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Investigating the dynamics of Lotka−Volterra model with disease in the prey and predator species

Atena Ghasemabadi[1], Mohammad Hossein Rahmani Doust[1]

[a] *Esfarayen University of Technology, Esfarayen, North Khorasan, Iran*
[b] *Department of Mathematics, University of Neyshabur, Adib BLVD, Neyshabur, Iran*

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Abstract

In this paper, a predator−prey model with logistic growth rate in the prey population was proposed. It included an SIS infection in the prey and predator population. The stability of the positive equilibrium point, the existence of Hopf and transcortical bifurcation with parameter a were investigated, where a was regarded as predation rate. It was found that when the parameter a passed through a critical value, stability changed and Hopf bifurcation occurred. Biologically, the population is positive and bounded. In the present article, it was also shown that the model was bounded and that it had the positive solution. Moreover, the current researchers came to the conclusion that although the disease was present in the system, none of the species would be extinct. In other words, the system was persistent. Important thresholds, R_0, R_1 and R_2, were identified in the study. This theoretical study indicated that under certain conditions of R_0, R_1 and R_2, the disease remained in the system or disappeared.

Keywords: Differential Equations, Threshold, Prey−Predator Model, Global Stability, SIS Disease.
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1. Introduction

Mathematical modeling is an important tool in analyzing the spread and control of infectious diseases. Recently, researchers have paid more attention to epidemiological models.

*Atena Ghasemabadi
Email addresses: ghasemabadi.math@gmail.com (Mohammad Hossein Rahmani Doust), mh.rahmanidoust@neyshabur.ac.ir (Mohammad Hossein Rahmani Doust)

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In 1926, Lotka−Volterra introduced a model of interaction between species. This model is applied not only in the ecology, but also in complex systems such as computer games [9], genetics [16], marriage [8], biochemical reactors [3], war [2] and urban sewage purification systems.

Anderson and May [1] studied a model of Lotka−Volterra prey−predator in which the diseased prey was hunted to a greater degree and infected prey didn't have any reproduction. They inferred that the disease destabilizes the interactions between the prey and predator. Hadeler and Freedman [10] investigated a Rosenzweig prey−predator model with higher predation on diseased prey. In this model, predators spread parasites into the environment. In this way, the prey contracted the disease. Predators came down with disease only by eating the infected prey. Hadeler and Freedman got a threshold at which an infected periodic solution or an infected equilibrium appeared. Venturino [19] also examined the effect of SIS diseases on Lotka-Volterra systems.

A delay prey-predator model with Holling type II functional response was examined by Mukhopadhyaya and Bhattacharyya [13]. In their research, they considered the dynamics of the system with the effect of delay and diffusion on the stability and persistence of the model.

Pielou [17] formulated the following Lotka−Volterra model

$$\begin{cases} \dot{N}_1 = \left[r_1 \left(1 - \dfrac{N_1}{K_1} \right) - a\, N_2 \right] N_1, \\ \dot{N}_2 = \left[k\, a\, N_1 - d_2 \right] N_2, \end{cases} \tag{1.1}$$

where N_1 and N_2 are the density of prey and predator population respectively. Ghasemabadi [7] examined a more general form of the system (1.1). In her research, she studied the stability and Hopf bifurcation of the system.

In the current study, a model of prey and predator diseases was investigated based on the system (1.1). Disease was considered in both prey and predator species. In comparison to the previous models, this model was based on a predator-prey model with an attractive equilibrium point. Moreover, disease can persist in the predator's species and it is possible for the predators to contract the disease during the predation process.

In section 2, the SIS predator−prey with the standard incidence and the mass action incidence has been introduced. Population is always positive. In the second section, the present researchers have proved this point. Then, it has been shown that the population is bounded.

In section 3, the researchers have investigated the local and global stability of equilibrium points.

In Section 4, we prove that the system is persistent and show that all species survive, and none of them will be extinct.

In the final section, the researchers obtain conditions for Hopf and Transcritical bifurcations.

2. Mathematical model

In this section, the present researchers consider a model in which prey and predator are suffering from a disease and predators feed on both the susceptible and infectious prey. The total size of prey and predator population is

$$N_1 = S_1 + I_1, \qquad N_2 = S_2 + I_2,$$

where S_1 , S_2, I_1 and I_2 are the numbers of susceptible prey, susceptible predator, infectious prey and infectious predator respectively.

