Int. J. Nonlinear Anal. Appl. 12(2021) No. 1, 726-734
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/IJNAA.2021.4934

# Fuzzy co-even domination of strong fuzzy graphs

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(Communicated by Mohammmad Bagher Ghaemi)

**Abstract**

The aim of this research is to initiate a new concept of domination in fuzzy graphs which is called a fuzzy co-even domination number denoted by $\gamma_{fco}(G)$. We will touch only a few aspects of the theory to of this definition. Some properties and boundaries of this definition are introduced. The fuzzy co-even domination number of fuzzy certain graphs as fuzzy complete, fuzzy complete bipartite, fuzzy star, fuzzy cycle, fuzzy null, fuzzy path, and fuzzy star are determined. Additionally, this number is computed for the complement of mentioned above fuzzy certain graphs. Finally, this number is also determined for the join to mentioned above fuzzy certain graphs with itself.

*Keywords:* Fuzzy co-even dominating set, fuzzy co-even domination number, Join of fuzzy graphs and complement of fuzzy graphs.
*2010 MSC:* Primary 90C33; Secondary 26B25.

## 1. Introduction

In the last few years, there has been a growing interest to bring together two areas in which there is a relationship between them, which is the fuzzy set [24] and graph theory [8]. Generate from this merge a new branch called a fuzzy graph was introduced in 1975 by Rosenfeld [17]. So, a fuzzy graph is a modern branch and our viewpoints shed some new light on this branch with important another branch that is a domination number in graphs [9]. The domination number played an important role via theoretical study in various fields such as a fuzzy graph [25, 26], topological graph [10, 11], labeled graph [5, 6] and others [1]-[4], [7, 13, 16, 18, 21, 22]. Moreover, several applications in problems life were solved by these branches. So, the study of this number is worth to care. A.Somasundram, S.Somasundram [19-20] are the first used the concept of this number. There are some definition for

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*Received:* October 2020 *Revised:* February 2021

this number, for example Mahioub and Soner [12] defined it by taking the minimum fuzzy cardinality to an all-dominating set and Xavior et al. [23] by determining the minimum dominating set and taking the sum of its all members. In this paper, by adding some hypotheses we work with the definition of Xavior et al. because it seems to be the best adapted to our theory.

Consider G(V, E) is an undirected, simple, and finite graph. A mapping $\sigma : \mathrm{V} \to [0,1]$ where $V$ is a nonempty set of vertices called a fuzzy subset and $G = (\sigma, \mu)$ where $\mu : \mathrm{V} \times \mathrm{V} \to [0,1]$ and $\mu(\mathrm{u},\mathrm{v}) \leq \sigma(\mathrm{u}) \wedge \sigma(\mathrm{v})$ is called a fuzzy graph. An edge (u, v) is called effective if $\mu(\mathrm{u},\mathrm{v}) = \sigma(\mathrm{u}) \wedge \sigma(\mathrm{v})$ and the fuzzy graph is called strong if its edge is effective. The open effective neighborhood is $N_E(v) = \{u; (u,v) \text{ i an effective edge }\}$ and $N_E[v] = N_E(v) \cup \{v\}$. Degree of $v$ is the number of effective edges in $N_E(v)$ and denoted by $\deg_E(v)$. In this work, we deal with the strong fuzzy graph, so every two pairwise vertices are adjacent by an effective edge. Furthermore, A fuzzy null graph is called strong if it has no edges and a strong fuzzy star graph denoted by $S_n; S_n \equiv K_{1,n-1}$

**Definition 1.1.** *Consider $G = (\sigma, \mu)$ is a fuzzy graph of a graph $G(V,E)$, if there is a set $D \subseteq V$ and $\forall v \in V - D$ there is a vertex $u; u$ and $v$ are adjacent by an effective edge ( $u$ dominates $v$ ) and the $\deg_E(v)$ is even, then $D$ is called a fuzzy co-even dominating set ($FCOD$) on $G$.*

