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Numerical Approach for Reconstructing an Unknown Source Function in Inverse Parabolic Problem

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Abstract

The inverse problem considered in this paper is devoted to reconstruction of the unknown source term in parabolic equation from additional information which is given by measurements at final time. The cost functional is introduced and existence of the minimizer for this functional is established. The numerical algorithm to solve the inverse problem is based on the Ritz-Galerkin method with shifted Legendre polynomials as basis functions. Finally, some numerical results are presented to demonstrate the accuracy and efficiency of the proposed method for test example.

Keywords: Inverse source Problem, Cost Functional, Ill-Posed Problem, Regularization Method, Ritz-Galerkin Method.
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1. Introduction

Inverse problems of identification coefficients in parabolic partial differential equations have been become increasingly popular in recent years due to its important role in various fields of applications such as, Mathematics, heat transfer process, chemistry, biology and engineering; etc [1, 3, 4, 5, 11, 12]. The problems receive considerable attention from a lot of authors by using different stable numerical and analytical approaches. Some considerations in this area can be found in [10, 11, 13, 14, 17, 18]. The inverse problems are very sensitive to random errors in input data due to measurements, thus requiring the special methods to overcome this difficultly is necessary. So, the inverse problem involves its reformulation as a well-posed problem and make use of regularization strategy. The selection of the

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efficient numerical method for these kind of problem is one of the challenge for numerical simulation. Different numerical approaches have been employed for solving inverse problems of identifying source term, such as the meshless method [16], mollification method [18], homotopy analysis method [13] and etc. The design of the stable numerical methods for these kind of inverse problems is important because it is crucial to recover the unknown coefficients. The algorithm in this paper is based on the Ritz-Galerkin method.
The Ritz-Galerkin is a numerical approach for solving prabolic problems but there is a constraint for selecting the satisfier function in the method [17]. In Ritz-Galerkin method, a set of base functions is choosen such that this set is dense in function space and each base function satisfies the given homogenous boundary conditions, then the numerical approximation is defined by linear combination of base functions. Employing the approximate solution in the Ritz-Galerkin method usually results a linear system for finding the unknown coefficients. The main goal of this work is devoted to determine heat source function in parabolic equation which is depends on space variable with overspecified condition which is given by observations at final time.
The outline of this article is arranged as follows:
The mathematical formulation of the problem with appropriate conditions is given in Section 2. The cost functional is defined in Section 3 and we prove that a minimizer for this functional exists. The Ritz-Galerkin method is introduced in Section 4. In Section 5, the Ritz-Galerkin approach is implemented for the inverse problem via appropriate general base functions. Since the resultant linear system by proposed method is ill-conditioned, regularization technique should be employed to acquire stable solution and a threshold given by the L-curve criterion. In Section 6, some properties of Legendre and shifted Legendre polynomials are introduced to use as a basis set in the Ritz method. In Section 7, the obtained results for test example are reported and discussed to illustrate the ability and efficiency of the proposed approach. Conclusions are drawn in Section 8.

2. Mathematical Formulation

In this section, we consider the following inverse problem of recovering the temperature $u(x,t)$ and the source term $f(x)$ in the parabolic equation

$$\rho(x,t)u_t(x,t) = a(x)u_{xx}(x,t) + b(x)u_x(x,t) + c(x)u(x,t) + f(x), \qquad 0 < x < l,\ 0 < t \leq T, \tag{1}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \qquad 0 \leq x \leq l, \tag{2}$$

$$u(0,t) = p(t), \qquad 0 \leq t \leq T, \tag{3}$$

$$u(l,t) = q(t), \qquad 0 \leq t \leq T, \tag{4}$$

and an overdetermination condition which is given by measurments at final time as follows:

$$u(x,T) = u_T(x), \qquad 0 \leq x \leq l, \tag{5}$$

where $\rho(x,t), a(x), b(x), c(x), p(t), q(t), u_T(x)$ and $u_0(x)$ are considered as known smooth functions, l and T are positive constants and $f(x)$ is an unknown source function to be estimated.
The unique solvability of this problem under suitable assumptions is investigated in [6]. The unknown source function in this model arise in various applications such as in hydrology, heat transfer problems and ... [1, 3, 4].
For a give function $f(x)$, the problem (1)-(4) is called forward problem which is well-posed problem, so that for $f(x) \in C^{\alpha}(0,l), \alpha > 0$ there exists a unique solution $u(x,t)$. However, the inverse problem of determination $u(x,t)$ and $f(x)$ from overspecifiesd condition (5) is ill-posed according to Hadamard definition [10, 15].

