Fixed point theorems for single valued mappings satisfying the ordered nonexpansive conditions on ultrametric and non-Archimedean normed spaces

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Abstract

In this paper, some fixed point theorems for nonexpansive mappings in partially ordered spherically complete ultrametric spaces are proved. In addition, we investigate the existence of fixed points for nonexpansive mappings in partially ordered non-Archimedean normed spaces. Finally, we give some examples to discuss the assumptions and support our results.

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1. Introduction and preliminaries

The well-known fixed-point theorem of Banach [3] is a very important tool for solving existence problems in many branches of mathematics and physics. There are a large number of generalizations of the Banach contraction principle in the literature. One can generalize this theorem by altering the action spaces. In one of these ways, the action spaces are replaced by metric spaces endowed with an ordered or partially ordered structure. Ran and Reurings [11], O'Regan and Petruşel [9] and others started the investigations concerning a fixed point theory in ordered metric spaces. Later, many authors followed this concept by introducing and investigating the different types of contraction mappings. Some interesting fixed point theorems concerning partially ordered metric spaces can also be found in [1, 5, 6].

In this paper, motivated by the work of Ran and Reurings [11], Kirk and Shahzad [4], we introduce two new conditions for nonexpansive mappings on spherically complete ultrametric spaces and non-Archimedean normed spaces and, using these conditions, obtain some fixed point theorems. That's why, we first recall some basic notions in ultrametric spaces and non-Archimedean normed spaces. For more details the reader is referred to [12].

Let (X, d) be a metric space. The metric space (X, d) is called an ultrametric space if the metric d satisfies the strong triangle inequality, i.e., for all $x, y, z \in X$:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\},$$

in this case d is said to be ultrametric. We denote by $B(x, r)$, the closed ball

$$B(x, r) = \{y \in X : d(x, y) \leq r\},$$

where $x \in X$ and $r \geq 0$ $(B(x, 0) = \{x\})$. A known characteristic property of ultrametric spaces is the following:

If $x, y \in X$, $0 \leq r \leq s$ and $B(x, r) \cap B(y, s) \neq \emptyset$, then $B(x, r) \subset B(y, s)$.

An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection [12]. Let $\mathbb{K}$ be a non-Archimedean valued field. A norm on a vector space X over $\mathbb{K}$ is a map $\|\cdot\|$ from X into $[0, \infty)$ with the following properties:

1) $\|x\| \neq 0$ if $x \in E \setminus \{0\}$;

2) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, for all $x, y \in X$);

3) $\|\alpha x\| = |\alpha|\|x\|$, for all $\alpha \in \mathbb{K}, x \in X$).

In 1993, Petalas and Vidalis in [8] presented a generalization of a well-known fixed point theorem for the class of spherically complete non-Archimedean normed spaces, and in 2000 Priess-Crampe and Ribenboim in [10] obtained similar results in ultrametric space, but the proofs of these theorems weren't constructive. In 2012 Kirk and Shahzad in [4] gave more constructive proofs of these theorems and strengthened the conclusions as follow:

Theorem 1.1. *[4] Suppose that (X, d) is a spherically complete ultrametric space and $T : X \longrightarrow X$ is a nonexpansive mapping. Then every closed ball of the form*

$$B(x, d(x, Tx)) \qquad (x \in X)$$

contains either a fixed point of T or a minimal T-invariant closed ball. Where a ball $B(x, r)$ in X is called T-invariant if $T(B(x, r)) \subset B(x, r)$ and is called minimal T-invariant if $B(x, r)$ is T-invariant and $d(u, Tu) = r$, for all $u \in B(x, r)$.

We also recall that a partial order on a nonempty set X is a binary relation $\preceq$ over X which satisfies the following conditions:

1) $x \preceq x$, for all $x \in X$ (reflexivity);
2) $x \preceq y$ and $y \preceq x$ imply $x = y$, for all $x, y \in X$ (antisymmetry);
3) $x \preceq y$ and $y \preceq z$ imply $x \preceq z$, for all $x, y, z \in X$ (transitivity).

The set X with a partial order $\preceq$ is called a partially ordered set and it is denoted by the pair $(X, \preceq)$. If $(X, \preceq)$ is a partially ordered set and $x, y \in X$, then x and y are said to be comparable elements of X if either $x \preceq y$ or $y \preceq x$.