**Definition 1.2.** *A fuzzy co-even dominating set $D$ in a fuzzy graph $G = (\sigma, \mu)$ is called minimum fuzzy coeven dominating (MFCOD) if $|D|$ is the smallest number of all cardinal of FCOD.*

**Definition 1.3.** *Consider $W(D_i) = \{\sum \sigma(v); \forall v \in D_i; D_i$ is a minimum fuzzy co - even dominating set$\}$, then the fuzzy domination number of a fuzzy graph is $\gamma_{fco}(G) = \min \{W(D_i); D_i\, isaminimumco\text{−}evendominatingset\}$*

**Example 1.4.** *In a strong fuzzy cycle $G$ as shown in Figure 1 , there are six MFCOD that are $D_1 = \{v_1, v_2\}$ $D_2 = \{v_1, v_3\}, D_3 = \{v_1, v_4\}, D_4 = \{v_2, v_3\}, D_5 = \{v_2, v_4\}$, and $D_6 = \{v_3, v_4\}$.*

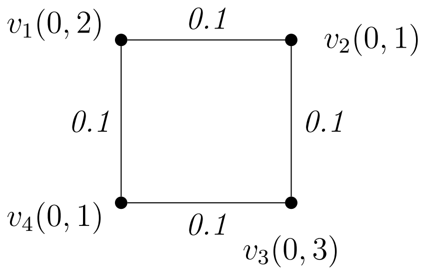


Figure 1: The strong fuzzy cycle

*Then* $|D_1| = 0.3, |D_2| = 0.5, |D_3| = 0.3, |D_4| = 0.4, |D_5| = 0.2$, *and* $|D_6| = 0.4$. *Thus,* $\gamma_{fco}(G) = 0.2$.

## 2. Some properties of FCOD and MFCOD.

**Proposition 2.1.** *Suppose that $G = (\sigma, \mu)$ is a strong fuzzy graph has $n$ vertices and has $\gamma_{fco}(G)$,*

*(1) If $G$ has vertices which have zero or odd degrees, then all these vertices belong to every MFCOD and $\gamma_{fco}(G) \geq \sum \sigma(v)$, where $v$ has zero or odd degree.*

*(2) if $D$ is a fuzzy co-even dominating set, then $V - D$ doesn't necessarily a fuzzy co- even dominating set.*

*(3)* $\min\left(\sigma\left(v_j\right)\right) \leq \gamma_{fco}(G) \leq \sum_{i=1}^{n} \sigma\left(v_i\right)$, *where* $v_j$ *the vertex that dominates all vertices in* $G$.

**Proof .**

(1) It is clear that from Definition 1.1 and Definition 1.3.

(2) If $G$ contains a vertex that has zero or odd degree, then all of these vertices belong to every FCOD set $D$. Thus, $V - D$ is not FCOD set, since $V - (V - D) = D$ and $D$ contains a vertex has zero or odd degree.

(3) If the number of vertices of a minimum fuzzy co-even dominating set is one, then $\min\left(\sigma\left(v_i\right)\right) = \gamma_{fco}\left(K_{n,m}\right)$, where $v_j$ the vertex that dominates all vertices in G.

Now, if each vertex in $G$ has zero or odd degree, then $\gamma_{fco}(G) = \sum_{i=1}^{n} \sigma\left(v_i\right)$. As a consequence of the cases above, $\min\left(\sigma\left(v_i\right), i = 1, \ldots, r\right) \leq \gamma_{fco}(G) \leq \sum_{i=1}^{n} \sigma\left(v_i\right)$ □

**Proposition 2.2.** *If* $G$ *is a strong path has* $n$ *vertices;* $n \equiv 1(\bmod 3)$, *then* $\gamma_{fco}\left(P_n\right) = \sigma\left(v_1\right) + \sigma\left(v_n\right) + \sum_{i=1}^{(n-4)/3} \sigma\left(v_{4+3i}\right)$.

