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A discrete problem involving the $p(k)-$ Laplacian operator with three variable exponents

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Abstract

In this paper, we determine the different intervals of a positive parameters λ, for which we prove the existence and non existence of a non trivial solutions for the discrete problem (1.1). Our technical approach is based on the variational principle and the critical point theory.

Keywords: Discrete boundary value problem, Anisotropic problem, Critical point theory, Eigenvalue.
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1. Introduction

Let $T > 2$ be a positive integer and $[1,T]_{\mathbb{Z}}$ denote a discrete interval given by $[1,T]_{\mathbb{Z}} := \{1,2,3,.....,T\}$.

We consider the discrete Dirichlet anisotropic problem as follows:

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right)+|u(k)|^{p(k)-2}u(k)+|u(k)|^{q(k)-2}u(k)= \\ \qquad = \lambda g(k)|u(k)|^{r(k)-2}u(k), \quad for \quad k\in[1,T]_{\mathbb{Z}}, \\ \qquad u(0)=u(T+1)=0, \end{cases} \tag{1.1}$$

where Δ denotes the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$, $g : [1,T]_{\mathbb{Z}} \to (0,+\infty)$ is a given function, λ is a real parameter such that $\lambda > 0$ and $p : [0,T]_Z \to [2,+\infty)$ and $q, r : [1,T]_Z \to [2,+\infty)$ are given functions.

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In the last years, the study of boundary value problems for finite difference equations has captured special attention, for example view the recent results in the references [1, 2, 3, 4, 5, 20, 19]. This type of problems have an important role in different domains of research, such as mechanical engineering, control systems, economics, computer science, physics, artificial or biological neural networks, cybernetics, ecology and many others. The important tools employed to study this kind of problem are critical point theory and variational methods [7, 14].

However, there is an increasing interest to the existence results to boundary value problems for difference equations with $p(k)-$Laplacian operator, because of their applications in many fields. To the best of our knowledge, discrete problems involving anisotropic exponents have been discussed for the first time by Mihăilescu, Rădulescu and Tersain in[22], the authors proved, by using the critical point theory, the existence of a continuous spectrum of eigenvalues for the problem:

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right)=\lambda|u(k)|^{q(k)-2}u(k), & for \quad k\in[1,T]_{\mathbb{Z}}, \\ u(0)=u(T+1)=0, \end{cases} \tag{1.2}$$

and for the second time by Koné and Ouaro in [15, 18]. After Marek Galewski in ([9, 10, 11, 12]) and G.Molica Bisci and all in [21], have studied the existence of at least one solution, multiplicity and a sequences of solutions for the problem

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right)=\lambda f(k,u(k)), & for \quad k\in[1,T]_{\mathbb{Z}}, \\ u(0)=u(T+1)=0, \end{cases}$$

where $f:[1,T]_{\mathbb{Z}}\times\mathbb{R}\longmapsto\mathbb{R}$ is a continuous function that checks some conditions.

More recently, in ([24, 6, 16, 17, 8, 13]) the authors have been investigated the existence and multiplicity of solutions for nonlinear discrete boundary value problems involving $p(.)-$Laplacian operator using variational methods.

We note that problem (1.1) is the discrete variant of a kind of the variable exponent anisotropic problem ([23])

$$\begin{cases} -div\left(|\nabla u|^{p(x)-2}\nabla u\right)+|u|^{p(x)-2}u+|u|^{q(x)-2}u=\lambda g(x)|u|^{r(x)-2}u, & x\in\Omega, \\ u(x)=0 \quad for \quad x\in\partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N(N\geq 3)$ with Lipschitz boundary. The mappings $p,q,r:\bar{\Omega}\to[2,+\infty)$ are Lipschitz continuous functions, while $g:\bar{\Omega}\to(0,+\infty)$ is a measurable function.

Our analysis mainly concern the existence and the nonexistence of a weak solutions to problem (1.1) more general than (1.2), with three variable exponents and the weight g under appropriate assumptions (2.2) below, between the functions exponents $p(k), q(k)$ and $r(k)$. Our aim is to determine the concerts intervals for the parameter λ for which problem (1.1) has, or not has, a nontrivial solutions. More precisely, we prove the existence of two positive constants λ_* and λ^* with $\lambda_*\leq\lambda^*$ such that for each $\lambda\in[\lambda^*,+\infty)$ the problem (1.1) has at least one nontrivial solution, while for any $\lambda\in(0,\lambda_*)$ problem (1.1) has no nontrivial solution. For these results, we use some known tools such as the direct variational methods and the critical point theory.