3. The Cost Functional and its Minimizer

The unique solution for the forward problem (1)-(4) under suitable conditions was proved by Schauder's theory for parabolic differential equations. If for a given $f(x)$ the solution of forward problem is denoted by $u(x,t,f)$, then we consider the following minimization problem.
Find $f(x) \in \mathcal{A} = \{h(x) | \|h\| \leq M, \quad h \in H^1(0,l)\}$ such that for functional $J(f)$ which is defined as follows,

$$J(f) = \frac{1}{2}\int_0^l |u(x,T,f) - u_T(x)|^2 dx + \lambda \int_0^l |\nabla f|^2 dx,$$

the following relation is holded

$$J(f) = \min_{h(x)\in\mathcal{A}} J(h(x)),$$

where $u(x,t,f)$ is a solution of Eq. (1) for $f(x) \in \mathcal{A}$ and M is positive constant and λ is the regularization parameter. Also, we assume that $u_T(x) \in L^2(0,l)$. In the following theorem, the existence of minimizer for $J(f)$ is considered.
Theorem 1. There exists $\widetilde{f}(x) \in \mathcal{A}$ such that $J(\widetilde{f}) = \min_{f(x)\in\mathcal{A}} J(f(x))$.

Proof . Proof. Since $J(f) \geq 0$ for every $f \in \mathcal{A}$, the quantity $Inf J(f)$ axists, the infimum begin over all admissible elements of $\mathcal{A}$.
There exists a minimizing sequence $\{u_n\}$ and $\{f_n\} \in \mathcal{A}$ such that $\lim_{n\to\infty} J(f_n) = J(f)$. Obviously, we may assume that for every $n = 1, 2, \cdots$

$$\inf_{f\in\mathcal{A}} J(f) \leq J(f_n) \leq \inf_{f\in\mathcal{A}} J(f) + \frac{1}{n},$$

Since $J(f_n) \leq Q$ so we have

$$\| \bigtriangledown f_n\|_{L^2} \leq Q.$$

By sobolev embedding theorem, we have

$$\|f_n\|_{C^{1/2}(0,l)} \leq Q,$$

and so

$$\|u_n(x,t)\|_{C^{1/2,1/4}([0,l]\times[0,T])} \leq Q.$$

It can be easily see that $\mathcal{A}$ is a closed subset of $H^1(0,l)$ and so $\mathcal{A}$ is a compact set. The sequences $\{f_n\}$ and $\{u_n\}$ have subsequences $\{f_{n_k}\}$ and $\{u_{n_k}\}$ such that converge with respect to the appropriate metric spaces to the some elements $\widetilde{f}(x)$ and $\widetilde{u}(x,t)$. By the continuity of the $J(f)$, we obtain

$$\inf_{f\in\mathcal{A}} J(f) = \lim_{n\to\infty} J(f_n) = \lim_{n_k\to\infty} J(f_{n_k}) = J(\widetilde{f}).$$

Hence $J(\widetilde{f}) = \min_{f\in\mathcal{A}} J(f)$.
This complete the proof of the theorem. □

In the following section a numerical approach for recovering the unknown source term $f(x)$ based on the Ritz method is introduced.

4. A Review of the Ritz-Galerkin Approach

We consider the following differential equation

$$L[y(x)] + f(x) = 0, \quad a < x < b. \tag{6}$$

Multiplying both side of Eq. (6) by arbitrary weight function $w(x)$, then:

$$\int_b^a w(x)(L[y(x)] + f(x))dx = 0. \tag{7}$$

Since, $w(x)$ is any arbitrary function then Eqs. (6) and (7) are equivalent. The approximate solution $u(x)$ for Eq.(6) is introduced as a linear combination of base functions as follows:

$$u(x) = \varphi_0(x) + \sum_{j=1}^{n} c_j \varphi_j(x), \tag{8}$$

where $\varphi_j(x)$ are basis functions. By replacing the exact solution $y(x)$ with the approximate solution $u(x)$ in the Eq.(6), the residual term is defined as follows:

$$res(x) = L[u(x)] + f(x). \tag{9}$$

Our goal is to choose $u(x)$ so that the Eq.(7) holds for approximate solution $u(x)$ instead of $y(x)$ and for some selection of weight functions. In the Galerkin method, the weight functions are choosen from the basis functions $\varphi_i(x)$. It is necessary that the following equations hold

$$\int_a^b \varphi_i(x)(L[u(x)] + f(x))dx = 0, \quad i = 1, \cdots, n. \tag{10}$$

