal{P}_0$ is an $X_1$-parametrizations collection of $M$ and $\mathcal{F}_0$ is a family of $X_2$-functions on $M$.*

- *If $\mathcal{F} := \Phi^*\mathcal{P}_0$, then $\mathcal{F}$ has $\Phi$-reflexive property.*
- *If $\mathcal{P} := \Phi_*\mathcal{F}_0$, then $\mathcal{P}$ has $\Phi$-reflexive property.*

**Proof .** We only prove the first statement. It is sufficient to show that $\Phi^*\Phi_*\mathcal{F} \subseteq \mathcal{F}$, by the Lemma 3.5. Fix $f_0 \in \Phi^*\Phi_*\mathcal{F}$. Then $f_0 \circ p \in \Phi$ for all $p \in \Phi_*\mathcal{F}$. By the Lemma 3.5, we obtain $\Phi_*\mathcal{F} = \Phi_*\Phi^*\mathcal{P}_0 \supseteq \mathcal{P}_0$. It follows immediately that $f_0 \circ p \in \Phi$ for any $p \in \mathcal{P}_0$. We conclude that $f_0 \in \Phi^*\mathcal{P}_0 = \mathcal{F}$. Therefore $\Phi_*\Phi^*\mathcal{F} \subseteq \mathcal{F}$. By similar arguments, the second statement is proved. □

**Proposition 3.9.** *Suppose that $\Phi$ is a continuous $X_1$-parametrizations collection of a topological space $X_2$. Then the collection $\Phi$ has $\Phi$-reflexive property.*

**Proof .** By the Lemma 3.5, it is sufficient to show that $\Phi_*\Phi^*\Phi \subseteq \Phi$. Get an arbitrary element $p_0 \in \Phi_*\Phi^*\Phi$. By definition, for any $f \in \Phi^*\Phi$, we have $f \circ p_0 \in \Phi$. Clearly, $id_{X_2} \in \Phi^*\Phi$. Therefore $p_0 = id_{X_2} \circ p_0 \in \Phi$. This completes the proof. □

**Proposition 3.10.** *Assume that $\Phi$ is a continuous $X_1$-parametrizations collection of a topological space $X_2$. Suppose that $\mathcal{F}_0$ is a family of $X_2$-functions on $M$ and the set $\mathcal{P}_0$ is a collection of $X_1$-parametrizations of $M$. We will denote the weakest topology on $M$ by $\mathcal{T}_{\mathcal{F}_0}$ where any members of $\mathcal{F}_0$ are continuous maps. We will denote the strongest topology on $M$ by $\mathcal{T}_{\mathcal{P}_0}$ where all members of $\mathcal{P}_0$ are continuous maps. $\mathcal{T}_{\mathcal{P}_0}$. If for all elements $f \in \mathcal{F}_0$ and for all members $p \in \mathcal{P}_0$, we have $f \circ p \in \Phi$, then $\mathcal{T}_{\mathcal{F}_0} \subseteq \mathcal{T}_{\mathcal{P}_0}$.*

**Proof .** By the definition of weakest topology, the following set $\mathcal{S}_{\mathcal{F}_0}$ is a sub-basis of $\mathcal{T}_{\mathcal{F}_0}$

$$\mathcal{S}_{\mathcal{F}_0} = \{f^{-1}(W) \subseteq M | f \in \mathcal{F}_0, W \subseteq X_2 \text{ is an open subset}\}.$$

From the definition of strongest topology, a subset $U$ of $M$ is a member of $\mathcal{T}_{\mathcal{P}_0}$ if and only if for all $p \in \mathcal{P}_0$, $p^{-1}(U) \subset X_1$ is an open subset.

Now, we complete the proof by using above notes. Fix $f^{-1}(W) \in \mathcal{S}_{\mathcal{F}_0}$ for some $f \in \mathcal{F}_0$. Get an arbitrary element $p$ in $\mathcal{P}$. Because $f \circ p \in \Phi$, therefore the set $\big(f \circ p\big)^{-1}(W) = p^{-1}\big(f^{-1}(W)\big)$ is open. It is easily seen that, $f^{-1}(W)$ is an element of $\mathcal{T}_{\mathcal{P}_0}$. Hence $\mathcal{T}_{\mathcal{F}_0} \subseteq \mathcal{T}_{\mathcal{P}_0}$. □

**Definition 3.11.** *Suppose that $X$ is a topological space. Let the sets $M_1$, $M_2$ be non-empty sets and let $\zeta : M_1 \to M_2$ be a map.*

- *Two $X$-parametrizations collections $\mathcal{P}_1$ of $M_1$ and $\mathcal{P}_2$ of $M_2$ are $\zeta$-**related**, (written $\mathcal{P}_1 \sim_\zeta \mathcal{P}_2$) provided $\zeta_\#(\mathcal{P}_1) := \{\zeta \circ p | p \in \mathcal{P}_1\} \subseteq \mathcal{P}_2$.*

- *Two $X$-functions families $\mathcal{F}_1$ of $M_1$ and $\mathcal{F}_2$ of $M_2$ are $\zeta$-**related**, (written $\mathcal{F}_1 \sim_\zeta \mathcal{F}_2$) provided $\zeta^\#(\mathcal{F}_2) := \{f \circ \zeta | f \in \mathcal{F}_2\} \subseteq \mathcal{F}_1$.*