The rest of this paper is organised as follows, the second section is devoted to mathematical preliminaries and statement of main results. In the third section we give the mains results and thier proofs.

2. Preliminaries

Define the T-dimensional Hilbert space (see, [1])

$$E = \{u : [0, T+1]_{\mathbb{Z}} \to \mathbb{R} \quad , \quad u(0) = u(T+1) = 0\},$$

with the inner product

$$(u, v) = \sum_{k=0}^{T} \Delta u(k)\Delta v(k), \qquad \forall u, v \in E.$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=0}^{T} |\Delta u(k)|^2\right)^{\frac{1}{2}}.$$

Moreover, it is useful to introduce other norm on E, denoted by $|u|_m$ and is namely,

$$|u|_m = \left(\sum_{k=1}^{T} |u(k)|^m\right)^{\frac{1}{m}}, \qquad \forall m \geq 2. \tag{2.1}$$

For any function $h : [0, T]_Z \to [2, +\infty)$, we use the following notations:

$$h^- = \min_{k\in[0,T]_{\mathbb{Z}}} h(k) \quad and \quad h^+ = \max_{k\in[0,T]_{\mathbb{Z}}} h(k).$$

In this paper, we study the boundary value problem (1.1) assuming that the functions p, q and r satisfy the following assumptions:

$$2 \leq p^- \leq p^+ < r^- \leq r^+ < q^- \leq q^+. \tag{2.2}$$

We start with the following auxillary result, which will be used later.

Lemma 2.1. *([22])*

(a) *For any $m \geq 2$ there exists a positive constant C_m such that,*

$$\sum_{k=1}^{T} |u(k)|^m \leq C_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall u \in E$$

(b) *There exists two positive constant C_1 and C_2 such that,*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1\|u\|^{p^-} - C_2, \quad \forall u \in E \quad with \quad \|u\| > 1$$

(c) *There exists a positive constant C_3 such that,*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_3\|u\|^{p^+}, \quad \forall u \in E \quad \textit{with} \quad \|u\| < 1$$

(d) $$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq (T+1)\left(\|u\|^{p^+} + 1\right), \quad \forall u \in E.$$

Definition 2.2. *We say that $\lambda > 0$ is an eigenvalue of problem (1.1) if there exists $u \in E$, such that $u \neq 0$ and*

$$\begin{aligned}\sum_{k=1}^{T+1} \quad &|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) + \sum_{k=1}^{T}|u(k)|^{p(k)-2}u(k)v(k)\\ &+ \sum_{k=1}^{T}|u(k)|^{q(k)-2}u(k)v(k) = \lambda\sum_{k=1}^{T} g(k)|u(k)|^{r(k)-2}u(k)v(k).\end{aligned}$$

for any $v \in E$.

If $\lambda > 0$ is an eigenvalue of problem (1.1), then the corresponding eigenfunction $u_\lambda \in E$ is a weak solution for the problem (1.1).

To study the boundary value problem (1.1), we define the following functionals, for $u \in E$, we put:

$$\varphi_0(u) = \sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)} + \sum_{k=1}^{T}|u(k)|^{p(k)} + \sum_{k=1}^{T}|u(k)|^{q(k)}, \tag{2.3}$$

$$\psi_0(u) = \sum_{k=1}^{T} g(k)|u(k)|^{r(k)}, \tag{2.4}$$

$$\varphi_1(u) = \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T} \frac{|u(k)|^{p(k)}}{p(k)} + \sum_{k=1}^{T} \frac{|u(k)|^{q(k)}}{q(k)}, \tag{2.5}$$

$$\psi_1(u) = \sum_{k=1}^{T} g(k)\frac{|u(k)|^{r(k)}}{r(k)}, \tag{2.6}$$

and for any $\lambda > 0$ and $u \in E$, we define the functional I_λ as follows:

$$I_\lambda(u) = \varphi_1(u) - \lambda\psi_1(u). \tag{2.7}$$

With any fixed $\lambda > 0$ the functionals I_λ is differentiable in the sense of Gâteaux,(see [22, 12]), and its Gâteaux derivatives at u reads

$$\left(I_\lambda^{'}(u), v\right) = \left(\varphi_1^{'}(u), v\right) - \lambda\left(\psi_1^{'}(u), v\right), \tag{2.8}$$

for any $v \in E$,

where

$$\left(\varphi_1'(u), v\right) = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) + \sum_{k=1}^{T} \left(|u(k)|^{p(k)-2} + |u(k)|^{q(k)-2}\right) u(k) v(k), \tag{2.9}$$

and

$$\left(\psi_1'(u), v\right) = \sum_{k=1}^{T} g(k) |u(k)|^{r(k)-2} u(k) v(k). \tag{2.10}$$