Susceptible individuals become infectious by their contact with infectious individuals. When an individual is susceptible to disease again after recovery, this model is called a SIS disease. Bacterial infections are SIS diseases. When an individual has permanent immunity after recovery, this model is called a SIR disease. Viral infections are SIR diseases. Based on the assumptions, model (1.1) is formulated as the following SIS model:

$$\begin{cases} S_1 = \left[b_1 - \dfrac{a_1\, r_1\, N_1}{K_1}\right]\, N_1 - \left[d_1 + (1-a_1)\dfrac{r_1\, N_1}{K_1}\right] S_1 - a\, N_2\, S_1 - \beta_1 \dfrac{S_1 I_1}{N_1} + \gamma_1 I_1 \\ \dot{I}_1 = \beta_1 \dfrac{S_1 I_1}{N_1} - \gamma_1 I_1 - \left[d_1 + (1-a_1)\dfrac{r_1\, N_1}{K_1}\right] I_1 - a\, N_2\, I_1 \\ \dot{N}_1 = r_1 \left(1 - \dfrac{N_1}{K_1}\right)\, N_1 - a\, N_1\, N_2 = \left[r_1\left(1 - \dfrac{N_1}{K_1}\right) - a\, N_2\right]\, N_1, \\ \dot{S}_2 = k\, a\, N_1\, N_2 - \alpha \dfrac{S_2 I_1}{N_2} - d_2 S_2 - \beta_2 \dfrac{S_2 I_2}{N_2} + \gamma_2 I_2, \\ \dot{I}_2 = \left[\beta_2 \dfrac{S_2}{N_2} - d_2 - \gamma_2\right]\, I_2 + \alpha \dfrac{S_2 I_1}{N_2} \\ \dot{N}_2 = k\, a\, N_1\, N_2 - d_2\, N_2 = [k\, a\, N_1 - d_2]\ N_2, \end{cases} \tag{2.1}$$

The six differential equations of system (2.1) can be reduced to the following four differential equations:

$$\begin{cases} \dot{I}_1 = \left[\beta_1 \dfrac{N_1 - I_1}{N_1} - \gamma_1 - d_1 - (1-a_1)\dfrac{r_1\, N_1}{K_1} - a\, N_2\right] I_1 \\ \dot{N}_1 = \left[r_1\left(1 - \dfrac{N_1}{K_1}\right) - a\, N_2\right]\, N_1, \\ \dot{I}_2 = \left[\beta_2 \dfrac{N_2 - I_2}{N_2} - d_2 - \gamma_2\right]\, I_2 + \alpha \dfrac{(N_2 - I_2) I_1}{N_2} \\ \dot{N}_2 = [k\, a\, N_1 - d_2]\ N_2, \end{cases} \tag{2.2}$$

All the parameters with their biological meanings are given in the following table.

a_1	convex combination constant of prey
b_1	natural birth rate constant of prey
d_1	natural death rate constant of prey
d_2	natural death rate constant of predator
$r_1 = b_1 - d_1$	growth rate constant of prey
K_1	carrying capacity of the environment of prey
β_1	daily contact rate of prey
β_2	daily contact rate of predator
γ_1	recovery rate constant of prey
γ_2	recovery rate constant of predator
k	efficiency in turning predation into new predators
α	average number of contacts
a	predation rate

Table 1: Description of parameters for system (2.1)

We show that system (2.2) is well−defined. For the well-defined system of (2.2), we need to show that the system has a positive solution and the system is bounded. Biologically, the population of each species is always positive and bounded. By using [7], we can easily prove the following theorem:

Theorem 2.1. *[7] Let* (I_1, N_1, I_2, N_2) *be any solution of system* (2.2) *with the initial conditions* $I_1(0) > 0, N_1(0) > 0, I_2(0) > 0, N_2(0) > 0$ *then*

$$I_1(t) > 0, \quad N_1(t) > 0, \quad I_2(t) > 0, \quad N_2(t) > 0 \quad for\ all \ \ t > 0$$

Proposition 2.2. *Total population of prey,* N_1*, is bounded.*

Proof . From the second equation of system (2.2), the following inequality holds:

$$\frac{dN_1}{dt} \le r_1\, N_1 \left(1 - \frac{N_1}{K_1}\right)$$

For equation

$$\frac{dN_1}{dt} = r_1\, N_1 \left(1 - \frac{N_1}{K_1}\right),$$

the equilibrium point $N_1 = K_1$ is globally asymptotically stable. Hence, for any $\epsilon > 0$, when $t \to +\infty$ the following relation holds:

$$N_1(t) \le K_1 + \epsilon$$

therefore, N_1 is bounded. □

Prey population is always less than the capacity of the environment. The above proposition confirms this matter.