**Proposition 3.12.** *Consider the above assuming.*

- *If $\mathcal{P}_1 \sim_\zeta \mathcal{P}_2$, then the map $\zeta : (M_1, \mathcal{T}_{\mathcal{P}_1}) \to (M_2, \mathcal{T}_{\mathcal{P}_2})$ is continuous.*

- *If $\mathcal{F}_1 \sim_\zeta \mathcal{F}_2$, then the map $\zeta : (M_1, \mathcal{T}_{\mathcal{F}_1}) \to (M_2, \mathcal{T}_{\mathcal{F}_2})$ is continuous.*

**Proof .** First, let $U$ be an element of $\mathcal{T}_{\mathcal{P}_2}$ and let $\mathcal{P}_1 \sim_\zeta \mathcal{P}_2$. Get an arbitrary element $p \in \mathcal{P}_1$. The $X$-parametrization $\zeta \circ p$ is an element of $\mathcal{P}_2$, because $\mathcal{P}_1 \sim_\zeta \mathcal{P}_2$. Therefore, $p^{-1}\big(\zeta^{-1}(U)\big) = (\zeta \circ p)^{-1}(U) \subseteq X$ is an open subset. Of course, the subset $\zeta^{-1}(U) \subseteq M_1$ is open. It shows that $\zeta$ is a continuous map.

Now, assume that $\mathcal{F}_1 \sim_\zeta \mathcal{F}_2$. We show that $\zeta$ is continuous. Let $f^{-1}(V)$ be an element of sub-basis of $\mathcal{T}_{\mathcal{F}_2}$ where the set $V \subseteq X$ is open and $f \in \mathcal{F}_2$. Because $\mathcal{F}_1 \sim_\zeta \mathcal{F}_2$, the function $f \circ \zeta$ is an element of $\mathcal{F}_1$. Therefore the set $\zeta^{-1}\big(f^{-1}(V)\big) = (f \circ \zeta)^{-1}(V) \subseteq M_1$ is an open subset. This completes the proof. $\square$

**Proposition 3.13.** *Suppose that $\mathcal{P}_i$ is an $X$-parametrizations collection of $M_i$ and let $\mathcal{F}_i$ be an $X$-function family on $M_i$ ($i = 1, 2, 3$). Suppose that $\zeta_1 : M_1 \to M_2$ and $\zeta_2 : M_2 \to M_3$ are two maps.*

- *If $\mathcal{P}_1 \sim_{\zeta_1} \mathcal{P}_2$ and $\mathcal{P}_2 \sim_{\zeta_2} \mathcal{P}_3$, then $X$-parametrizations collections $\mathcal{P}_1$ and $\mathcal{P}_3$ are $\zeta_2 \circ \zeta_1$-related.*

- *If $\mathcal{F}_1 \sim_{\zeta_1} \mathcal{F}_2$ and $\mathcal{F}_2 \sim_{\zeta_2} \mathcal{F}_3$, then $X$-functions families $\mathcal{F}_1$ and $\mathcal{F}_3$ are $\zeta_2 \circ \zeta_1$-related.*

**Proposition 3.14.** *Assume that $\Phi$ be a continuous $X_1$-parametrizations collection of a topological space $X_2$. Let $M_1$, $M_2$ be non-empty sets and let $\zeta : M_1 \to M_2$ be a map.*

*i) Let $X_1$-parametrizations collections $\mathcal{P}_1$ of $M_1$ and $\mathcal{P}_2$ of $M_2$ be $\zeta$-related. Then $\Phi^*\mathcal{P}_1$ on $M_1$ and $\Phi^*\mathcal{P}_2$ on $M_2$ are $\zeta$-related, too.*

*ii) Let $X_2$-functions families $\mathcal{F}_1$ on $M_1$ and $\mathcal{F}_2$ on $M_2$ be $\zeta$-related. Then $\Phi_*\mathcal{F}_1$ of $M_1$ and $\Phi_*\mathcal{F}_2$ of $M_2$ are $\zeta$-related, too.*

*iii) Let $\mathcal{P}_0$ be a collection of $X_1$-parametrizations on $M_1$. Then $\zeta^\# \phi^* \zeta_\# \mathcal{P}_0 \subseteq \phi^*\mathcal{P}_0$.*

*iv) Let $\mathcal{F}_0$ be a family of $X_2$-functions on $M_2$, then $\zeta_\# \phi_* \zeta^\# \mathcal{F}_0 \subseteq \phi_*\mathcal{F}_0$.*

**Proof .**

i) Assume that $g \in \zeta^\#(\Phi^*\mathcal{P}_2)$. There is $f \in \Phi^*\mathcal{P}_2$ such that $g = f \circ \zeta$. Suppose that $p_0$ is an arbitrary element of $\mathcal{P}_1$. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are $\zeta$-related. It shows that $X_1$-parametrization $\zeta \circ p_0$ is a member of $\mathcal{P}_2$. Therefore, the continuous map $g \circ p_0 = f \circ \zeta \circ p_0$ is an element of $\Phi$. Hence $g$ is in $\Phi^*\mathcal{P}_1$. It shows that $\zeta^\#(\Phi^*\mathcal{P}_2) \subseteq \Phi^*\mathcal{P}_1$.

ii) By similar arguments of i), we can prove this part ii).