2. Main results

In this section, first, we give two theorems that investigate the existence of a fixed point for nonexpansive mappings defined on partially ordered ultrametric spaces and non-Archimedean normed spaces. In general, these theorems do not hold in metric spaces.

Definition 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists an ultrametric d in X such that (X, d) is an ultrametric space. We would say that a $B(x, r)$ is partially f-invariant if for any $u \in B$, that u compare with x,*

$$fu \in B(x, r).$$

Also, the ball $B(x, r)$ is minimal partially f-invariant if $fu \in B(x, r)$ and $d(u, fu) = r$ for any $u \in B$ that u compare with x.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists an ultrametric d in X such that (X, d) is an ultrametric space, and $f : X \to X$ satisfying the following statements:*

(M_1) *if $x, y \in X$ and $x \preceq y$, then $fx \preceq fy$;*
(M_2) *$d(fx, fy) \leq d(x, y)$ for all $x, y \in X, x \preceq y$;*
(M_3) *if $\{x_n\}$ is a nonincreasing sequence in X and $\{B(x_n, r_n)\}$ is a descending collection of closed balls in X, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an upper bound $z \in X$ of sequence $\{x_{n_k}\}$ in $\bigcap_{n=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that $z \preceq fz$.*

Then for any $x \in X$ with $x \preceq fx$, $B(x, d(x, fx))$ contains either a fixed point of f or a minimal partially f-invariant ball.

Proof . First, we assert that for all $z \in X$, the ball $B(z, d(z, fz))$ is partially f- invariant. To see this, let $z \in X$, $r = d(z, fz)$ and let $u \in B(z, r)$ such that u and z are comparable, then

$$\begin{aligned} d(fu, z) &\leq \max\{d(fu, fz), d(z, fz)\} \\ &\leq \max\{d(u, z), d(z, fz)\} \\ &= r. \end{aligned}$$

Now let $x_0 \in X$ and $x_0 \preceq fx_0$, and let $x_1 = x_0$, $r_1 = d(x_1, fx_1)$, and

$$\lambda_1 = \inf\{d(x, fx) \mid x \in B(x_1, r_1) : x_1 \preceq x \preceq fx\}.$$

$f^n x_1$ belongs to $B(x_1, r_1)$ and $f^n x_1 \preceq f^{n+1} x_1$ for all $n \in \mathbb{N}$, so $d(f^n x_1, f^{n+1} x_1) \leq r_1$. Therefore, $\lambda_1 \leq r_1$. Now, if $\lambda_1 = r_1$, then $B(x_1, r_1)$ is a minimal partially f-invariant ball. Otherwise, let ε_n be a sequence of positive numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$. We can choose $x_2 \in B(x_1, r_1)$ such that

$$x_1 \preceq x_2 \preceq fx_2, \ and \ r_2 = d(x_2, fx_2) < \min\{r_1, \lambda_1 + \varepsilon_1\}.$$

Let

$$\lambda_2 = \inf\{d(x, fx) \mid x \in B(x_2, r_2) : x_2 \preceq x \preceq fx\}.$$

As seen above, we have $\lambda_2 \leq r_2$. Again, if $\lambda_2 = r_2$, then $B(x_2, r_2)$ is a minimal partially f-invariant ball. Otherwise, we select $x_3 \in B(x_2, r_2)$ with

$$x_2 \preceq x_3 \preceq fx_3, \qquad r_3 := d(x_3, fx_3) < \min\{r_2, \lambda_2 + \varepsilon_2\}.$$

Having defined $x_n \in X$, let

$$\lambda_n = \inf\{d(x, fx) \mid x \in B(x_n, r_n) : x_n \preceq x \preceq fx\}.$$

In a similar way, we have $\lambda_n \le r_n$, if $\lambda_n = r_n$, then $B(x_n, r_n)$ is a minimal partially f-invariant ball. Otherwise, we select $x_{n+1} \in B(x_n, r_n)$ with

$$x_n \preceq x_{n+1} \preceq fx_{n+1}, \qquad r_{n+1} := d(x_{n+1}, fx_{n+1}) < \min\{r_n, \lambda_n + \varepsilon_n\}.$$