Theorem 2.3. *All the solutions of system* (2.2) *are uniformly bounded.*

Proof . Let $X = k\, N_1 + N_2$. Time derivative of X is given as

$$\frac{dX}{dt} = k\frac{dN_1}{dt} + \frac{dN_2}{dt}$$

then

$$\begin{aligned}\frac{dX}{dt} &= k\, r_1 \left(1 - \frac{N_1}{K_1}\right) N_1 - k\, a\, N_1\, N_2 + k\, a\, N_1\, N_2 - d_2\, N_2 \\ &= k\, r_1\, N_1 - k\, r_1 \frac{N_1^2}{K_1} - d_2\, N_2 \\ &\le k\, r_1\, K_1 - d_2\, N_2,\end{aligned}$$

by using proposition (2.2) and the above inequality, we have

$$\begin{aligned}\frac{dX}{dt} + \mu X &\le k\, r_1 K_1 - d_2\, N_2 + k\, \mu\, N_1 + \mu N_2 \\ &\le k K_1 (r_1 + \mu) + (\mu - d_2)\, N_2,\end{aligned}$$

choosing $\mu \le d_2$, then

$$\frac{dX}{dt} + \mu X \le k\, K_1 (r_1 + \mu) = M \tag{2.3}$$

A solution of inequality (2.3) is given as

$$X(t) \le Ce^{-\mu t} + \frac{M}{\mu}$$

and

$$\begin{aligned} X(t) &\le X(0)e^{-\mu t} + \frac{M}{\mu}(1 - e^{-\mu t}) \\ &\le \max\left(X(0), \frac{M}{\mu}\right). \end{aligned}$$

Therefore, $\limsup X(t) \le \frac{M}{\mu}$ as $t \to +\infty$ independent of the initial conditions. □

3. Equilibria

System (2.2) has the following equilibrium points:

$$\begin{aligned} &E_0 = (0,0,0,0), && E_1 = (0, K_1, 0, 0), \\ &E_2 = \left(K_1\left(1 - \frac{1}{R_0}\right), K_1, 0, 0\right), && E_3 = (0, N_1^*, 0, N_2^*), \\ &E_4 = (0, N_1^*, I_2^*, N_2^*), && E_5 = (I_1^{**}, N_1^*, I_2^{**}, N_2^*), \end{aligned}$$

where

$$\begin{aligned} &N_1^* = \frac{d_2}{k\,a}, && N_2^* = \frac{r_1}{a}\left[1 - \frac{d_2}{k\,a\,K_1}\right], \\ &I_1^{**} = N_1^*\left(1 - \frac{1}{R_1}\right), && I_2^* = N_2^*\left(1 - \frac{1}{R_2}\right) \end{aligned}$$

and I_2^{**} is the positive root of

$$I_2^2 + A\,I_2 + B = 0 \tag{3.1}$$

where

$$A = \frac{(d_2 + \gamma_2)\,N_2^*}{\beta_2} - N_2^* + \alpha\frac{I_1^{**}}{\beta_2}, \qquad B = -\alpha\,\frac{N_2^*\,I_1^{**}}{\beta_2}, \tag{3.2}$$

The positive solution of equation (3.1) is obtained as

$$I_2^{**} = \frac{-A + \sqrt{A^2 - 4B}}{2}$$

Three epidemiological thresholds are

$$\begin{aligned} R_0 &= \frac{\beta_1}{\gamma_1 + b_1 - a_1 r_1}, \\ R_1 &= \frac{\beta_1}{\gamma_1 + d_1 + (1 - a_1) r_1 d_2 / k\,a\,K_1 + a\,N_2^*}, \\ R_2 &= \frac{\beta_2}{d_2 + \gamma_2}. \end{aligned}$$

The following definition and lemma are used later in this paper.

Definition 3.1. *Suppose that functions f and g are continuous and locally Lipschitz in $x \in \mathbb{R}^n$, the following equation*

$$\dot{x} = f(t, x), \tag{3.3}$$

is called asymptotically autonomous with the following limit equation

$$\dot{y} = g(y), \tag{3.4}$$

if $f(t, x) \to g(x)$ as $t \to \infty$ uniformly for x in $\mathbb{R}^n$.

Lemma 3.2. *Suppose the equilibrium point E of the limit system* (3.4) *is globally asymptotically stable and solutions of system* (3.3) *are bounded. Then any solution $x(t)$ of system* (3.3) *satisfies $x(t) \to E$ as $t \to \infty$.*

The prey will survive when prey is not contracted with disease and no predator is available. The following theorem confirms this matter.

Theorem 3.3. *The equilibrium point $E_1 = (0, K_1, 0, 0)$ is globally attractive provided*

$$\frac{d_2}{k\,a\,K_1} > 1, \qquad R_0 \leq 1, \qquad I_0 > 0.$$

Proof . We consider the fourth equation of system (2.2). Since $N_1 < K_1$, we have:

$$\dot{N_2} \leq [k\,a\,K_1 - d_2]N_2$$

The solution of the above inequality is

$$N_2 = N_0\,exp\left[k\,a\,K_1\left(1 - \frac{d_2}{k\,a\,K_1}\right)t\right]$$

Therefore $N_2 \to 0$ when $t \to +\infty$ provided $\frac{d_2}{k\,a\,K_1} > 1$. Since $I_2 \leq N_2$, therefore $I_2 \to 0$ when $t \to +\infty$ provided $\frac{d_2}{k\,a\,K_1} > 1$. Hence for every $\epsilon > 0$ there exists a $T > 0$ such that for every $t > T$ we have $N_2(t) < \epsilon$.