iii) We use the above definitions for this proof: $\zeta^{\#}\phi^{*}\zeta_{\#}\mathcal{P}_0 = \{f\circ\zeta |\ f\in\Phi^{*}\zeta_{\#}\mathcal{P}_0\} = \{f\circ\zeta |\ \forall p_0\in\zeta_{\#}\mathcal{P}_0, f\circ p_0\in\Phi\} = \{f\circ\zeta |\ \forall p\in\mathcal{P}_0, f\circ\zeta\circ p\in\Phi\}\subseteq\Phi^{*}\mathcal{P}_0$.

iv) In the same arguments of iii), we show that this item: $\zeta_{\#}\phi_{*}\zeta^{\#}\mathcal{F}_0 = \{\zeta\circ p |\ p\in\Phi_{*}\zeta^{\#}\mathcal{F}_0\} = \{\zeta\circ p | f_0\circ p\in\Phi, \forall f_0\in\zeta^{\#}\mathcal{F}_0\} = \{\zeta\circ p | f\circ\zeta\circ p\in\Phi, \forall f\in\mathcal{F}_0\}\subseteq\Phi_{*}\mathcal{F}_0$.

□

**Definition 3.15.** *Let $\Phi$ be a continuous $X_1$-parametrizations collection of a topological space $X_2$. Assume that $V\subseteq X_1$ is an open subset and $\mathcal{F}_0$ is a family of $X_2$-functions on $M$. Suppose that*

$$\Phi_{*}|_{V}\mathcal{F}_0 := \{p\in\Phi_{*}\mathcal{F}_0 |\ dom(p)\subseteq V\},\quad \Phi_{*}^{V}\mathcal{F}_0 := \{p\in\Phi_{*}\mathcal{F}_0 | dom(p) = V\}.$$

*If $\Phi_{*}|_{V}\Phi^{*}\mathcal{P}_0 = \mathcal{P}_0$ (or $\Phi^{*}\Phi_{*}|_{V}\mathcal{F}_0 = \mathcal{F}_0$), then we say that the $X_1$-parametrizations collection $\mathcal{P}_0$ (or the $X_2$-functions family $\mathcal{F}_0$) has* $\Phi|_V$**-reflexive property**. *Similarly, an $X_1$-parametrizations collection $\mathcal{P}_0$ (or $X_2$-functions family $\mathcal{F}_0$) has* $\Phi^V$**-reflexive property** *if and only if $\Phi_{*}^{V}\Phi^{*}\mathcal{P}_0 = \mathcal{P}_0$ (or $\Phi^{*}\Phi_{*}^{V}\mathcal{F}_0 = \mathcal{F}_0$).*

**Example 3.16.** *Suppose that $\Gamma_n$ is a pseudomonoid on $\mathbb{R}^n$ consists of all local diffeomorphisms of $\mathbb{R}^n$. If $\mathcal{P}$ is a collection of $\mathbb{R}^n$-parametrizations on $M$ where $M = \cup_{p\in\mathcal{P}}dom(p)$. Then the collection $\mathcal{P}$ has $\Gamma_n$-reflexive property if and only if $(M,\mathcal{P})$ is a smooth $n$-manifold where the collection $\mathcal{P}$ is a maximal atlas on $M$.*

**Example 3.17.** *Assume that the pseudomonoid $\Gamma$ contains all smooth real-value functions on all open subsets of $\mathbb{R}$.*

- *Assume that $\mathcal{C}$ (or $\mathcal{F}$) is an $\mathbb{R}$-parametrizations collection of $M$ (or a real-value functions family on $M$). Then the collection $\mathcal{C}$ (or $\mathcal{F}$) has $\Gamma^{\mathbb{R}}$-reflexive property if and only if the triple $(M,\mathcal{C},\Gamma^{*}\mathcal{C})$ (or $(M,\Gamma_{*}^{\mathbb{R}}\mathcal{F},\mathcal{F})$) is a Frölicher space.*

**Theorem 3.18.** *Let $\Phi$ be a continuous $X_1$-parametrizations collection of a topological space $X_2$ and assume that $V\subseteq X_1$ is an open subset.*

*i) Suppose that $\mathbb{P}^V$ and $\mathbb{F}^V$ are all collections of $X_1$-parametrizations and all families of $X_2$-functions on $M$(resp.) such that they have $\Phi^V$-reflexive property. Then two maps $\Phi_{*}^{V}:\mathbb{F}^V\to\mathbb{P}^V$ and $\Phi^{*}|_{\mathbb{P}^V}:\mathbb{P}^V\to\mathbb{F}^V$ are inverses of each other.*

*ii) Suppose that $\mathbb{P}|_V$ and $\mathbb{F}|_V$ are all collections of $X_1$-parametrizations and all families of $X_2$-functions on $M$(resp.) such that they have $\Phi|_V$-reflexive property. Then two maps $\Phi_{*}|_{\mathbb{F}|_V}:\mathbb{F}|_V\to\mathbb{P}|_V$ and $\Phi^{*}|_V:\mathbb{P}|_V\to\mathbb{F}|_V$ are inverses of each other.*