**Proof .** The vertices $v_1$ and $v_n$ belong to every FCOD set according to Proposition 2.1.(1). It is clear that these vertices dominate their adjacent vertices $v_2$ and $v_{n-1}$ respectively. Thus, the set of remaining vertices which are not dominated by the two vertices $v_1$ and $v_n$ is $\{v_i, i = 3, 4, \ldots, n-2\}$. Let $F = \left\{v_{4+i}, i = 0, 1, \ldots, \frac{n-4}{3}\right\}$, one can easily show that $F$ is the FCOD set. The maximum neighborhood of each vertex in $G$ is two, so $F$ is FCOD set with minimum cardinality. Also, $F$ is unique, since $|F| = n - 4 \equiv 0(\bmod 3)$, $\gamma_{fco}\left(P_n\right) = \sigma\left(v_1\right) + \sigma\left(v_n\right) + \sum_{i=1}^{(n-4)/3} \sigma\left(v_{4+3i}\right)$. □

**Proposition 2.3.** *If* $G$ *is a fuzzy complete graph has* $n$ *vertices, then*

$$\gamma_{fco}\left(K_n\right) = \left\{\begin{array}{ll} \min\left(\sigma\left(v_i\right), i = 1, \ldots, n\right), & \text{if } n \text{ is odd} \\ \sum_{i=1}^{n} \sigma\left(v_i\right), & \text{if } n \text{ is even} \end{array}\right\}$$

**Proof .** Two possible cases are as the following.
Case 1. Suppose that $n$ is odd, then all vertices of $G$ have even degrees. Thus, a vertex can dominate all other vertices. Therefore, $\gamma_{fco}\left(K_n\right) = \min\left(\sigma\left(v_i\right), i = 1, \ldots, n\right)$.
Case 2. Suppose that $n$ is even, then all vertices have odd degrees. Thus, all these vertices belong to each FCOD set according to Proposition 2.1.(1). Therefore, $\gamma_{fco}\left(K_n\right) = \sum_{i=1}^{n} \sigma\left(v_i\right)$. Thus, from the two cases above, the required is obtained. □

**Proposition 2.4.** *If* $G$ *is a fuzzy complete bipartite graph* $K_{n,m}$ *contains two partite sets* $V_1$ *and* $V_2$ *suth that* $V_1$ *has* $n$ *vertices* $\{u_1, u_2, \ldots, u_n\}$ *and* $V_2$ *has* $m$ *vertices* $\{v_1, v_2, \ldots, v_m\}$; $n, m \geq 3$, *then*

$$\gamma_{fco}\left(K_{n,m}\right) = \left\{\begin{array}{lr} \min\left(\sigma\left(u_i\right), i = 1, \ldots, n\right) + \min\left(\sigma\left(v_j\right), j = 1, \ldots, m\right), & \text{if } m \text{ and } n \text{ are even} \\ \sum_{j=1}^{m} \sigma\left(v_j\right), & \text{if } n \text{ is odd and } m \text{ is even} \\ \sum_{i=1}^{n} \sigma\left(u_i\right), & \text{if } n \text{ is even and } m \text{ is odd} \\ \sum_{i=1}^{n} \sigma\left(u_i\right) + \sum_{j=1}^{m} \sigma\left(v_j\right), & \text{if } n \text{ and } m \text{ are odd} \end{array}\right\}$$

**Proof .** Four possible cases are as follows.
Case 1. Suppose that $m$ and $n$ are even, then all vertices have even degrees in this graph. Thus, the minimum F COD set occurs when taking only one vertex from each partite sets $V_1$ and $V_2$. Therefore, $\gamma_{sco}\left(K_{n,m}\right) = \min\left(\sigma\left(u_i\right), i = 1, \ldots, n\right) + \min\left(\sigma\left(v_j\right), j = 1, \ldots, m\right)$.
Case 2. Suppose that $n$ is odd and $m$ is even, then all vertices in the set $V_2$ belong to each FCOD set by Proposition 2.1.(1). Furthermore, these vertices dominate all vertices in the graph $G$ and each