The second equation of system (2.2) is converted in the following way:

$$N_1 > r_1\,N_1\left(1 - \frac{N_1}{K_1}\right) - a\,\epsilon\,N_1$$

For every $\epsilon > 0$, the equilibrium $K_1\left(1 - \frac{a\,\epsilon}{r_1}\right)$ is globally asymptotically stable for the following equation:

$$\dot{N_1} = r_1\,N_1\left(1 - \frac{N_1}{K_1} - \frac{a\,\epsilon}{r_1}\right)$$

Thus, when t is large enough, the following inequality is obtained:

$$N_1(t) \geq K_1 - \epsilon,$$

On the other hand, $N_1 \leq K_1$, therefore we have

$$\lim_{t \to +\infty} N_1(t) = K_1.$$

The first equation of system (2.2) is asymptotically autonomous to

$$\dot{I}_1 = \beta_1\, I_1 \left[1 - \frac{1}{R_0}\right] - \frac{\beta_1}{K_1}\, I_1^2.$$

The solution of the above equation is

$$I_1(t) = \left(\frac{K_1(1-\frac{1}{R_0})}{1 + CK_1(1-\frac{1}{R_0})exp(-\beta_1(1-\frac{1}{R_0})t)}\right),$$

therefore $I_1 \to 0$ when $t \to +\infty$ provided $R_0 < 1$. Hence the equilibrium point $E_1 = (0, K_1, 0, 0)$ is globally attractive. □

In the following theorem, we show that the equilibria E_3, E_4 and E_5 are locally asymptotically stable.

Theorem 3.4. *The local behavior of equilibria is as follows:*
(1) $E_3 = (0, N_1^*, 0, N_2^*)$ *is locally asymptotically stable provided* $R_1 < 1$ *and* $R_2 < 1$.
(2) $E_4 = (0, N_1^*, I_2^*, N_2^*)$ *is locally asymptotically stable provided* $R_1 > 1$ *and* $R_2 > 1$.
(3) The equilibrium point $E_5 = (I_1^{**}, N_1^*, I_2^{**}, N_2^*)$ *is locally asymptotically stable provided* $R_2 < 1$.

Proof . (1) The Jacobian matrix at equilibrium point $E_3 = (0, N_1^*, 0, N_2^*)$ is

$$J_3 = \begin{bmatrix} \beta_1\left(1-\frac{1}{R_1}\right) & 0 & 0 & 0 \\ 0 & \frac{-r_1\,N_1^*}{K_1} & 0 & -a\,N_1^* \\ \alpha & 0 & \beta_2\left(1-\frac{1}{R_2}\right) & 0 \\ 0 & k\,a\,N_2^* & 0 & 0 \end{bmatrix}$$

The characteristic equation at equilibrium point $E_3 = (0, N_1^*, 0, N_2^*)$ is

$$(\lambda - A_{11})(\lambda - A_{12})(\lambda^2 - A_{13}\,\lambda - A_{14}) = 0$$

where

$$A_{11} = \beta_1\left(1-\frac{1}{R_1}\right), \qquad A_{12} = \beta_2\left(1-\frac{1}{R_2}\right)$$
$$A_{13} = \frac{-r_1\,N_1^*}{K_1}, \qquad A_{14} = -k\,a^2\,N_1^*\,N_2^*.$$

According to Routh-Hurwitz criterion E_3 is locally asymptotically stable provided $R_1 < 1$ and $R_2 < 1$. The Jacobian matrix at equilibrium point $E_4 = (0, N_1^*, I_2^*, N_2^*)$ is

$$J_4 = \begin{bmatrix} \beta_1\left(1-\frac{1}{R_1}\right) & 0 & 0 & 0 \\ 0 & \frac{-r_1\,N_1^*}{K_1} & 0 & -a\,N_1^* \\ \alpha & 0 & \beta_2\left(1-\frac{1}{R_1}\right) & 0 \\ 0 & k\,a\,N_2^* & 0 & 0 \end{bmatrix}$$