*iii) Suppose that $\mathbb{P}_\Phi$ and $\mathbb{F}_\Phi$ are all collections of $X_1$-parametrizations and all families of $X_2$-functions on $M$ such that they have $\Phi$-reflexive property. Then two maps $\Phi_{*}|_{\mathbb{F}_\Phi}:\mathbb{F}_\Phi\to\mathbb{P}_\Phi$ and $\Phi^{*}|_{\mathbb{P}_\Phi}:\mathbb{P}_\Phi\to\mathbb{F}_\Phi$ are inverses of each other.*

**Proof .** The proof is straightforward immediate. □

**Corollary 3.19.** *Let $\Phi$ be the collection of all smooth functions $f:U\subseteq\mathbb{R}^n\to\mathbb{R}$ where the subset $U$ is open, for all $n\in\mathbb{N}$. Assume that $\mathscr{D}_\Phi$ is all diffeology structures on $M$ and $\mathscr{S}_\Phi$ is all differential structures on $M$, such that they have $\Phi$-reflexive property. Then two maps $\phi^{*}|_{\mathscr{D}_\Phi}:\mathscr{D}_\Phi\to\mathscr{S}_\Phi$ $\Phi_{*}|_{\mathscr{S}_\Phi}:\mathscr{S}_\Phi\to\mathscr{D}_\Phi$ are the inverses each other.*

## 4. Φ-reflexive property and $\mathcal{P}$-tangent space

Throughout this section, $X$ is a smooth manifold. Let $\mathcal{P}$ be a collection of $X$-parametrizations of $M$. Let $\mathcal{T}_{\mathcal{P}}$ denote the strongest topology on $M$. A function $h : U \to \mathbb{R}$ from an open subset of $M$ to $\mathbb{R}$ is said locally $\mathcal{P}$-smooth if and only if for any member $p \in \mathcal{P}$, the function $h \circ p$ is locally smooth. The set $C^{\infty}_{\mathcal{P}}(m)$ denotes all local $\mathcal{P}$-smooth functions on the neighborhoods of $m \in M$.

Assume that $\mathcal{F}$ is an $X$-functions family on $M$. Consider $\mathcal{T}_{\mathcal{F}}$ the weakest topology on the set $M$ such that any members of $\mathcal{F}$ are continuous. A function $h : U \to \mathbb{R}$ from an open subset of $M$ to $\mathbb{R}$ is said locally $\mathcal{F}$-smooth if and only if for any $m \in U$, there are the local smooth functions $\tilde{f}_m : W \subseteq X \to \mathbb{R}$ and $f_m \in \mathcal{F}$ such that $h = \tilde{f}_m \circ f_m$ on some neighbourhood of $m$. The notation $C^{\infty}_{\mathcal{F}}(m)$ denotes all local $\mathcal{F}$-smooth functions on neighbourhoods of $m \in M$.

Now, we define an equivalence relation on $C^{\infty}_{\mathcal{P}}(m)$ $\big($or $C^{\infty}_{\mathcal{F}}(m)\big)$. Let $f, g \in C^{\infty}_{\mathcal{P}}(m)$ $\big($or $f, g \in C^{\infty}_{\mathcal{F}}(m)\big)$. Then $f \sim g$ if and only if there is an element $W \in \mathcal{T}_{\mathcal{P}}$ $\big($or $W \in \mathcal{T}_{\mathcal{F}}\big)$ contains $m$ such that $f|_W = g|_W$. The equivalence classes of this relation is denoted by $\mathfrak{C}^{\infty}_{\mathcal{P}}(m)$ $\big($or $\mathfrak{C}^{\infty}_{\mathcal{F}}(m)\big)$.

**Definition 4.1.** *We define* $\mathcal{P}$**-tangent space** *at* $m \in M$, $T^{\mathcal{P}}_m(M)$, *be all maps* $V^{\mathcal{P}}_m : \mathfrak{C}^{\infty}_{\mathcal{P}}(m) \to \mathbb{R}$ *satisfying two following conditions:*

*i)* $V^{\mathcal{P}}_m(\alpha h + \beta g) = \alpha V^{\mathcal{P}}_m(h) + \beta V^{\mathcal{P}}_m(g)$, *(linearity),*

*ii)* $V^{\mathcal{P}}_m(hg) = V^{\mathcal{P}}_m(h)g(m) + h(m)V^{\mathcal{P}}_m(g)$, *(Leibniz rule),*

*for any* $h, g \in \mathfrak{C}^{\infty}_{\mathcal{P}}(m)$ *and for all* $\alpha, \beta \in \mathbb{R}$.

**Proposition 4.2.** *Let* $h \in \mathfrak{C}^{\infty}_{\mathcal{P}}(m)$ *be a constant function on some neighborhood of* $m$. *Then* $V^{\mathcal{P}}_m(h) = 0$ *for any* $V^{\mathcal{P}}_m \in T^{\mathcal{P}}_m(M)$.

**Proof .** First, assume that $h \equiv 1$ on a neighborhood of $m$. Then we have $V^{\mathcal{P}}_m(1) = V^{\mathcal{P}}_m(1.1) = V^{\mathcal{P}}_m(1).1 + 1.V^{\mathcal{P}}_m(1) = 2V^{\mathcal{P}}_m(1)$. Therefore $V^{\mathcal{P}}_m(1) = 0$. Now, consider $h \equiv c$ is a constant function on some neighborhood of $m$. Then $V^{\mathcal{P}}_m(c) = c.V^{\mathcal{P}}_m(1) = c.0 = 0$. This completes the proof. □ In the above definition, if we replace $\mathcal{P}$ by $\mathcal{F}$, we obtain the definition of $T^{\mathcal{F}}_m M$. Therefore, we can prove a similar proposition for constant functions at $\mathfrak{C}^{\infty}_{\mathcal{F}}(m)$.