The above n equations leads the linear or nonlinear system of equations for the unknown coefficients c_j. For solving a boundary value problem with this method, it is required that the functions $\{\varphi_i(x)\}_{i=1}^n$ satisfy the homogeneous form of the defined boundary conditions and $\varphi_0(x)$ must satisfy the defined essential boundary conditions.

5. Implementation of the Ritz-Galerkin Approach

Our strategy in this section is to convert the inverse problem (1)-(5) to the direct problem.

For convenience, we assume $l = 1$. By applying Eqs. (1) and (5) we have

$$\rho(x,t)u_t(x,T) = a(x)u_{xx}(x,T) + b(x)u_x(x,T) + c(x)u(x,T) + f(x).$$

thus

$$f(x) = \rho(x,t)u_t(x,T) - a(x)u_{xx}(x,T) - b(x)u_x(x,T) - c(x)u(x,T). \tag{11}$$

Substituting (11) into Eq. (1) yields

$$\begin{aligned}\rho(x,t)u_t(x,t) &= a(x)u_{xx}(x,t) + b(x)u_x(x,t) + c(x)u(x,t)\\ &\quad + \rho(x,t)u_t(x,T) - a(x)u''_T(x) - b(x)u'_T(x) - c(x)u_T(x)\end{aligned} \tag{12}$$

By setting

$$w(x,t) = u(x,t) - [(1-x)p(t) + xq(t)].$$

It's obvious that

$$w(0,t) = w(1,t) = 0,$$

and

$$w(x,0) = u(x,0) - [(1-x)p(0) + xq(0)].$$

Now Eqs. (1)-(4) are equivalent to

$$\begin{aligned}\rho(x,t)[w_t + (1-x)p'(t) + xq'(t)] =& a(x)w_{xx} + b(x)[w_x - p(t) + q(t)] \\ &+ c(x)[w + (1-x)p'(t) + xq'(t)] \\ &+ \rho(x,t)[w_t(x,T) + (1-x)p'(T) + xq'(T)] \\ &- a(x)u_T''(x) - b(x)u_T'(x) - c(x)u_T(x),\end{aligned} \tag{13}$$

with homogenous boundary conditions

$$w(0,t) = w(1,t) = 0, \tag{14}$$

and initial condition

$$w(x,0) = u(x,0) - [(1-x)p(0) + xq(0)]. \tag{15}$$

Let

$$\begin{aligned}F(w) =& \rho(x,t)[w_t + (1-x)p'(t) + xq'(t)] - a(x)w_{xx} \\ &- b(x)[w_x - p(t) + q(t)] - c(x)[w + (1-x)p'(t) + xq'(t)] \\ &- \rho(x,t)[w_t(x,T) + (1-x)p'(T) + xq'(T)] \\ &+ a(x)u_T''(x) + b(x)u_T'(x) + c(x)u_T(x).\end{aligned} \tag{16}$$

Now, we apply the Ritz-Galerkin method to construct an approximate solution for Eq.(16). The numerical approximation $\hat{w}(x,t)$ is defined in the form

$$\widehat{w}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} x(x-1)t(t-T)c_{ij}\phi_{ij}(x,t) + s(x,t), \tag{17}$$

where $\phi_{ij}(x,t)$ are known basis functions and $s(x,t)$ is an interpolating function as follows:

$$\begin{aligned}s(x,t) =& u_0(x) - [(1-x)p(0) + xq(0)] \\ &+ \frac{t}{T}[u_T(x) - ((1-x)p(T) + xq(T)) - u_0(x) + (1-x)p(0) + xq(0)].\end{aligned} \tag{18}$$

The approximate solution $\widehat{w}(x,t)$ satisfies the Eqs.(14), (15). The unknown coefficients c_{ij} are determined by the following equations:

$$< F(\hat{w}), \phi_{ij}(x,t) >= 0, \quad i = 0,\cdots,n, \quad j = 0,1,\cdots,m, \tag{19}$$

where the inner product $< . >$ is defined by

$$< F(\hat{w}), \phi_{ij}(x,t) >= \int_0^1\int_0^T F(\hat{w}(x,t))\phi_{ij}(x,t)dtdx.$$

From Eq. (19) a linear system of equations is derived for the unknown elements $c_{ij}, i = 0,\cdots,n, j = 0,1,\cdots,m$ which is represented by $\mathbf{AC} = \mathbf{b}$ where $\mathbf{A}$ is coefficients matrix. Since the resulting matrix is ill-conditioned so applying the regularization technique is necessary.