The Jacobian matrix at equilibrium point $E_5 = (I_1^{**}, N_1^*, I_2^{**}, N_2^*)$ is

$$J_5 = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ 0 & a_{42} & 0 & 0 \end{bmatrix}$$

where

$$\begin{aligned}
a_{11} &= -\beta_1 \left(1 - \frac{1}{R_1}\right), \\
a_{12} &= \left(1 - \frac{1}{R_1}\right) \left[\beta_1 \left(1 - \frac{1}{R_1}\right) - (1 - a_1)\frac{r_1 d_2}{k\, a\, K_1}\right], \\
a_{14} &= \frac{-d_2}{k} \left(1 - \frac{1}{R_1}\right), \\
a_{22} &= \frac{-r_1\, d_2}{k\, a\, K_1}, \\
a_{24} &= \frac{-d_2}{k}, \\
a_{33} &= \beta_2 \left(1 - \frac{1}{R_2}\right) - \frac{\alpha\, I_1^{**}}{N_1^*} - 2\beta_2 I_2^{**}, \\
a_{34} &= \beta_2 \frac{I_2^{**}}{N_2^{*2}} + \alpha\, \frac{I_2^{**}\, I_2^{**}}{N_2^{*2}}, \\
a_{42} &= k\, a\, N_2^*.
\end{aligned}$$

By using Routh-Hurwitz criterion, the units of (2) and (3) are proved. □

To prove the next theorem, we need the following lemma.

Lemma 3.5. *Let be $N_1(0) > 0$, $N_2(0) > 0$ and $d_2/(k\, a\, K_1) < 1$, then*

$$N_1 \to N_1^* \quad N_2 \to N_2^* \qquad \textit{as } t \to \infty$$

Proof . The characteristic equation at positive equilibrium point $E_{13} = (N_1^*, N_2^*)$ of system (1.1) is

$$\lambda^2 - B_1\lambda + D_1 = 0$$

where

$$B_1 = \frac{-r\, d_2}{k\, a\, K_1}, \quad D_1 = r\, d_2\, [1 - \frac{d_2}{k\, a\, K_1}],$$

hence $B_1 < 0, D_1 > 0$. According to Routh-Hurwitz criterion, E_{13} is locally asymptotically stable.

Now, Liapunov function V is considered:

$$V(N_1, N_2) = k(N_1 - N_1^* - N_1^* Ln(\frac{N_1}{N_1^*})) + (N_2 - N_2^* - N_2^* Ln(\frac{N_2}{N_2^*})),$$

the derivative of V with respect to time along the solution of system (1.1) is computed as

$$\frac{dV}{dt} \leq \frac{-r\, k}{K_1}(N_1 - N_1^*)^2$$

From the Liapunov-Lasalle invariance principal, it can be understood that E_{13} is globally attractive. Since it has been shown that, if $\frac{d_2}{kaK_1} < 1$,the equilibrium point E_{13} is locally asymptotically stable. Hence E_{13} is globally asymptotically stable. □

Theorem 3.6. *Assume $R_1 > 1$ and $R_2 < 1$. Then equilibrium point $E_5 = (I_1^{**}, N_1^*, I_2^{**}, N_2^*)$ is globally asymptotically stable.*

Proof . Since $N_2^* > 0$, condition $d_2/(k\,a\,K_1) < 1$ holds. By using lemma (3.5) we have

$$N_1 \to N_1^* \quad N_2 \to N_2^* \quad as \quad t \to \infty$$

The first and third equations of system (2.2) are asymptotically autonomous to

$$\dot{I}_1 = \beta_1 \left[1 - \frac{1}{R_1} - \frac{I_1}{N_1^*}\right] I_1, \tag{3.5}$$

$$\dot{I}_2 = \left[\beta_2 - \frac{\beta_2 I_2}{N_2^* - d_2 - \gamma_2}\right] I_2 + \frac{\alpha(N_2^* - I_2)I_1}{N_2^*} \tag{3.6}$$

The solution of equation (3.5) is obtained as follows:

$$I_1 = \frac{N_1^*\left(1 - \frac{1}{R_1}\right)}{1 + C\,N_1^*\left(1 - \frac{1}{R_1}\right) exp(-\beta_1\left(1 - \frac{1}{R_1}\right)t)},$$

if $R_1 > 1$ then

$$\lim_{t\to\infty} I_1(t) = I_1^*$$

In this case, equation (3.6) is asymptotically autonomous to

$$\dot{I}_2 = \left[\beta_2 - \frac{\beta_2\,I_2}{N_2^*} - d_2 - \gamma_2\right] I_2 + \alpha\,I_1^* - \frac{\alpha I_1^*\,I_2}{N_2^*}$$

the above equation is equivalent to

$$\frac{\dot{I}_2}{I_2^2 + A\,I_2 + B} = \frac{-\beta_2}{N_2^*} \tag{3.7}$$

where A and B are defined in (3.2). The solution of equation (3.7) is

$$I_2 = \frac{-A}{2} + \frac{1 + l_0\,exp(\frac{-\beta_2\sqrt{\Delta}t}{N_2^*})}{2(1 - exp(\frac{-\beta_2\sqrt{\Delta}t}{N_2^*}))}\sqrt{\Delta}$$

hence,

$$\lim_{t\to\infty} I_2(t) = \frac{-A + \sqrt{\Delta}}{2} = I_2^{**}.$$

Therefore, E_5 is global attractive and since $R_2 < 1$, it can be conluded that E_5 is locally asymptotically stable, hence E_5 is globally asymptotically stable. □

4. Persistence of the system

In the following theorem, we show that system (2.2) is persistent. Biologically, the persistence of a system means that all species survive and none of them will be extinct.