**Definition 4.3.** *Suppose that* $X_1$ *and* $X_2$ *are two smooth manifolds.*

- *Let* $\mathcal{P}_i$ *be an* $X_i$*-parametrizations collection on a non-empty set* $M_i (i = 1, 2)$. *A map* $\Psi : M_1 \to M_2$ *is said* $(\mathcal{P}_1, \mathcal{P}_2)$**-smooth** *if and only if for any* $p_1 \in \mathcal{P}_1$ *and for all* $p_1(x) \in M_1$ *there exists* $p_2 \in \mathcal{P}_2$ *and a local smooth map* $\tilde{\Psi} : U \subseteq X_1 \to X_2$ *such that* $\Psi \circ p_1 = p_2 \circ \tilde{\Psi}$, *on some neighborhood of* $x$.

- *Let* $\mathcal{F}_i$ *be an* $X_i$*-functions family on a non-empty set* $M_i (i = 1, 2)$. *A map* $\Psi : M_1 \to M_2$ *is said* $(\mathcal{F}_1, \mathcal{F}_2)$**-smooth** *if and only if for any* $f_2 \in \mathcal{F}_2$ *and for all* $\Psi(m) \in domain(f_2)$ *there exists* $f_1 \in \mathcal{F}_1$ *and a local smooth map* $\tilde{\Psi} : U \subseteq X_1 \to X_2$ *such that* $f_2 \circ \Psi = \tilde{\Psi} \circ f_1$, *on some neighborhood of* $m$.

**Proposition 4.4.** *If the function* $\Psi : M_1 \to M_2$ *is* $(\mathcal{P}_1, \mathcal{P}_2)$*-smooth or* $(\mathcal{F}_1, \mathcal{F}_2)$*-smooth map. Then* $\Psi$ *is continuous.*

**Proof .** Assume that the function $\Psi$ is a $(\mathcal{P}_1,\mathcal{P}_2)$-smooth. Let $W$ be an arbitrary element of $\mathcal{T}_{\mathcal{P}_2}$. It is sufficient to show that the set $\Psi^{-1}(W)\subseteq M_1$ is an open subset, i.e. for all $p_1\in\mathcal{P}_1$, $p_1^{-1}\big(\Psi^{-1}(W)\big)\subseteq X_1$ is an open set.

Get an arbitrary element $x\in p_1^{-1}\big(\Psi^{-1}(W)\big)$. According to the definition of $(\mathcal{P}_1,\mathcal{P}_2)$-smooth. We have $p_2\in\mathcal{P}_2$ and a local smooth map $\tilde{\Psi}:U\to X_2$ such that $\Psi\circ p_1=p_2\circ\tilde{\Psi}$ on some neighborhood $V_x$ of $x$. Furthermore, We have $p_1^{-1}\big(\Psi^{-1}(W)\big)\cap V_x=(\Psi\circ p_1)^{-1}(W)\cap V_x=(p_2\circ\tilde{\Psi})^{-1}(W)\cap V_x=\tilde{\Psi}^{-1}\big(p_2^{-1}(W)\big)\cap V_x$.
By the definition of $\mathcal{T}_{\mathcal{P}_2}$ and smoothness of $\tilde{\Psi}$, the set $\tilde{\Psi}^{-1}\big(p_2^{-1}(W)\big)$ is an open set. Hence $p_1^{-1}\big(\Psi^{-1}(W)\big)$ is an open set.

$(\mathcal{F}_1,\mathcal{F}_2)$-smooth case is proved by applying similar arguments. This completes the proof $\square$

**Definition 4.5.** *Suppose that $X_1$ and $X_2$ are two smooth manifolds.*

- *Let $\mathcal{P}_i$ be an $X_i$-parametrizations collection of a non-empty set $M_i (i=1,2)$. If $\Psi$ is $(\mathcal{P}_1,\mathcal{P}_2)$-smooth. Then* **derivation** *of $\Psi$ at $m$ or $\Psi_{*m}:T_m^{\mathcal{P}_1}M_1\to T_{H(m)}^{\mathcal{P}_2}M_2$ is defined by this equation: $\Psi_{*m}(V_m^{\mathcal{P}_1})h=V_m^{\mathcal{P}_1}(h\circ\Psi)$ for all $V_m^{\mathcal{P}_1}\in T_m^{\mathcal{P}_1}M_1$ and for any $h\in\mathfrak{C}^\infty_{\mathcal{P}_2}\big(H(m)\big)$.*
- *Let $\mathcal{F}_i$ be an $X_i$-functions family on a non-empty set $M_i (i=1,2)$. If $\Psi$ is an $(\mathcal{F}_1,\mathcal{F}_2)$-smooth. Then* **derivation** *of $\Psi$ at $m$ or $\Psi_{*m}:T_m^{\mathcal{F}_1}M_1\to T_{H(m)}^{\mathcal{F}_2}M_2$ is defined by this equation: $\Psi_{*m}(V_m^{\mathcal{F}_1})h=V_m^{\mathcal{F}_1}(h\circ\Psi)$ for all $V_m^{\mathcal{F}_1}\in T_m^{\mathcal{F}_1}M_1$ and for any $h\in\mathfrak{C}^\infty_{\mathcal{F}_2}\big(\Psi(m)\big)$.*