vertex in $V_1$ has even degree. Therefore, $\gamma_{sco}(K_{n,m}) = \sum_{j=1}^{m} \sigma(v_j)$.
Case 3 . Suppose that $n$ is even and $m$ is odd, the result is obtained by a similar technique in Case 2 .
Case 4. Suppose that $m$ and $n$ are odd, all vertices have odd degrees in this graph. Thus, by Proposition 2. 1.(1), all vertices in $G$ belong to all FCOD sets. Therefore, $\gamma_{sco}(K_{n,m}) = \sum_{i=1}^{n} \sigma(u_i) + \sum_{j=1}^{m} \sigma(v_j)$ Thus, from the four cases above, the required is obtained. □

**Proposition 2.5.** *If* G *is a strong fuzzy cycle has* $n$ *vertices* ; $n \equiv 0(\mathrm{mod} 3)$, *then*

$$\gamma_{fco}(C_n) = \min \left\{ \sum_{i=1}^{n/3} \sigma(v_{i+j}) \,; j = 0, 1, 2 \right\}.$$

**Proof .** Since $n \equiv 0(\mathrm{mod} 3)$, there are only three MFCOD sets which are

$$D_1 = \left\{ v_{1+3i}, i = 0, \ldots, \frac{n}{3} - 1 \right\},\ D_2 = \left\{ v_{2+3i}, i = 0, \ldots, \frac{n}{3} - 1 \right\},\ D_3 = \left\{ v_{3+3i}, i = 0, \ldots, \frac{n}{3} - 1 \right\}.$$

It is clear that the sets are MFCOD, since each vertex has maximum neighborhood and with even degree. Also, these sets have the same cardinality where $|D_1| = |D_2| = |D_3| = \frac{n}{3}$. Therefore, $\gamma_{sco}(C_n) = \min \left\{ \sum_{i=1}^{n/3} \sigma(v_{i+j}) \,; j = 0, 1, 2 \right\}$. □

**Proposition 2.6.** *If* $G$ *is a strong fuzzy star graph has* $n$ *vertices, then*

$$\gamma_{fco}(S_n) = \left\{ \begin{array}{ll} \sum_{i=0}^{n-1} \sigma(v_i), & \text{if } n \text{ is even} \\ \sum_{i=1}^{n-1} \sigma(v_i), & \text{if } n \text{ is odd} \end{array} \right\},$$

*where* $v_0$ *is the center of the star.*

**Proof .** Two possible cases are as the following.
Case 1. Suppose that $n$ is even, then all vertices have odd degrees. Thus, by Proposition 2.1.(1), $\gamma_{sco}(S_n) = \sum_{i=0}^{n-1} \sigma(v_i)$.
Case 2. Suppose that $n$ is odd, the center vertex has even degree and the other vertices have odd degrees. Thus, according to Proposition 2.1(1), $\gamma_{sco}(S_n) = \sum_{i=1}^{n-1} \sigma(v_i)$ . Thus, from the two cases above, the proof is done.
□

**Observation:** Consider $G$ as a fuzzy null graph has $n$ vertices, then

$$\gamma_{fco}(N_n) = \sum_{i=1}^{n} \sigma(v_i).$$

**Proof .** One can easily show that the proof is straightforward according to Proposition 2.1(1) . □

## 3. The complement of the certain graphs.

**Proposition 3.1.** *If* $G$ *is a strong fuzzy path has* $n$ *vertices, then*

$$\gamma_{fco}\left(\overline{P_n}\right) = \min \left\{ \sigma(v_i) + \sigma(v_j) \,; d(v_i, v_j) \neq 2, \forall i \neq j \ \textit{ in the } \text{graph}\, G \right\}.$$