Remark 2.3. *According to definition (2.2) and from equality (2.8), it is clear that u_λ is a weak solution of (1.1) if and only if u_λ is a critical point of the functional I_λ.*

3. Main results and their proofs

Throughout this paper, we study the boundary value problem (1.1) assuming that the functions $p(.)$,$q(.)$ and $r(.)$ satisfy the hypotheses given in (2.2) and we have the following results

Theorem 3.1. *Assume that the hypothesis (2.2) holds, then there exists a positive constant $\lambda_\star$ such that for any $\lambda \in (0, \lambda_\star)$ problem (1.1) has no solutions.*

Proof .

Put

$$\lambda_\star = \inf_{u \in E-\{0\}} \frac{\varphi_0(u)}{\psi_0(u)}, \tag{3.1}$$

where φ_0 and ψ_0 are given by (2.3) and (2.4).

Firstly, we show that $\lambda_\star > 0$. By (2.2) we infer that for all $k \in [1, T]_{\mathbb{Z}}$

$$p(k) < r(k) < q(k),$$

then for any $u \in E$ and $k \in [1, T]_{\mathbb{Z}}$, we have

$$|u(k)|^{r(k)} \leq |u(k)|^{p(k)} + |u(k)|^{q(k)} \tag{3.2}$$

then

$$\sum_{k=1}^{T} \left(|u(k)|^{p(k)} + |u(k)|^{q(k)}\right) \geq \sum_{k=1}^{T} |u(k)|^{r(k)},$$

so

$$\sum_{k=1}^{T} \left(|u(k)|^{p(k)} + |u(k)|^{q(k)}\right) \geq \frac{1}{|g|_\infty} \sum_{k=1}^{T} g(k) |u(k)|^{r(k)} \tag{3.3}$$

where,

$$|g|_\infty = \max_{k\in[1,T]_{\mathbb{Z}}} g(k), \tag{3.4}$$

then we deduce that

$$\varphi_0(u) \geq \frac{1}{|g|_\infty}\psi_0(u), \quad \forall u \in E,$$

or

$$\lambda_\star \geq \frac{1}{|g|_\infty} \qquad \textit{then} \qquad \lambda_\star > 0.$$

Secondly, assuming by contradiction that there exists $\lambda \in (0, \lambda_\star)$ is an eigenvalue of problem (1.1), which means that there exists $u_\lambda \in E$ such that $u_\lambda \neq 0$ and $I'_\lambda(u_\lambda) = 0$, so

$$\left(\varphi'_1(u_\lambda), v\right) = \lambda\left(\psi'_1(u_\lambda), v\right), \quad \forall v \in E.$$

In particular, for $v = u_\lambda$ we get

$$\varphi_0(u_\lambda) = \lambda\psi_0(u_\lambda).$$

Since $u_\lambda \neq 0$, it follows that $\varphi_0(u_\lambda) > 0$ and $\psi_0(u_\lambda) > 0$, then by definition of $\lambda_\star$ and the fact that $\lambda < \lambda_\star$, we infer that:

$$\varphi_0(u_\lambda) \geq \lambda_\star\psi_0(u_\lambda) > \lambda\psi_0(u_\lambda) = \varphi_0(u_\lambda),$$

the above inequality leads to a contradiction and the proof is completed.
□

Theorem 3.2. *Assume that the hypotheses (3.2) holds, then there exists a positive constant $\lambda^\star$ such that $\lambda_\star \leq \lambda^\star$ and for any $\lambda \in [\lambda^\star, +\infty)$ the problem (1.1) has at least one non trivial solution.*

To show theorem (3.2), we need to prove the following lemmas

Lemma 3.3. *If the condition (2.3) is true, then*

$$\lim_{\|u\|\longrightarrow 0} \frac{\varphi_0(u)}{\psi_0(u)} = +\infty.$$

Proof .

For any $k \in [1,T]_{\mathbb{Z}}$, we have $r^- \leq r(k) \leq r^+$, then for any $u \in E$, we get:

$$|u(k)|^{r(k)} \leq |u(k)|^{r^-} + |u(k)|^{r^+},$$

then

$$g(k)|u(k)|^{r(k)} \leq |g|_\infty \left(|u(k)|^{r^-} + |u(k)|^{r^+}\right),$$

where $|g|_\infty$ is given by (3.4).