**Proposition 4.6 (Chain rule).** *Suppose that $X_1$ and $X_2$ are two smooth manifolds*

- *Let $\mathcal{P}_i$ be an $X_i$-parametrizations collection of a non-empty set $M_i$ $(i=1,2,3)$. If $\Theta:M_1\to M_2$ and $\Psi:M_2\to M_3$ are $(\mathcal{P}_1,\mathcal{P}_2)$-smooth map and $(\mathcal{P}_2,\mathcal{P}_3)$-smooth map(resp.). Then the map $\Psi\circ\Theta:M_1\to M_3$ is a $(\mathcal{P}_1,\mathcal{P}_3)$-smooth map. For any $m\in M_1$, we obtain $(\Psi\circ\Theta)_{*m}=\Psi_{*\Theta(m)}\Theta_{*m}$.*
- *Let $\mathcal{F}_i$ be an $X_i$-functions family on a non-empty set $M_i$ $(i=1,2,3)$. If $\Theta:M_1\to M_2$ and $\Psi:M_2\to M_3$ are $(\mathcal{F}_1,\mathcal{F}_2)$-smooth map and $(\mathcal{F}_2,\mathcal{F}_3)$-smooth map(resp.). Then $\Psi\circ\Theta:M_1\to M_3$ is an $(\mathcal{F}_1,\mathcal{F}_3)$-smooth map. For any $m\in M_1$, we obtain $(\Psi\circ\Theta)_{*m}=\Psi_{*\Theta(m)}\Theta_{*m}$.*

**Proof .** We only prove the first case, the second case (with similar arguments) are left to the reader.

First, we show that $\Theta\circ\Psi$ is a $(\mathcal{P}_1,\mathcal{P}_2)$-smooth. Assume that the arbitrary elements $p_1\in\mathcal{P}_1$ and $p_1(x)\in M_1$. Because $\Psi$ is a $(\mathcal{P}_1,\mathcal{P}_2)$-smooth map. Hence, there exists the functions $p_2\in\mathcal{P}_2$, local smooth map $\tilde{\Psi}:U\subseteq X_1\to X_2$ and $\Psi\circ p_1=p_2\circ\tilde{\Psi}$ on some neighborhood $V_x$ of $x$. By the $(\mathcal{P}_1,\mathcal{P}_2)$-smoothness of $\Theta$. We conclude that, there exists a local smooth map $\tilde{\Theta}:V\subseteq X_2\to X_3$ and an element $p_3\in\mathcal{P}_3$ such that $\Theta\circ p_2=p_3\circ\tilde{\Theta}$ on some neighborhood $V_{\tilde{\Psi}(x)}$ of $\tilde{\Psi}(x)$. Now, we have $\Theta\circ\Psi\circ p_1=\Theta\circ p_2\circ\tilde{\Psi}=p_3\circ\tilde{\Theta}\circ\tilde{\Psi}$ on $\tilde{\Theta}^{-1}(V_{\tilde{\Psi}(x)})\cap V_x$ ,some neighborhood of $x$. Therefore, $\Theta\circ\Psi$ is a $(\mathcal{P}_1,\mathcal{P}_3)$-smooth map.

We prove the equation of derivations. Get $V_m^{\mathcal{P}_1}\in T_m^{\mathcal{P}_1}M_1$ and $h\in\mathfrak{C}^\infty_{\mathcal{P}_3}\Big(\Theta\big(\Psi(m)\big)\Big)$. We have $\big((\Theta\circ\Psi)_{*m}V_m^{\mathcal{P}_1}\big)h=V_m^{\mathcal{P}_1}(h\circ\Theta\circ\Psi)=\Psi_{*m}V_m^{\mathcal{P}_1}(h\circ\Theta)=\Theta_{*\Psi(m)}\big(\Psi_{*m}V_m^{\mathcal{P}_1}h\big)$. This completes the proof. $\square$

The new definitions proposed in this section are natural generalizations of smoothness and tangent space of manifold theory. The following proposition proves this assertion.

**Proposition 4.7.** *Suppose that $(X,\mathscr{A})$ is a smooth n-manifold. If $C^\infty(X)$ denote all smooth functions on $X$. Then, we have $T_xX=T_x^{C^\infty(X)}X=T_x^{\mathscr{A}}X$, for any $x\in X$.*

**Proof .** It is sufficient to show that $C^{\infty}_{\mathscr{A}}(x) = C^{\infty}(x)$ for all $x \in X$. Let $h \in C^{\infty}_{\mathscr{A}}(x)$ be an arbitrary element. By the definition of $C^{\infty}_{\mathscr{A}}(x)$, the map $h \circ \phi$ is a smooth function for all element of $\phi \in \mathscr{A}$. The function $h$ is a member of $C^{\infty}(x)$, by the smoothness definition in manifold theory.

Conversely, choose $h \in C^{\infty}(x)$. Because $h$ is a smooth function. Furthermore all elements of $\mathscr{A}$ are smooths. Therefore the function $h \circ \phi$ is smooth. So, we have $h \in C^{\infty}_{\mathscr{A}}(x)$, by the definition of $C^{\infty}_{\mathscr{A}}(x)$. Therefore, $C^{\infty}(x) = C^{\infty}_{\mathscr{A}}(x)$. It shows that $T_xX = T^{\mathscr{A}}_x X$.