5.1. The Regularization Method for linear system

Due to the ill-conditionary of the coefficients matrix A, our strategy is based on the regularization techniques. The sensitivity of solution with respect to noise in measured data is investigated. The numerical approaches for solving ill-conditioned linear system with some type of regularization have been considered by many authors [7, 15]. In this work Tikhonov regularization method is applied to solve ill-condition system to investigate the sensitivity of solution to random error in data.
Let λ be given constant. The regularized solution X_λ for linear system $AX = \mathbf{b}$ is the minimize of the functional Tikhonov which is defined as follows:

$$J_\lambda(X) = \| AX - \mathbf{b} \|^2 + \lambda \| X \|^2, \tag{20}$$

where λ is called regularization parameter and $\| . \|$ denotes the usual Euchlidean norm by default. The selection of optimum value for λ is crucial in this approach and still under research since it's defined the amount of regularization. The regularization parameter balance the data fidelity and the regularity of the solution. In this paper, the choice of suitable λ is based on L-curve criterion.
L-curve: The L-curve criterion is a well-know technique for choosing the regularization parameter. The criteria is based on the graph of the norm of residual term versus the norm of regularized solution as log-log scale and is sketched in the following.

$$L = \{ \left(log(\| X_\lambda \|^2), log(\| AX_\lambda - \mathbf{b} \|^2) \right), \lambda > 0 \},$$

The obtained curve is L-shape and so the method is called L-curve and the optimum value of λ corresponds to its corner [8, 9].
In the next section a brief introduction for the Legendre polynomials are stated.

6. The Legendre and Shifted Legendre Polynomials

The Legendre polynomials are defined as regular solutions of the following differential equation which is called Legendre's equation [2],

$$(1 - x^2)u'' - 2xu' + n(n+1)u = 0, \quad n = 0, 1, 2, \cdots,$$

The solution of the above equation is denoted by $p_n(x)$ as a polynomial of degree n for some special value of n. Also, an analytical form of these polynomials can be defined as follows :

$$p_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)! x^{n-2k}}{2^n k! (n-k)! (n-2k)!}.$$

The Legendre's polynomials can also constructed by using the Rodrigue's formula:

$$p_n(x) = \frac{1}{(2^n n!)} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

The orthogonality of these polynomials with respect to weight function $w(x) = 1$ in the interval [-1,1] is defined by

$$\langle p_n(x), p_m(x) \rangle = \int_{-1}^{1} p_n(x) p_m(x) dx = \begin{cases} 0, & n \neq m, \\ \frac{2}{2n+1}, & n = m. \end{cases}$$

In order to use these polynomials on the interval $x \in [0, 1]$, we defined the shifted Legendre polynomials by introducing the change of variable $x \to 2x - 1$. Let the shifted Legendre polynomial

$p_n(2x-1)$ is denoted by $\tilde{p_n}(x)$. Then the polynomials $\tilde{p_n}(x)$ are orthogonal on $[0,1]$ which means that:

$$\int_0^1 \tilde{p_n}(x)\tilde{p_m}(x)dx = \frac{1}{2n+1}\delta_{m,n}.$$

An explicit expression for the shifted Legendre polynomials is given by:

$$\tilde{p_n}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-x)^k.$$

In our computation the basis functions $\phi_{ij}(x,t)$ are chosen as follows:

$$\phi_{ij}(x,t) = \tilde{p_i}(x)\tilde{p_j}(t).$$

In the next section a test example is given in order to investigate the efficiency and validity of the present method.