**Proof .** In the complement of $G(\bar{G})$, each vertex is adjacent to all vertices except that adjacent vertices to it in $G$. So, there is no any vertex that dominates the vertex set $\bar{G}$, since there is no vertex in $\bar{G}$ has degree equal to $n-1$. Thus, the minimum number of vertices which dominate all vertices in $\bar{G}$ is greater than or equal two. Now, every two vertices in $\bar{G}$ can dominate all other vertices provided that the distance between these vertices is not equal to two. Since, if $d\left(v_i, v_j\right)=2$, then there is a vertex between the vertices $v_i$ and $v_j$. This vertex cannot be dominated by the vertices $v_i$ and $v_j$. Therefore, the proof is done. □

**Proposition 3.2.** *If $G$ is a strong fuzzy cycle has $n$ vertices, then*
$\gamma_{fco}\left(\overline{C_n}\right)=\min\left\{\sigma\left(v_i\right)+\sigma\left(v_j\right); d\left(v_i, v_j\right)\neq 2, \forall i\neq j \text{ in the graph } G\right\}$

**Proof .** It is similar to proof in the Proposition 3.1. □

**Observation:** If $G$ is a fuzzy complete or null graph has $n$ vertices, then

$$1)\ \gamma_{fco}\left(\overline{K_n}\right)=\gamma_{sco}\left(N_n\right)=\sum_{i=1}^{n}\sigma\left(v_i\right)$$
$$2)\ \gamma_{fco}\left(\overline{N_n}\right)=\gamma_{sco}\left(K_n\right)=\left\{\begin{array}{ll}\min\left(\sigma\left(v_i\right), i=1,\ldots,n\right), & \text{if } n \text{ is odd}\\ \sum_{i=1}^{n}\sigma\left(v_i\right), & \text{if } n \text{ is even}\end{array}\right\}.$$

**Proof .** It is obvious from Proposition 2.3 and Observation 2.7. □

**Proposition 3.3.** *If $G$ is a fuzzy complete bipartite $K_{n,m}$ contains two partite sets $V_1$ and $V_2$ suth that $V_1$ has $n$ vertices $\{u_1, u_2, \ldots, u_n\}$ and $V_2$ has $m$ vertices $\{v_1, v_2, \ldots, v_m\}$; $n, m\geq 3$, then*

$$\gamma_{fco}\left(\overline{K_{n,m}}\right)=\left\{\begin{array}{ll}\min\left(\sigma\left(u_i\right), i=1,\ldots,n\right)+\min\left(\sigma\left(v_j\right), j=1,\ \ldots, m\right), & \textit{if } m \textit{ and } n \textit{ are odd}\\ \min\left(\sigma\left(v_j\right), j=1,\ldots,m\right)+\sum_{i=1}^{n}\sigma\left(u_i\right), & \textit{if } n \textit{ is odd and } m \textit{ is even}\\ \min\left(\sigma\left(u_i\right), i=1,\ldots,n\right)+\sum_{j=1}^{m}\sigma\left(v_j\right), & \textit{if } n \textit{ is even and } m \textit{ is odd}\\ \sum_{i=1}^{n}\sigma\left(u_i\right)+\sum_{j=1}^{m}\sigma\left(v_j\right), & \textit{if } n \textit{ and } m \textit{ are even}\end{array}\right\}$$