Now, if there exists $n_0 \in \mathbb{N}$ such that $\lambda_{n_0} = r_{n_0}$, then $B(x_{n_0}, r_{n_0})$ is a minimal partially f-invariant ball. Otherwise, we obtain a nonincreasing sequence $\{x_n\}$ in X and a descending sequence of balls $\{B(x_n, r_n)\}$. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a $z \in \bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that $z \preceq fz$ and $x_{n_k} \preceq z$ for all $k \in \mathbb{N}$. Now since $\{r_n\}$ is nonincreasing, $r := \lim_{n \to \infty} r_n$ exists. Also λ_n is nondecreasing and bounded above, so $\lambda := \lim_{n \to \infty} \lambda_n$ also exists. Then for each n,

$$d(z, fz) \le \max\{d(z, x_{n_k}), d(x_{n_k}, fz)\} \le r_{n_k}.$$

Moreover,
$$\lambda_{n_k} \le d(z, fz) \le r \le r_{n_{k+1}} \le \lambda_{n_k} + \varepsilon_{n_k}, \qquad (k \in \mathbb{N}).$$
Letting $k \to \infty$ we see that $d(z, fz) = \lambda = r$. Set

$$a = \inf\{d(x, fx) \mid x \in B(z, d(z, fz)) : z \preceq x \preceq fx\}.$$

Since $z \in B(x_{n_k}, r_{n_k})$ and $x_{n_k} \preceq z$, for all $k \in \mathbb{N}$, we conclude that if $x \in B(z, d(z, fz))$ and $z \preceq x \preceq fx$, then

$$d(x, fx) \le d(z, fz) \le r_{n_k},$$

hence, $a \le r_{n_k}$. Moreover, $\lambda_{n_k} \le a$ since every closed ball in X is partially f-invariant. Thus

$$a = \inf\{d(x, fx) \mid x \in B(z, d(z, fz)) : z \preceq x \preceq fx\} = r = d(z, fz).$$

Now, if $r = 0$, then z is a fixed point of f in $B(x_0, d(x_0, fx_0))$. Otherwise, $B(z, d(z, fz))$ is minimal partially f-invariant. $\square$

Corollary 2.3. *Theorem 2.2 remains valid if the ultrametric space (X, d) is replaced by a non-Archimedean normed space $(X, \|\cdot\|)$.*

Now we give two examples to support our results.

Definition 2.4. *Let X be the space l^∞ over a non-Archimedean valued field $\mathbb{K}$ and $x, y \in X$. We say y is a sub-member of x, if $y = (0, 0, \ldots, 0, x_n, 0, \ldots, 0, x_m, \ldots)$, where $x = (x_1, x_2, x_3, \ldots)$. If y is a sub-member of x, then we denote, $y \subset x$.*

Example 2.5. *Let X be the space c_0 over $\mathbb{K}$ with the valuation of $\mathbb{K}$ discrete. Let $e \in \mathbb{K}$ with $0 < |e| < 1$ and $u = (e, 0, 0, 0, \ldots)$. Fix $z \in B(u, |e|)$. For all $x, y \in X$, define*

$$x \preceq y \iff \{x, y \in B(u, |e|), P_M(y) \subset (e, e, e, \ldots), P_N(x) \subset (e, e, e, \ldots)\}$$

and

$$\{(x_N \subset y_M \subset z) \vee (x = y)\},$$

where N and M are the smallest positive integers such that $x_N = (x_{N+1}, \ldots) \subset z$, $y_M = (y_{M+1}, \ldots) \subset z$ and $P_N(x) = (x_1, x_2, x_3, \ldots, x_N, 0, 0, 0, \ldots)$, $P_M(y) = (y_1, y_2, y_3, \ldots, y_M, 0, 0, 0, \ldots)$. Suppose that $f : c_0 \longrightarrow c_0$ is the mapping defined by

$$f(x) = \begin{cases} (e, x_1, x_2, x_3, \ldots) & x \in B(u, |e|), \\ 2x & therwise. \end{cases}$$

If $x \preceq y$, then $x = y$ or $x, y \in B(u, |e|)$, thus $\|fx - fy\| = \|x - y\|$ and since x is a sub-member of y, hence fx is a sub-member of fy, hence $fx \preceq fy$. Therefore (M_1) and (M_2) hold. Now, we want to show that the condition (M_3) holds. Let $\{x^n\}$ be a sequence of nonincreasing points in X and $\{B(x^n, r_n)\}$ be a collection of nonincreasing balls. If for each $n \in \mathbb{N}$, $x^n = x^{n+1}$, then we are done. Otherwise, set

$$N_0 = \min\{N_n \in \mathbb{N} : N_n \text{ is the smallest positive integer that } x_{N_n}^n \subset z\}.$$