7. Numerical Results

A selected test example is presented in this section to study the ability of the proposed numerical approach. The numerical method in this work is employed with different values of parameters n, m to solve the inverse problem (1)-(5). By Eq. (17), the approximate solution based on the strategy of the Ritz method is assumed in the form

$$\widehat{w}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} x(x-1)t(t-T)c_{ij}\phi_{ij}(x,t) + s(x,t).$$

By imposing the following conditions:

$$< F(\hat{w}), \phi_{ij}(x,t) >= \int_0^1\int_0^T F(\hat{w})\phi_{ij}(x,t)dtdx, i = 0,\cdots,n, j = 0,1,\cdots,m.$$

The resultant linear algebraic equations can be represented by

$$\mathbf{AC} = \mathbf{b}, \tag{21}$$

where the vectors $\mathbf{C}$, $\mathbf{b}$ denote the vectors of unknown constant coefficients and known right hand side respectively, and $\mathbf{A}$ is a known coefficients matrix.

The stability of the proposed method for test example is investigated when some perturbed approximation of overdetermination data (5) is in hand. The perturbed data at point x_i is produced by adding a noise ϵ_i to exact data $u_T(x)$ as follows:

$$u_\epsilon(x_i,T) = u_T(x_i) + \epsilon_i, \qquad i = 0,1,...,N,$$

where ϵ_i are random variables which are generated from a Gaussian normal distribution with zero mean and standard deviation σ given by

$$\sigma = \delta \times Max_{x\in[0,1]} \mid u_T(x) \mid,$$

where δ is represent the percentage of noise. The *normrand* command in MATLAB is used to generate the random variables $(\epsilon_i)_{i=0,1,\ldots,N}$. We investigate the effect of this perturbed function on the

solution of problem. By employing the Tikhonov regularization method, the ill-conditioned system $\mathbf{AC} = \mathbf{b}$ is replaced by system $(A^T A + \lambda \mathbf{I})\mathbf{C} = A^T \mathbf{b}$, where A^T is the transpose of $\mathbf{A}$ and λ is found by L-curve criteria [8, 9].
We presents the root mean square error (RMSE) defined by:

$$RMSE(f(x)) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[f_T(x_i) - f_{approx}(x_i)]^2},$$

to measure the values of error in our computations.

We implement the proposed method with Matlab 2017b software in a personal computer.

7.1. Example

Consider the following inverse problem (1)-(5) with input data

$$\rho(x,t) = 1, \quad a(x) = 2, \quad b(x) = 0, \quad c(x) = -\frac{\pi^2}{2}, \quad u_0(x) = sin(\frac{\pi}{2}x),$$
$$p(t) = 0, \quad q(t) = 2 - e^{-\pi^2 t}, \quad u(x,T) = u_T(x) = (2 - e^{-\pi^2})sin(\frac{\pi}{2}x),$$

and $T = 1$. The exact solutions of the inverse problem (1)-(5) are as follows:

$$u(x,t) = (2 - e^{-\pi^2 t})sin(\frac{\pi}{2}x), \qquad f(x) = 2\pi^2 sin(\frac{\pi}{2}x).$$

Fig. 1 shows the comparison between the exact and numerical solution of $u(x,t)$ when the input data (5) is exact. The optimal value of λ by the L-curve is represented in Fig. 1.
Fig. 2 displays the behaviour of exact and numerical solution of $f(x)$ when the overdetermined data (5) is contaminated by $\delta \in \{1,3,5\}\%$ noise. These figures show that when the over determination condition (5) is perturbated by noise then the numerical approximation becomes unstable due to the noise. Fig. 3 shows the absolute errors of $u(x,t)$ for different values of δ.
The results for comparison between the exact source term $f_{exact}(x)$ and the numerical solution $f_\epsilon(x)$ for different values of noise level δ along with their RMSE is shown in Table 1. Table 2 lists the computing results for exact source term $f_{exact}(x)$ and approximate solution $f_{approx}(x)$ for different values of m, n along with their RMSE.
From figures and tables for this example, it can be been seen that when the noise level decrease to zero, the computed solution goes to the exact solution and the obtained numerical approximations are stable in the presence of noise in input data.

8. Conclusion

In this work a numerical approach based upon the Ritz-Galerkin method combined with the Tikhonov regularization technique for solving an inverse parabolic source problem is presented. The key of the method is to employ the shifted Legender polynomials as basis functions in approximate solution of the proposed method and so the original problem reduce to solve an ill-conditioned linear system. The numerical approach is applied to solve a test example in one dimensional case and the numerical results show that the method can be applied for parabolic inverse problem due to its efficiency. The obtained numerical approximations are accurate for noisy data and so the numerical method produces stable results.