**Proof .** It is known that $\overline{K_{n,m}}\equiv K_n\cup K_m$, so the fuzzy co-even domination number of $\overline{K_{n,m}}$ is equal to the fuzzy co-even domination number of $K_n$ plus the fuzzy co-even domination number of $K_m$. Thus, there are four cases as the following.
Case 1. Suppose that $m$ and $n$ are odd, then each vertex in $K_n$ or $K_m$ has even degree. Thus, each vertex in $K_n$ can dominate all vertices in $K_m$. Similarly, each vertex in $K_m$ can dominate all vertices in $K_m$. Therefore, $\gamma_{fco}\left(\overline{K_{n,m}}\right)=\min\left(\sigma\left(u_i\right), i=1,\ldots,n\right)+\min\left(\sigma\left(v_j\right), j=1,\ldots,m\right)$.
Case 2. Suppose that $n$ is odd and $m$ is even, then every vertex in $K_n$ has even degree and all vertices in $K_m$ have odd degrees. Thus, all vertices in $K_m$ belong to every FCOD set according to Proposition 2.1(1). Furthermore, in the same manner in case 1 each vertex in $K_n$ dominates all the other vertices in $K_n$. Therefore, $\gamma_{fco}\left(\overline{K_{n,m}}\right)=\min\left(\sigma\left(u_i\right), i=1,\ldots,n\right)+\sum_{j=1}^{m}\sigma\left(v_j\right)$.
Case 3. Assume that $n$ is even and $m$ is odd, then by the same technique in Case 2, the result is obtained. Case 4. Assume that $n$ and $m$ are even, then each vertex in $K_n$ or $K_m$ has odd degree. Therefore, by Proposition 2.1(1), $\gamma_{fco}\left(\overline{K_{n,m}}\right)=\sum_{i=1}^{n}\sigma\left(u_i\right)+\sum_{j=1}^{m}\sigma\left(v_j\right)$.

Thus, from the four cases above, the proof is done. □

**Corollary 3.4.** *Let $G$ be a strong fuzzy star has $n$ vertices, then $\gamma_{fco}\left(\overline{S_n}\right)=\sigma\left(v_0\right)+\min\left(\sigma\left(v_i\right)\right)$, $i=1,\ldots,n-1$, where $v_0$ is the center of the star.*

**Proof .** It is clear that $S_n\equiv K_{1,n-1}$, thus $\overline{S_n}\equiv K_1\cup K_{n-1}$, and $K_1=\{v_0\}$. Therefore, by Proposition 3.3, the result is obtained. □

## 4. The join of two graphs from certain graphs

**Definition 4.1.** *The join (addition) $G_1 + G_2$ of $G_1$ and $G_2$ is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. (as an example, see Figure 2):*

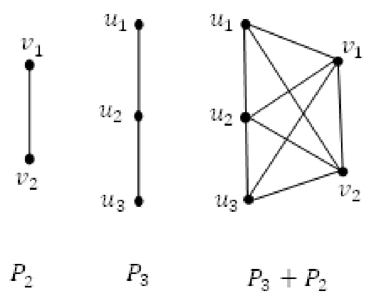


Figure 2. The join of $P_3$ and $P_2$

In this section, the join of a graph with itself $(G + G)$ is determined. In order to prevent any confusion, we labeled the vertices of the first graph $G$ by $\{v_1, v_2, \ldots, v_n\}$ and the copy of graph $G$ by $\{u_1, u_2, \ldots, u_n\}$ and obvious that $\sigma(v_i) = \sigma(u_i); i = 1, 2, \ldots, n$.

**Proposition 4.2.** *Let $G$ be a strong fuzzy path has $n$ vertices; $n \geq 2$, then*

$$\gamma_{fco}(P_n + P_n) = \left\{ \begin{array}{ll} 2\sum_{i=2}^{n-1} \sigma(v_i), & \text{if } n \text{ is odd} \\ 2(\sigma(v_1) + \sigma(v_n)), & \text{if } n \text{ is even} \end{array} \right\}.$$

**Proof .** Two possible cases are as the following.
Case 1. Suppose that $n$ is odd, then the degree of vertices $\{v_i$ and $u_i; i = 2, \ldots, n-1\}$ in the graph $P_n + P_n$ are odd, then by Proposition 2.1(1), each vertex $v_i$ belongs to every FCOD set. Also, these $v_i$ are dominate the set $\{v_1, v_n, u_1, u_n\}$ which have even degrees. Therefore, $\gamma_{fco}(P_n + P_n) = 2\sum_{i=2}^{n-1} \sigma(v_i)$.
Case 2. Suppose that $n$ is even, then the degrees of vertices $\{v_1, v_n, u_1, u_n\}$ in $P_n + P_n$ are odd, then by Proposition 2.1(1), each vertex $v_i$ belong to every FCOD set. Also, these $v_i$ are dominate the set $\{v_i$ and $u_i; i = 2, \ldots, n-1\}$ which have even degrees. Therefore, $\gamma_{fco}(P_n + P_n) = 2(\sigma(v_1) + \sigma(v_n))$. Thus, from the two cases above, the proof is done. □