Thus, summing for k from 1 to T, we get, for any $u \in E$

$$\psi_0(u) \leq |g|_\infty \left(\sum_{k=1}^{T} |u(k)|^{r^-} + \sum_{k=1}^{T} |u(k)|^{r^+} \right),$$

using lemma (2.1) (a), we infer that

$$\psi_0(u) \leq |g|_\infty \left(C_{r^-} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{r^-} + C_{r^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{r^+} \right),$$

again by lemma 2.1 (d), we deduce that

$$\psi_0(u) \leq (1+T)|g|_\infty \left(C_{r^-}(1 + \|u\|^{r^-}) + C_{r^+}(1 + \|u\|^{r^+}) \right). \tag{3.5}$$

Next, for any $u \in E$, with $\|u\| < 1$, by (2.3) and lemma 2.1 (c), we deduce that

$$\varphi_0(u) \geq C_3 \|u\|^{p^+}, \tag{3.6}$$

then for any $u \in E$ with $\|u\| < 1$, small enough, from the inequalities (3.5) and (3.6), we get

$$\frac{\varphi_0(u)}{\psi_0(u)} \geq \frac{C_3}{(1+T)|g|_\infty} \frac{\|u\|^{p^+}}{C_{r^-}(1 + \|u\|^{r^-}) + C_{r^+}(1 + \|u\|^{r^+})}.$$

Since $r^+ \geq r^- > p^+$, passing to the limit as $\|u\| \longrightarrow 0$, in the above inequality we prove that $\lim\limits_{\|u\| \longrightarrow 0} \dfrac{\varphi_0(u)}{\psi_0(u)} = +\infty$, and thus lemma (3.3) holds. □

Lemma 3.4. *If the condition (2.2) is true, then for any $\lambda > 0$, I_λ is coercive.*

$$ie : \lim_{\|u\| \longrightarrow \infty} (\varphi_1(u) - \lambda \psi_1(u)) = +\infty$$

Proof .

For any $u \in E$, from (2.5) we have

$$\begin{aligned} \varphi_1(u) &= \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T} \frac{|u(k)|^{p(k)}}{2p(k)} + \sum_{k=1}^{T} \left(\frac{|u(k)|^{p(k)}}{2p(k)} + \frac{|u(k)|^{q(k)}}{q(k)} \right) \\ &\geq \frac{1}{p^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} + \frac{1}{max(2p^+, q^+)} \sum_{k=1}^{T} \left(|u(k)|^{p(k)} + |u(k)|^{q(k)} \right). \end{aligned} \tag{3.7}$$

Let s fix such that $r^+ < s < q^-$, then for any $u \in E$ and $k \in [1, T]_{\mathbb{Z}}$, we get that

$$|u(k)|^{p(k)} + |u(k)|^{q(k)} \geq |u(k)|^s,$$

then by (3.7), we get

$$\varphi_1(u) \geq \frac{1}{p^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} + \frac{1}{max(2p^+, q^+)} |u|_s^s. \tag{3.8}$$

Next, since $g(k)|u(k)|^{r(k)} \leq |g|_\infty(|u(k)|^{r^-} + |u(k)|^{r^+})$, then from (2.6) we infer that

$$\psi_1(u) \leq \frac{|g|_\infty}{r^-}\left(\sum_{k=1}^{T}|u(k)|^{r^-} + \sum_{k=1}^{T}|u(k)|^{r^+}\right). \tag{3.9}$$

By Hölder inequality we prove that, for any $u \in E$

$$\sum_{k=1}^{T}|u(k)|^{r^-} \leq T^{\frac{s-r^-}{s}}\left(\sum_{k=1}^{T}\left(|u(k)|^{r^-}\right)^{\frac{s}{r^-}}\right)^{\frac{r^-}{s}} = A|u|_s^{r^-}, \tag{3.10}$$

and

$$\sum_{k=1}^{T}|u(k)|^{r^+} \leq T^{\frac{s-r^+}{s}}\left(\sum_{k=1}^{T}\left(|u(k)|^{r^+}\right)^{\frac{s}{r^+}}\right)^{\frac{r^+}{s}} = B|u|_s^{r^+}, \tag{3.11}$$

where

$$A = T^{\frac{s-r^-}{s}} > 0 \quad and \quad B = T^{\frac{s-r^+}{s}} > 0. \tag{3.12}$$