Definition 4.1. *System* (2.2) *is persistent if there exist positive constants* m_1, $m_2\, m_3, m_4$ *and* M_1, M_2, M_3, M_4 *such that every positive solution* $(I_1(t),\, N_1(t),\, I_2(t),\, N_2(t))$ *of system* (2.2) *satisfies:*

$$\begin{aligned} m_1 &\le \liminf_{t\to\infty} I_1(t) \le \limsup_{t\to\infty} I_1(t) \le M_1 \\ m_2 &\le \liminf_{t\to\infty} N_1(t) \le \limsup_{t\to\infty} N_1(t) \le M_2 \\ m_3 &\le \liminf_{t\to\infty} I_2(t) \le \limsup_{t\to\infty} I_2(t) \le M_3 \\ m_4 &\le \liminf_{t\to\infty} N_2(t) \le \limsup_{t\to\infty} N_2(t) \le M_4 \end{aligned}$$

Theorem 4.2. *let be* $\frac{k\,a\,K_1}{d_2} < 1$ *then system* (2.2) *is persistent.*

Proof . We consider the forth equation of system (2.2). Let be $\phi(t) = \frac{1}{N_2(t)}$, we get

$$\frac{d\phi}{\phi} = [d_2 - k\,a\,N_1]dt.$$

Since $N_1 \le K_1$, we have

$$\frac{d\phi}{\phi} \ge [d_2 - k\,a\,K_1]dt,$$

the answer of the above inequality is

$$N_2(t) \le N_2(0)\, exp[k\,a\,K_1 - d_2]t,$$

by using the inequality $\frac{k\,a\,K_1}{d_2} < 1$, there is $M_1 > 0$ such that

$$\limsup_{t\to\infty} N_2(t) \le M_1$$

Since $I_2 \le N_2$, the following inequality holds:

$$\limsup_{t\to\infty} I_2(t) \le \limsup_{t\to\infty} N_2(t) \le M_1$$

By using proposition (2.2), we have

$$\limsup_{t\to\infty} N_1(t) \le K_1$$

Since $I_1 \le N_1$, the following inequality holds:

$$\limsup_{t\to\infty} I_1(t) \le \limsup_{t\to\infty} N_1(t) \le K_1$$

The first equation of system (2.2) can be written in the following inequality

$$\begin{aligned}
\dot{I}_1 &= \left[\beta_1 \frac{N_1 - I_1}{N_1} - \gamma_1 - d_1 - (1-a_1)\frac{r_1\, N_1}{K_1} - a\, N_2\right] I_1 \\
&\geq \left[\beta_1 - \gamma_1 - d_1 - (1-a_1)\, r_1 - a\, M_1 - \frac{\beta_1\, I_1}{N_1}\right] I_1 \\
&\geq \left[-\gamma_1 - d_1 - (1-a_1)\, r_1 - a\, M_1\right] I_1
\end{aligned}$$

The solution of the above inequality is

$$I_1(t) \geq C\, exp(-\gamma_1 - d_1 - (1-a_1)\, r_1 - a\, M_1)t$$

therefore, there exists a $m_2 > 0$ such that

$$\liminf_{t\to\infty} I_1(t) \geq m_2$$

Since $I_1 \leq N_1$, the following inequality holds:

$$\liminf_{t\to\infty} N_1(t) \geq \liminf_{t\to\infty} I_1(t) \geq m_2$$

Now, we consider the third equation of system (2.2), we get

$$\begin{aligned}
\dot{I}_2 &= \left[\beta_2 \frac{N_2 - I_2}{N_2} - \gamma_2 - d_2\right] I_2 + \frac{\alpha(N_2 - I_2)I_1}{N_2} \\
&\geq \left[\beta_2 \frac{N_2 - I_2}{N_2} - \gamma_2 - d_2\right] I_2 \\
&\geq [-\gamma_2 - d_2] I_2
\end{aligned}$$

The solution of the above inequality is

$$I_2(t) \geq C_1\, exp(-\gamma_2 - d_2)t$$

therefore, there exists a $m_3 > 0$ such that

$$\liminf_{t\to\infty} I_2(t) \geq m_3$$

Since $I_2 \leq N_2$, the following inequality holds:

$$\liminf_{t\to\infty} N_2(t) \geq \liminf_{t\to\infty} I_2(t) \geq m_3,$$

therefore, system (2.2) is persistent. □

5. Bifurcation

The parameter a which is the predation rate is identified as a bifurcation parameter.