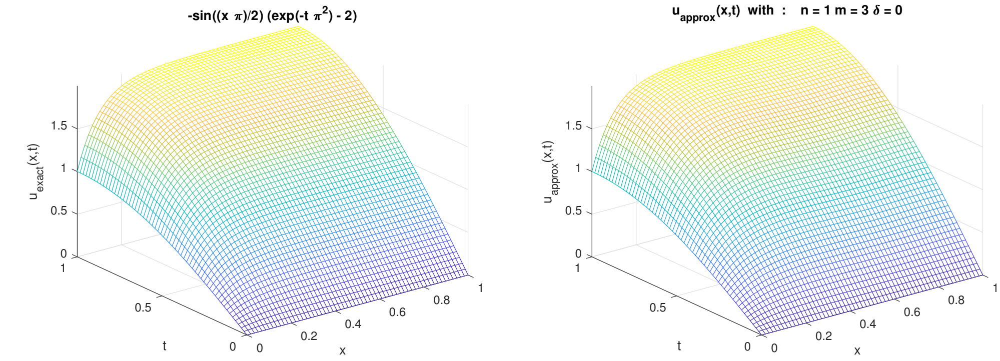

Figure 1: The comparison between the exact and numerical solution of $u(x,t)$ with n=1, m=3 and $\lambda = 7.15004 \times 10^{-4}$.

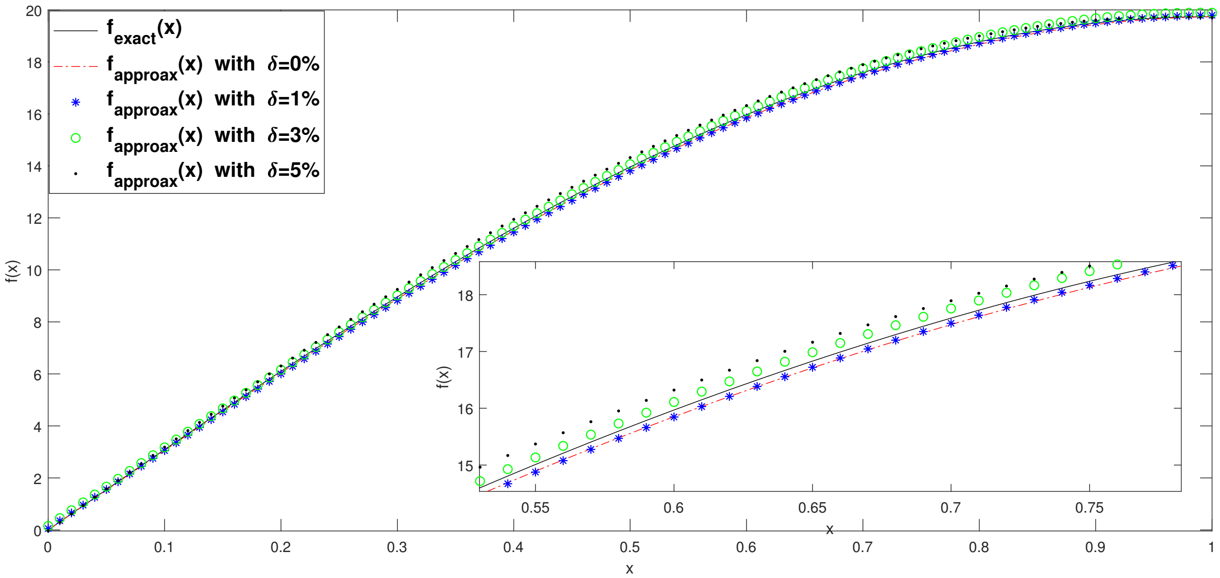

Figure 2: The comparison between the exact and numerical solution of f(x) for different values of noise level with n=1, m=3.