Therefore, for any $u \in E$ with $\|u\| > 1$, from (2.9) and the above inequalities ((3.8)-(3.11)) and applying lemma (2.1) (b), we deduce that: for any $\lambda > 0$, we have

$$\begin{aligned} I_\lambda(u) \;\geq\; & \frac{1}{p^+}\left(C_1\|u\|^{p^-} - C_2\right) + \frac{1}{\max(2p^+, q^+)}|u|_s^s - \lambda\frac{|g|_\infty}{r^-}\left(A|u|_s^{r^-} + B|u|_s^{r^+}\right) \\ \geq\; & \frac{C_1\|u\|^{p^-} - C_2}{p^+} + \frac{|u|_s^s}{2\max(2p^+, q^+)} - \lambda\frac{A|u|_s^{r^-}|g|_\infty}{r^-} \\ & + \frac{|u|_s^s}{2\max(2p^+, q^+)} - \lambda\frac{B|u|_s^{r^+}|g|_\infty}{r^-}, \end{aligned}$$

so

$$I_\lambda(u) \geq \frac{C_1\|u\|^{p^-} - C_2}{p^+} - \left(\alpha|u|_s^{r^-} - \beta|u|_s^s\right) - (\gamma|u|_s^{r^+} - \beta|u|_s^s), \tag{3.13}$$

where $\alpha = A\lambda\frac{|g|_\infty}{r^-} > 0$, $\gamma = B\lambda\frac{|g|_\infty}{r^-} > 0$ and $\beta = \dfrac{1}{2\max(2p^+, q^+)} > 0$.
Let $h_1, h_2 :]0, +\infty[\longrightarrow \mathbb{R}$ two real functions, given by

$$h_1(t) = \alpha t^{r^-} - \beta t^s \qquad and \qquad h_2(t) = \gamma t^{r^+} - \beta t^s \qquad \forall t > 0.$$

It is easy to show that h_1 and h_2 achieves its positive global maximums $M_1 = h_1(t_1)$ and $M_2 = h_2(t_2)$, where:

$$t_1 = \left(\frac{\alpha r^-}{\beta s}\right)^{\frac{1}{s-r^-}} > 0 \quad and \quad t_2 = \left(\frac{\gamma r^+}{\beta s}\right)^{\frac{1}{s-r^+}} > 0,$$

then we infer that $h_1(t) \leq M_1$ and $h_2(t) \leq M_2$, $\forall t > 0$.
Therefore, for any $u \in E$, with $\|u\| > 1$ and $\lambda > 0$, by (3.13), we get that

$$I_\lambda(u) \geq \frac{C_1\|u\|^{p^-} - C_2}{p^+} - M_1 - M_2, \tag{3.14}$$

passing to the limit as $\|u\| \longrightarrow \infty$ in (3.14), we infer that lemma (3.4) holds.
□

Proof of theorem (3.2).

Put:

$$\lambda^\star = \inf_{u \in E - \{0\}} \frac{\varphi_1(u)}{\psi_1(u)}. \tag{3.15}$$

Step(1).
We show that $\lambda^\star > 0$.

By (3.2) and from (3.3), we infer that for any $u \in E$,

$$\frac{|u(k)|^{p(k)}}{p(k)} + \frac{|u(k)|^{q(k)}}{q(k)} \geq \frac{|u(k)|^{r(k)}}{q(k)} \geq \frac{|u(k)|^{r(k)}}{q^+},$$

then

$$\sum_{k=1}^{T} \frac{|u(k)|^{p(k)}}{p(k)} + \sum_{k=1}^{T} \frac{|u(k)|^{q(k)}}{q(k)} \geq \frac{r^-}{q^+|g|_\infty} \sum_{k=1}^{T} g(k) \frac{|u(k)|^{r(k)}}{r(k)}$$

then

$$\varphi_1(u) \geq \frac{r^-}{q^+|g|_\infty} \psi_1(u), \quad \forall u \in E,$$

so

$$\lambda^\star \geq \frac{r^-}{q^+|g|_\infty} > 0.$$

Thus step (1) is verified.

Step(2).
We show that each $\lambda \in (\lambda^\star, +\infty)$ is an eigenvalue of the problem (1.1).