5.1. Hopf bifurcation

In this subsection, we investigate the Hopf bifurcation around the interior equilibrium point E_5. Hopf bifurcation occurs provided the Jacobian matrix $J(E_5)$ has a pair of purely imaginary eigenvalues and the other eigenvalues have negative real parts and $Re\left[\frac{d\lambda}{da}\right]|_{a=a_0} \neq 0$. Assume that the characteristic equation at the the interior equilibrium point E_5 is as follows:

$$\lambda^4 + A_1\,\lambda^3 + A_2\lambda^2 + A_3\,\lambda + A_4 = 0 \tag{5.1}$$

For purely imaginary eigenvalues, it is clear that coefficients of characteristic polynomial (5.1) must satisfy the following condition:

$$A_1\,A_2\,A_3 - A_1^2\,A_4 - A_3^2 = 0.$$

Suppose $\pm i\omega$ is a pair of purely imaginary eigenvalues corresponding to a_0. We derive from the characteristic equation (5.1) relative to a

$$[4\lambda^3 + 3A_1\lambda^2 + 2A_2\lambda + A_3]\frac{d\lambda}{da} + \left(\lambda^3\frac{dA_1}{da} + \lambda^2\frac{dA_2}{da} + \lambda\frac{dA_3}{da} + \frac{dA_4}{da}\right) = 0$$

hence

$$\frac{d\lambda}{da} = -\left(\frac{\lambda^3\frac{dA_1}{da} + \lambda^2\frac{dA_2}{da} + \lambda\frac{dA_3}{da} + \frac{dA_4}{da}}{4\lambda^3 + 3A_1\lambda^2 + 2A_2\lambda + A_3}\right) \tag{5.2}$$

we substitute $i\omega$ in to equation (5.2), we have

$$\frac{d\lambda}{da}|_{i\omega} = -\left(\frac{-i\omega^3\frac{dA_1}{da} - \omega^2\frac{dA_2}{da} + i\omega\frac{dA_3}{da} + \frac{dA_4}{da}}{-4i\omega^3 - 3A_1\omega^2 + 2A_2\omega i + A_3}\right)$$

hence

$$Re\left(\frac{d\lambda}{da}|_{i\omega}\right) = -\left(\frac{[A_3 - 3A_1\omega^2]\left[\frac{dA_4}{da} - \omega^2\frac{dA_2}{da}\right] + [2A_2\omega - 4\omega^3]\left[\omega^3\frac{dA_1}{da} - \omega\frac{dA_3}{da}\right]}{[A_3 - 3A_1\omega^2]^2 + [2A_2\omega - 4\omega^3]^2}\right)$$

Theorem 5.1. *Consider prameter a as bifurcation parameter. System* (2.2) *undergoes a Hopf-bifurcation provided*

$$\left([A_3 - 3A_1\omega^2]\left[\frac{dA_4}{da} - \omega^2\frac{dA_2}{da}\right] + [2A_2\omega - 4\omega^3]\left[\omega^3\frac{dA_1}{da} - \omega\frac{dA_3}{da}\right]\right) \neq 0$$

5.2. Transcritical bifurcation

The following theorem presents the conditions for the occurrence of transcritical bifurcation.

Theorem 5.2. *Consider system* (2.2)
(1) If $R_1 = 1$ and $R_2 \neq 1$ then system (2.2) *undergoes a transcritical bifurcation at the equilibrium point E_3 of system* (2.2) *but this system dosen't have saddle node bifurcation when the parameter a crosses the critical value*

$$a^* = \frac{r_1\,d_2\,a_1}{K_1\,a\,(\gamma_1 + d_1 + r_1 - \beta_1)}$$

(2) If $R_1 = 1$ then system (2.2) *undergoes a transcritical bifurcation at t equilibrium point E_4 of system* (2.2).
(3) If $R_1 = 1$ then system (2.2) *undergoes a transcritical bifurcation at the nontrivial positive equilibrium point E_5 of system* (2.2).