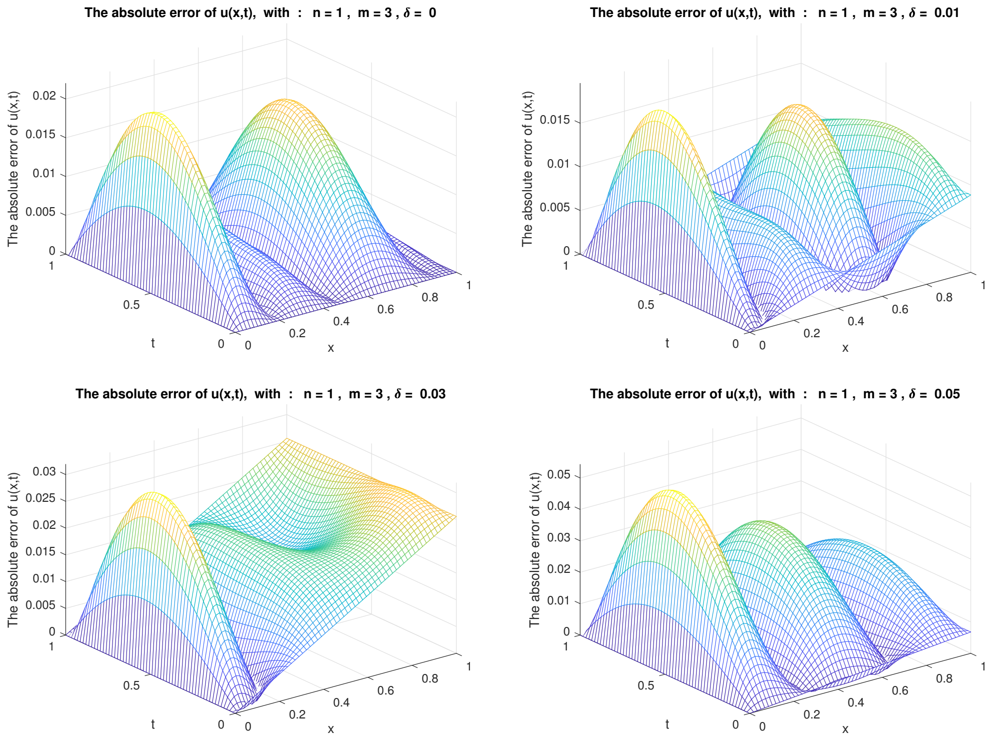

Figure 3: The absolute error of numerical solution of $u(x,t)$ with n=1, m=3 and differet values of δ.

x	Exact solution	Numerical solution δ=0%	Numerical solution δ=1%	Numerical solution δ=3%	Numerical solution δ=5%
0.0	0.0000	0.0000	0.0526	0.1518	-0.0403
0.1	3.0879	3.0595	3.0501	3.1742	3.1829
0.2	6.0998	6.0443	6.0013	6.1577	6.3049
0.3	8.9614	8.8816	8.8279	9.0194	9.2492
0.4	11.6024	11.5026	11.4556	11.6798	11.9432
0.5	13.9577	13.8440	13.8157	14.0654	14.3200
0.6	15.9694	15.8499	15.8468	16.1100	16.3201
0.7	17.5878	17.4735	17.4965	17.7563	17.8927
0.8	18.7731	18.6780	18.7226	18.9572	18.9976
0.9	19.4962	19.4377	19.4941	19.6768	19.6054
1.0	19.7392	19.7392	19.7918	19.8910	19.6989
$RMSE$		0.0072	0.0097	0.0169	0.0653

Table 1: The comparison between the results of exact and numerical solution of $f(x)$ for different values of noise level with $n = 1, m = 3$, with its RMSE.

x	Exact solution	Numerical solution $n{=}1,m{=}3$	Numerical solution $n{=}2,m{=}5$	Numerical solution $n{=}3,m{=}1$
0.0	0.0000	0.0000	0.0000	0.0000
0.1	3.0879	3.0595	3.0849	3.0159
0.2	6.0998	6.0443	6.0944	5.9656
0.3	8.9614	8.8816	8.9540	8.7763
0.4	11.6024	11.5026	11.5930	11.3793
0.5	13.9577	13.8440	13.9464	13.7124
0.6	15.9694	15.8499	15.9563	15.7205
0.7	17.5878	17.4735	17.5737	17.3577
0.8	18.7731	18.6780	18.7599	18.5881
0.9	19.4962	19.4377	19.4870	19.3865
1.0	19.7392	19.7392	19.7392	19.7392
$RMSE$		0.0072	$9.2714e-05$	0.0325

Table 2: The comparison between the results of exact and numerical solution of $f(x)$ for different values of m,n with its RMSE.

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