We fix $\lambda \in (\lambda^\star, +\infty)$.
By lemma (3.4), we have I_λ is coercive and is weakly lower semi-continuous. Applying theorem (theorem 1.2 in[25]) in order to prove that there exists $u_\lambda \in E$ as a global minimum point of I_λ and thus as a critical point of I_λ.

In order to finish the proof of step(2), it is enough to prove that u_λ is non trivial. Indeed, since $\lambda > \lambda^\star$ and by definition of $\lambda^\star$ there exists $v_\lambda \in E$ such that

$$\varphi_1(v_\lambda) < \lambda \psi_1(v_\lambda),$$

that is

$$I_\lambda(v_\lambda) < 0,$$

which shows that $u_\lambda \neq 0_E$ and we conclude that there exists a $u_\lambda \in E$ with $u_\lambda \neq 0_E$ who is a critical point of I_λ, or λ is an eigenvalue of the problem (1.1). Thus step(2) is true.

Step(3).

We show that $\lambda^\star$ is an eigenvalue of problem (3.4).
For this, we will prove that there exists $u^\star \in E$ such that $u^\star \neq 0$ and $I'_{\lambda^\star}(u^\star) = 0$.

Let $\lambda_n > 0$ be a minimizing sequence for $\lambda^\star$ ($i.e \quad \lambda_n > \lambda^\star$). By step(2), we deduce that for each n there exists a sequence $\{u_n\} \in E$ such that $u_n \neq 0$ and $I'_{\lambda_n}(u_n) = 0$,

so
$$\left(\varphi_1'(u_n), v\right) = \lambda_n \left(\psi_1'(u_n), v\right), \quad \forall v \in E. \tag{3.16}$$

Taking $v = u_n$, we find that

$$\varphi_0(u_n) - \lambda_n \psi_0(u_n) = 0, \tag{3.17}$$

passing to the limit as $n \longrightarrow +\infty$ in relation (3.17), we have:

$$\lim_{n \longrightarrow +\infty} \left(\varphi_0(u_n) - \lambda_n \psi_0(u_n)\right) = 0. \tag{3.18}$$

On the other hand, a similar argument as those used in proof of lemma (3.4), we get that

$$\lim_{\|u_n\| \longrightarrow +\infty} \left(\varphi_0(u_n) - \lambda_n \psi_0(u_n)\right) = +\infty. \tag{3.19}$$

Then, from (3.18) and (3.19) we show that the sequence $\{u_n\}$ is bounded in E, since E is a finite dimensional Hilbert space then there exists a subsequence, still denoted by $\{u_n\}$ and $u^\star \in E$ such that $u_n \longrightarrow u^\star$ as $n \longrightarrow +\infty$.

Therefore, passing to the limit as $n \longrightarrow +\infty$ in relation (3.16), we get that

$$\left(\varphi_1'(u^\star), v\right) = \lambda^\star \left(\psi_1'(u^\star), v\right), \qquad \forall v \in E.$$

or

$$\left(I'_{\lambda^\star}(u^\star), v\right) = 0, \quad \forall v \in E.$$

so $u^\star$ is a critical point of $I_{\lambda^\star}$.

It remains to show that $u^\star$ is non trivial. In fact, if not we have $u_n \longrightarrow 0$ in E as $n \longrightarrow +\infty$ or $\|u_n\| \longrightarrow 0$, then by lemma (3.3), we deduce that:

$$\lim_{n \longrightarrow +\infty} \left(\frac{\varphi_0(u_n)}{\psi_0(u_n)}\right) = +\infty.$$

Finally, the equality (3.17) implies that

$$\lim_{n \longrightarrow +\infty} \left(\frac{\varphi_0(u_n)}{\psi_0(u_n)}\right) = \lambda^\star;$$

which is a contradiction. Consequently $u^\star \neq 0$ and thus $\lambda^\star$ is an eigenvalue of the problem (1.1).

Step(4).

We prove that $\lambda_\star \leq \lambda^\star$. Since $\lambda^\star$ is an eigenvalue of the problem (3.4), then by theorem (3.1), we deduce that

$$\lambda^\star \quad \notin \quad]0; \lambda_\star[,$$

Saw That $0 < \lambda^\star$ therefore $\lambda_\star \leq \lambda^\star$. The proof of theorem (3.2) is now completed.

Remark 3.5. *We are not able deduce whether $\lambda_\star = \lambda^\star$ or $\lambda_\star < \lambda^\star$. Therefore, In the latter case, an interesting open problem conserns the existence of solutions of problem (1.1) in the interval $[\lambda_\star < \lambda^\star)$.*

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