For another equality, it sufficient to show that $C^{\infty}_{C(X)}(x) = C^{\infty}(x)$. The same reasoning applies proves this case. □

**Proposition 4.8.** *Assume that $\mathcal{P}$, $\mathcal{F}$ are an $X$-parametrizations collection and an $X$-functions family on $M$(resp.). If $\{m\}$ is an element of $\mathcal{T}_{\mathcal{P}}$ or $\mathcal{T}_{\mathcal{F}}$. Then $T^{\mathcal{P}}_m M \cong \{0\}$ or $T^{\mathcal{F}}_m M \cong \{0\}$ (resp.).*

**Proof .** If we prove that all members of $\mathfrak{C}^{\infty}_{\mathcal{P}}(m)\big(\text{or } \mathfrak{C}^{\infty}_{\mathcal{F}}(m)\big)$ are constant functions. The assertion follows by the proposition 4.2.

Since $\{m\}$ is a member of $\mathcal{T}_{\mathcal{P}}$(or $\mathcal{T}_{\mathcal{F}}$). It follows that any element of $C^{\infty}_{\mathcal{P}}(m)\big(\text{or } C^{\infty}_{\mathcal{F}}(m)\big)$ is equivalence to a constant function. Therefore, $\mathfrak{C}^{\infty}_{\mathcal{P}}(m)\big(\text{or } \mathfrak{C}_{\mathcal{F}}(m)\big)$ is only constant functions on $M$. □

**Corollary 4.9.** *If $m$ is not an element of $Im\ (p)$ for all $p \in \mathcal{P}$. Then we have $T^{\mathcal{P}}_m M \cong \{0\}$.*

**Proposition 4.10.**

- *Assume that $\mathcal{P}$ is an $X$-parametrizations collection of $M$. Let $N \subseteq M$ be an open subset. Then $\mathcal{P}|^V$ is an $X$-parametrizations collection on $N$, where $\mathcal{P}|^N := \{p|^N : p^{-1}(N) \cap U \to N |$the $X$-parametrization $p : U \to M$ is an element of $\mathcal{P}\}$. Also, we have $T^{\mathcal{P}|^N}_m N = T^{\mathcal{P}}_m M$ for all $m \in N$.*
- *Assume that that $\mathcal{F}$ is an $X$-functions family on $M$. Let $N \subset M$ be an open subset. Then $\mathcal{F}|_N$ is an $X$-functions family on $N$, where $\mathcal{F}|_N := \{f|_N : N \to X |$the $X$-function $f : M \to X$ is an element of $\mathcal{F}\}$. Also, we have $T^{\mathcal{F}|_N}_m N = T^{\mathcal{F}}_m N$ for all $m \in N$.*

**Proof .** Because the elements of $\mathfrak{C}^{\infty}_{\mathcal{F}}(m)$ and $\mathfrak{C}^{\infty}_{\mathcal{P}}(m)$ are defined locally and $N$ is an open subset. The proof is straightforward. □

**Proposition 4.11.** *Suppose that $(X, \mathscr{A})$ is a smooth manifold.*

- *Assume that $\mathcal{P}$ is an $X$-parametrization collection of $M$. Then any $p \in \mathcal{P}$ is an $(\mathscr{A}, \mathcal{P})$-smooth map. It induces a linear map $p_{*x} : T_xX \to T^{\mathcal{P}}_{p(x)} M$ for all $x \in domain(p)$.*
- *Assume that $\mathcal{F}$ is an $X$-functions family on $M$. Then any $X$-function $f \in \mathcal{F}$ is an $(\mathcal{F}, \mathscr{A})$-smooth map. It induces a linear map $f_{*m} : T^{\mathcal{F}}_m M \to T_{f(m)}X$ for all $m \in M$.*

**Proof .** Fix $\phi \in \mathscr{A}$ and $\phi(x) \in domain(p) \subseteq X$. If we get $\tilde{p} := \phi : \mathbb{R}^n \to X$ and $p_2 := p$. Then the conditions of $(\mathcal{P}_1, \mathcal{P}_2)$-smoothness definition are being held. Now, $p_{*x}$ induces a linear map from $T_xX$ to $T^{\mathcal{P}}_{p(x)} M$ because $T_xX = T^{\mathscr{A}}_x X$. Similar arguments apply to prove the other item. □

**Definition 4.12.** *Assume that $\Phi$ is a collection of smooth maps $\Phi : U \subseteq X_1 \to X_2$ where $X_1$ and $X_2$ are two smooth manifold and $U$ is an open subset. Therefore, there exists the family of maps $\Phi_* : TU \subseteq TX_1 \to TX_2$ with properties $\Phi_*(x, V_x) = \big(\Phi(x), \phi_{*x}(V_x)\big)$, where $\phi : U \to X_2 \in \Phi$ is said* **tangent to** $\Phi$ *and denoted by $T\Phi$.*