**Proposition 4.3.** *Let $G$ is a strong fuzzy star has $n$ vertices; $n \geq 3$, then*

$$\gamma_{fco}(S_n + S_n) = \left\{ \begin{array}{ll} 2\sum_{i=1}^{n} \sigma(v_i), & \text{if } n \text{ is even} \\ 2\sigma(v_0), & \text{if } n \text{ is odd} \end{array} \right\}.$$

**Proof .** Two possible cases are as follows.
Case 1. Suppose that $n$ is even, then the degree of each vertex in the graph $S_n + S_n$ is odd. Thus, according to the Proposition 21(1), $\gamma_{fc0}(S_n + S_n) = 2\sum_{i=1}^{n} \sigma(v_i)$.
Case 2. Suppose that $n$ is odd, then the degree of each vertex in the graph $S_n + S_n$ is even except the center of each star. Thus, according to Proposition 2.1(1), the center vertices of each star belong to each FCOD set. It is clear that the two centers dominate all other vertices in $S_n + S_n$. Therefore, $\gamma_{sco}(S_n + S_n) = 2\sigma(v_0)$.

As a consequence of the two cases above, the proof is done. □

**Proposition 4.4.** *Let $G$ be a strong fuzzy cycle has $n$ vertices; $n \geq 3$, then*

$$\gamma_{fco}\left(C_n + C_n\right) = \left\{ \begin{array}{c} 2\sum_{i=1}^{n} \sigma\left(v_i\right), \ \textit{if } n \textit{ is odd} \\ 2\min\left(\sigma\left(v_i\right)\right), \ \textit{if } n \textit{ is even} \end{array} \right\}.$$

**Proof .** Two possible cases are as the following.
Case 1. Suppose that $n$ is odd, then the degree of each vertex in $G$ is odd. Thus, according to Proposition 2.1(1), $\gamma_{fco}\left(C_n + C_n\right) = 2\sum_{i=1}^{n} \sigma\left(v_i\right)$.
Case 2. Suppose that $n$ is even, then the degree of each vertex in $G$ is even. Thus, a vertex of the minimum value of fuzzy in $C_n$ is chosen to dominate all vertices in the copy of $C_n$. Again, the corresponding vertex in the copy of $C_n$ dominates all vertices of $C_n$. Thus, $\gamma_{fco}\left(C_n + C_n\right) = 2\min\left(\sigma\left(v_i\right)\right)$. As a consequence of the two cases above, the proof is done. □

**Proposition 4.5.** *Consider $G$ as a fuzzy null has $n$ vertices; $n \geq 1$, then*

$$\gamma_{fco}\left(N_n + N_n\right) = \left\{ \begin{array}{c} 2\sum_{i=1}^{n} \sigma\left(v_i\right), \ \textit{if } n \textit{ is odd} \\ 2\min\left(\sigma\left(v_i\right)\right), \ \textit{if } n \textit{ is even} \end{array} \right\}.$$

**Proof .** It is similar to proof in the Proposition 4.4. □

**Note that** there is another proof of the graph $N_n + N_n$ by using Proposition 2.6, since $N_n + N_n \equiv K_{n,n}$.