Let $i \geq N_0$. If there exists $n \in \mathbb{N}$ such that i^{th} coordinate of x^n is non-zero, then define $v_i := x_i^n$ and otherwise $v_i = 0$. For $i < N_0$, take $v_i = e$. Put $v = (v_1, v_2, \ldots)$. For each $n \in \mathbb{N}_0$, $x^n \preceq v$, this means that v is an upper-bound for $\{x^n\}$, so it is enough to show that $v \in \cap B(x^n, r_n)$. Let $n \in \mathbb{N}_0$ be arbitrary. Select $m \in \mathbb{N}$ such that $m > n$, we have

$$\|x^n - x^m\| \leq \max\{\|x^n - x^{n+1}\|, \|x^n - x^{n+1}\|, \ldots, \|x^{m-1} - x^m\|\}$$
$$\leq \max\{r_n, r_{n+1}, \ldots, r_m\} = r_n.$$

Thus for each $n < m \in \mathbb{N}$, $\|x^n - x^m\| \leq r_n$. If there exists $n \in \mathbb{N}$ such that $\|x^n - v\| > r_n$, then there exists $i \in \mathbb{N}$ such that $|x_i^n - v_i| > r_n$. We have two case:

case 1 If $i < N_0$, then $v_i = e$ and $x_i^n = e$, which is a contradiction.

case 2 If $i \geq N_0$, then there exists $m \geq n$ such that $x_i^m = v_i$, therefore $|x_i^n - x_i^m| > r_n$. Which is a contradiction.

Hence, $\|x^n - v\| \leq r_n$ for each $n \in \mathbb{N}$, so $v \in \cap B(x^n, r_n)$. It is clear that $v \preceq fv$. Therefore, M_3 holds. f is fixed point free. Now we show that for each $x \in B(u, |e|)$, $d(x, fx) = |e|$. Note that $d(u, fu) = |e|$. Let $x \in B(u, |e|)$. Then

$$\|x - u\| = \sup\{|x_1 - e|, |x_2|, |x_3|, \ldots\} \leq |e| = d(u, fu).$$

Therefore $|x_2| \leq |e|, |x_3| \leq |e|, \ldots$ and $|x_1| \leq \max\{|x_1 - e|, |e|\} \leq |e|$. Since $\mathbb{K}$ is a valuation field, hence $|x_1| = |e|$ or $|x_1 - e| = |e|$. If $|x_1 - e| = |e|$, then $\|x - fx\| = |e| = d(u, fu)$, because

$$\|x - fx\| = \sup\{|x_1 - e|, |x_2 - x_1|, |x_3 - x_2|, \ldots\}.$$

Otherwise, $|x_1| = |e|$, similarity $|x_2 - x_1| = |e|$ or $|x_2| = |e|$. If $|x_2 - x_1| = |e|$ then $\|x - fx\| = |e| = d(u, fu)$ and otherwise $|x_2| = |e|$. Because $(x_n) \in c_0$, thus there exist $n \in \mathbb{N}$ such that $|x_n| < |e|$, let n_0 be the smallest positive integer such that $|x_{n_0}| < |e|$, therefore $|x_{n_0} - x_{n_0-1}| = |e|$, hence $\|x - fx\| = |e| = d(u, fu)$. So for each $x \in B(u, |e|)$,

$$d(x, fx) = d(u, fu).$$

Example 2.6. *Let X be the space c_0 over $\mathbb{K}$ with the valuation of $\mathbb{K}$ discrete. Define*

$$x \preceq y \iff \{x, y \in B(0, 1), P_M(y) \subset (0, 0, 0, \ldots), P_N(x) \subset (0, 0, 0, \ldots)\}$$

and

$$\{(x_N \subset y_M \subset z) \vee (x = y)\},$$

and let $e \in \mathbb{K}$ with $0 < |e| < 1$ and $u = (e, 0, 0, 0, \ldots)$. Consider the following mapping on c_0

$$f(x) = \begin{cases} (0, x_1, x_2, x_3, \ldots), & x \in B(u, |e|), \\ 2x, & \text{otherwise}., \end{cases}$$

f is a nonexpansive mapping with fixed point 0. One can show that (M_1), (M_2) and (M_3) hold.

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