Proof . To prove three parts of this theorem, we use the Sotomayor's theorem.
(1) If $R_1 = 1$ then the jacobian matrix J_3 has a zero eigenvalue and the corresponding right and left eigenvector of zero eigenvalue are
$v = \left(1, 0, \frac{-\alpha}{\beta_2}\left(1 - \frac{1}{R_2}\right)^{-1}, 0\right)$ and $w = (1, 0, 0, 0)$ respectively. Therefore we get

$$F_a(0, N_1^*, 0, N_2^*, a) = \begin{bmatrix} -N_2\, I_1 \\ -N_1\, N_2 \\ 0 \\ k\, N_1\, N_2 \end{bmatrix}_{|(0,N_1^*,0,N_2^*)} = \begin{bmatrix} 0 \\ -N_1^*\, N_2^* \\ 0 \\ k\, N_1^*\, N_2^* \end{bmatrix}$$

therefore

$$w^T\, F_a(0, N_1^*, 0, N_2^*, a) = 0$$

and

$$w^T\, (F_{x\,a}(0, N_1^*, 0, N_2^*, a).v) = -N_2^* \neq 0$$

where $x = (x_1, x_2, x_3, x_4) = (I_1, N_1, I_2, N_2)$ and $v = (v_1, v_2, v_3, v_4)$.

$$D^2F(0, N_1^*, 0, N_2^*, \beta_1)(v, v) = \begin{bmatrix} \sum_{j_1,j_2=1}^{4} \frac{\partial^2 f_1(0,N_1^*,0,N_2^*,\beta_1)}{\partial x_{j_1}\, \partial x_{j_2}} v_{j_1}\, v_{j_2} \\ \sum_{j_1,j_2=1}^{4} \frac{\partial^2 f_2(0,N_1^*,0,N_2^*,\beta_1)}{\partial x_{j_1}\, \partial x_{j_2}} v_{j_1}\, v_{j_2} \\ \sum_{j_1,j_2=1}^{4} \frac{\partial^2 f_3(0,N_1^*,0,N_2^*,\beta_1)}{\partial x_{j_1}\, \partial x_{j_2}} v_{j_1}\, v_{j_2} \\ \sum_{j_1,j_2=1}^{4} \frac{\partial^2 f_4(0,N_1^*,0,N_2^*,\beta_1)}{\partial x_{j_1}\, \partial x_{j_2}} v_{j_1}\, v_{j_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2\beta_1}{N_1^*} \\ 0 \\ \frac{-2\beta_1}{N_1^*}\left(\frac{\alpha}{\beta_2(1-\frac{1}{R_2})^{-1}} + \frac{\alpha}{N_2^*}\right)\left(\frac{\alpha}{\beta_2(1-\frac{1}{R_2})^{-1}}\right) \\ 0 \end{bmatrix}$$

therefore

$$w^T D^2F = \frac{-2\beta_1}{N_1^*} \neq 0$$

therefore by Sotomayor's theorem the equilibrium point E_3 is a transcritical bifurcation point.
2) We obtain $w^T\, F_a(0, N_1^*, I_2^*, N_2^*, a), w^T\, (F_{x\,a}(0, N_1^*, I_2^*, N_2^*, a).v)$ and
$D^2F(0, N_1^*, I_2^*, N_2^*, \beta_1)(v, v)$ similar to the above discussion.

$$w^T\, F_a(0, N_1^*, 0, N_2^*, a) = 0,$$
$$w^T\, (F_{x\,a}(0, N_1^*, 0, N_2^*, a).v) = -N_2^* \neq 0,$$
$$w^T D^2F = \frac{-2\beta_1}{N_1^*} \neq 0.$$

hence, the system (2.2) has a transcritical bifurcation in the equilibrium point E_4.
3) The proof of this part is similar to the two previous parts. □

6. Conclusion

In this article, the researchers proposed and analyzed a mathematical model which consisted of non-linear differential equations for six different populations, namely susceptible prey S_1, infected prey I_1, predator N_2, prey N_1, susceptible predator S_2 and infected predator I_2.

Having defined the system, the researchers showed that the system is well-defined, i.e. it always has the positive solution and is bounded.

The researchers came to the conclusion that there are three epidemiological threshold quantities for the model:

$$R_0 = \frac{\beta_1}{\gamma_1 + b_1 - a_1 r_1},$$
$$R_1 = \frac{\beta_1}{\gamma_1 + d_1 + (1-a_1) r_1 d_2 / k\, a\, K_1 + a\, N_2^*},$$
$$R_2 = \frac{\beta_2}{d_2 + \gamma_2}.$$

We showed that if $R_1 > 1$ and $R_2 < 1$, then the disease remains in the system and that it will not disappear. In other words, we showed that the interior equilibrium point E_5 is globally asymptotically stable. We obtained similar conditions for equilibrium points of E_3 and E_4.

By selecting parameter a as a bifurcation parameter, a sufficient condition was obtained for the existence of Hopf bifurcation. By using Sotomayor's theorem, We proved that under certain conditions, equilibrium points E_3, E_4 and E_5 have transcritical bifurcation.

Many researchers have studied prey$-$predator disease models. In the present paper, infectious disease may persist in the predator population and the predators may get the disease during the predation process.

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