**Proposition 4.6.** *If $G$ is a fuzzy complete has $n$ vertices, then*

$$\gamma_{fco}\left(K_n + K_n\right) = 2\sum_{i=1}^{n} \sigma\left(v_i\right).$$

**Proof .** One can easily show that $K_n + K_n \equiv K_{2n}$ thus, the degree of each vertex in this graph is odd. Therefore, according to Proposition 2.1(1), the proof is done. □

**Proposition 4.7.** *If $G$ is a fuzzy complete bipartite $K_{n,m}$ contain two partite sets $V_1$ and $V_2$ such that $V_1$ has $n$ vertices $\{u_1, u_2, \ldots, u_n\}$ and $V_2$ has $m$ vertices $\{v_1, v_2, \ldots, v_m\}$; $n, m \geq 3$, then*

$$\gamma_{fco}\left(K_{n,m} + K_{n,m}\right) = \left\{ \begin{array}{ll} 2\min\left( \begin{array}{c} \sigma\left(u_i\right), \sigma\left(v_j\right), i = 1, \ldots, n \\ j = 1, \ldots, m \end{array} \right), & \textit{if } m \textit{ and } n \textit{ are even} \\ 2\sum_{i=1}^{n} \sigma\left(u_i\right), & \textit{if } n \textit{ is odd and } m \textit{ is even} \\ 2\sum_{i=1}^{m} \sigma\left(v_i\right), & \textit{if } n \textit{ is even and } m \textit{ is odd} \\ 2\left(\sum_{i=1}^{n} \sigma\left(u_i\right) + \sum_{i=1}^{m} \sigma\left(v_i\right)\right), & \textit{if } n \textit{ and } m \textit{ are odd} \end{array} \right\}$$

**Proof .** Let $V_{11}$ and $V_{22}$ ($V_{11}$ has $n$ vertices and $V_{22}$ has $m$ vertices) are the partite sets of a copy of $G \equiv K_{n,m}$ Two possible cases are as the following.
Case 1. Assume that $m$ and $n$ are even, then the degree of each vertex in $K_{n,m} + K_{n,m}$ is even. Suppose that $\sigma(v) = \min\left(\sigma\left(u_i\right), \sigma\left(v_j\right); i = 1, \ldots, n \text{ and } j = 1, \ldots, m\right)$, without loss of generality assume that $\sigma(v) \in V_1$ Thus, the vertex $v$ dominates all vertices in the sets $V_{11}, V_2$ and $V_{22}$. Also, the corresponding vertex to $v$ in $V_{11}$ dominates all vertices in the sets $V_1, V_2$ and $V_{22}$. Therefore, $\gamma_{fco}\left(K_{n,m} + K_{n,m}\right) = 2\sigma(v)$.
Case 2. Assume that $n$ is odd and $m$ is even, then the degree of each vertex in $V_2$ is odd, since each vertex of them is adjacent to all vertices in the sets $V_1, V_{11}$, and $V_{22}$. Similarly, the degree of each vertex in $V_{22}$ is odd. Furthermore, the degree of each vertex in $V_1$ or $V_{11}$ is even, then according to Proposition 2.1(1) , $\gamma_{fco}\left(K_{n,m} + K_{n,m}\right) = 2\sum_{i=1}^{n} \sigma\left(u_i\right)$

Case 3. Assume that $n$ is even and $m$ is odd, then in the same manner in the previous case, the result is obtained.
Case 4. Assume that $m$ and $n$ are odd, in the same technique in Case 1 , one can easily to prove that the degree of each vertex in $K_{n,m} + K_{n,m}$ is odd. Thus, according to Proposition 2.1, $\gamma_{fco}\left(K_{n,m} + K_{n,m}\right) = 2\left(\sum_{i=1}^{n} \sigma\left(u_i\right) + \sum_{i=1}^{m} \sigma\left(v_i\right)\right)$.

As a consequence of the four cases above, the proof is done. □

## 5. Conclusion

Based on the results, some properties and boundaries of this definition are determined. The fuzzy co-even domination number of fuzzy certain graphs as fuzzy complete, fuzzy path, fuzzy star, fuzzy cycle, fuzzy null, fuzzy complete bipartite, and fuzzy star are calculated. Also, this definition was calculated to some certain graphs and it is some operations as complement and join (the addition) for any two graphs from some certain graphs.

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