**Definition 4.13.**
- *Let $\mathcal{F}$ be an $X$-functions family on the set $M$. The set $T^{\mathcal{F}}M = \cup_{m\in M} T^{\mathcal{F}}_m M$ is called* **$\mathcal{F}$-tangent bundle** *of $M$. The bundle $T^{\mathcal{F}}M$ admits a natural $TX$-functions family by maps $f_* : T^{\mathcal{F}}M \to TX$ with $f_*(m, V^{\mathcal{F}}_m) = \big(f(x), f_{*m}(V^{\mathcal{F}}_m)\big)$ where the $X$-function $f : M \to X$ is an element of $\mathcal{F}$.*
- *Suppose that $\mathcal{P}$ is an $X$-parametrizations collection of $M$. The set $T^{\mathcal{P}}M = \cup_{m\in M} T^{\mathcal{P}}_m M$ is called* **$\mathcal{P}$-tangent bundle** *of $M$. $T^{\mathcal{P}}M$ such that admits a natural $TX$-parametrizations collection by maps $p_* : TU \to T^{\mathcal{P}}M$ with $p_*(x, V_x) = \big(p(x), p_{*x}(V_x)\big)$ where the $X$-parametrization $p : U \to M$ is an element of $\mathcal{P}$.*

**Proposition 4.14.** *Consider $X_1$, $X_2$ are two smooth manifolds and $\Phi$ is a local smooth maps collection from $X_1$ to $X_2$. Let $\mathcal{P}$ and $\mathcal{F}$ be an $X_1$-parametrizations collection of $M$ and an $X_2$-functions family on $M$(resp.). If for any $X_2$-function $f \in \mathcal{F}$ and for any $X_1$-parametrization $p \in \mathcal{P}$ we have $f \circ p \in \Phi$. Then for all elements $m \in M$, we have $\mathfrak{C}^\infty_{\mathcal{F}}(m) \subseteq \mathfrak{C}^\infty_{\mathcal{P}}(m)$*

**Proof .** Get $[h] \in \mathfrak{C}^\infty_{\mathcal{F}}(m)$. So, there are $f_m \in \mathcal{F}$ and $\tilde{f}_m : U \subseteq X \to \mathbb{R}$ such that $h = \tilde{f}_m \circ f_m$ on some neighborhood $V_m$ of $m$. Without losing the totality, we can suppose that $domain(h) = V_m$. By the Proposition 3.10, $V_m \in \mathcal{T}_{\mathcal{P}}$. Because $f_m \circ p \in \Phi$ for any $f_m \in \mathcal{F}$. Thus $h \circ p = \tilde{f}_m \circ f_m \circ p$ is a smooth map for all $p \in \mathcal{P}$. Therefore $h \in C^\infty_{\mathcal{P}}(m)$ and $[h] \in \mathfrak{C}^\infty_{\mathcal{P}}(m)$. If $[f] = [g] \in \mathfrak{C}^\infty_{\mathcal{F}}(m)$. Then by the Proposition 3.10 $[f] = [g] \in \mathfrak{C}^\infty_{\mathcal{P}}(m)$. This completes the proof. $\square$

**Corollary 4.15.** *Suppose that the conditions of the previous proposition are being held. We define the following map: $\Delta : T^{\mathcal{P}}M \longrightarrow T^{\mathcal{F}}M$, by $\Delta(m, V^{\mathcal{P}}_m) = (m, V^{\mathcal{P}}_m|_{\mathfrak{C}^\infty_{\mathcal{F}}(m)})$ for all $m \in M$ and $V^{\mathcal{P}}_m \in T^{\mathcal{P}}_m M$. Then $\Delta$ is a smooth map. For any $X_1$-parametrization $p \in \mathcal{P}$ and for any $X_2$-function $f \in \mathcal{F}$, we have $(f \circ p)_* = f_* \circ \Delta \circ p_*$.*

**Definition 4.16.** *Assume that $\mathcal{P}$ is an $X$-parametrizations collection of $M$. The set of all derivation maps $p_* : TU \to T^{\mathcal{P}}M$ defined by $p_*(x, V_x) \mapsto \big(p(x), p_{*x}(V_x)\big)$ is said* **tangent to $\mathcal{P}$** *and denoted by $T\mathcal{P}$.*

*Let $\mathcal{F}$ be an $X$-functions family on $M$. The set of all derivation maps $f_* : T^{\mathcal{F}}M \to TX$ defined by $f_*(m, V^{\mathcal{F}}_m) \mapsto \big(f(m), f_{*m}(V^{\mathcal{F}}_m)\big)$ is said* **tangent to $\mathcal{F}$** *and denoted by $T\mathcal{F}$.*

**Theorem 4.17.** *Suppose that $\Phi$ is a collection of smooth map $\Phi : U \subseteq X_1 \to X_2$ where $X_1$ and $X_2$ are smooth manifolds and the subset $U$ is an open set. Suppose that $\mathcal{F}$ and $\mathcal{P}$ are an $X_1$-functions family of $M$ and an $X_1$-parametrizations collection of $M$. If the following conditions are being held:*

- $T^{\mathcal{P}}_m M = T^{\mathcal{F}}_m M$ *for all* $m \in M$,
- $T\mathcal{P} = (T\Phi)_*(T\mathcal{F})$,
- $T\mathcal{F} = (T\Phi)^*(T\mathcal{P})$.

*Then $\mathcal{P}$ and $\mathcal{F}$ have $\Phi$-reflexive property. Furthermore, we have $\mathcal{P} = \Phi_*\mathcal{F}$ and $\mathcal{F} = \Phi^*\mathcal{P}$.*

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