Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 838-846
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.4939

# Numerical simulation of arterial pulse propagation using autonomous models

Mayada G. Mohammed[a], Hassan Kamil Jassim[a,*]

[a] *Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq*

*(Communicated by Madjid Eshaghi Gordji)*

### Abstract

We present a model of the fluid flow between elastic walls simulating arteries actively interacting with the blood. The lubrication theory for the flow is coupled with the pressure and shear stress from the walls. The resulting nonlinear partial differential equation describes the displacement of the walls as a function of the distance along the flow and time.

*Keywords:* channel flow, elastic, pulses.

## 1. Introduction

The overwhelming majority of mathematical models of arterial blood flow treat arteries as passive material [9, 4, 11] . A popular approximation of the flow-artery interaction is the proportionality between the increments in the artery's cross-sectional area and the flow pressure [7, 11]

$$\sqrt{p - p_0} \sim A - \mathrm{p}_{A_0}$$

where $p_0$ and $A_0$ are the reference pressure and cross-sectional area respectively. Studies of the mechanics of pulse propagation through an artery typically focus on passive response to the time-pulsating boundary condition imposed at the artery's inlet [6].

However, the arteries have muscles which actively push the blood. There are a few models [8] where the arteries actively exert pressure, however, these models are non-autonomous, that is the active component of the pressure is introduced by an explicit function of time and coordinate [11]. This means that the pulses running through the artery are created and governed by a factor that is completely external to the artery.

*Corresponding author

*Email addresses:* mayadagassab20@utq.edu.iq (Mayada G. Mohammed), hassankamil@utq.edu.iq (Hassan Kamil Jassim)

*Received:* September 2020 *Revised:* November 2020

Earlier one of the authors formulated an autonomous pulse model [11] for the flow between hypothetically active elastic walls, and numerical solutions have been obtained [1] Our aim in the current paper to present new model has the same properties of [11] and we obtain numerical solution for it. It is so crucial to obtain different model has the same role as an autonomous pulse model and this is the focus of this paper.

Before presenting our model, we describe the previous model. For simplicity, the walls are assumed to be unbounded planes (rather than tubes) and elastic. The model is partly derived from physical principles and partly from phenomenological, that is to say it was required to incorporate a proper mathematical mechanism of generating pulse-like solutions. We emphasize that the model, at least at this stage, is not intended to claim direct relevance to actual arteries. Nevertheless, in the base of the model lies the physically reasonable notion of the balance between the mechanical push from the walls applied to the blood and the opposing viscous friction. Accordingly the model can be described as active- dissipative and the pulses classified as auto-pulses, that is self-supported dissipative structures. In distinction from conservative waves, for example waves in fluids, the auto-waves have unique characteristics such as velocity, size and amplitude.

Before describing the model we review the underlying idea originating from the models of spinning combustion fronts [10] and structures in reaction-diffusion systems [12, 13]. The latter models have the form of a 6th-order nonlinear partial differential equation

$$\frac{\partial F}{\partial t} = \frac{\partial^6 F}{\partial x^6} - \frac{\partial}{\partial x}\left[\left(\frac{\partial F}{\partial x}\right)^3\right] + \left(\frac{\partial F}{\partial x}\right)^4 \tag{1.1}$$

Here $F(x,t)$ stands for the coordinate (along, say, $Z$ axis ) of the front subject to periodic boundary conditions. The front is understood as a surface (in 2D case ) or a line (as in 1D case of (1.1)) separating hot burned products from cold fresh composition. It is important to note that the combustion system is active and dissipative: its active character is due to heat generation in a chemical reaction, and its dissipative character is due to heat conductivity. These two features unite the combustion fronts and the arterial blood flows in the same category of active dissipative systems.

By differentiating w. r. t$x$ we rewrite equation (1.1) in terms of the derivative $\partial F/\partial x \equiv W$,

$$\frac{\partial W}{\partial t} = \frac{\partial^6 W}{\partial x^6} - \frac{\partial^2}{\partial x^2}\left(W^3\right) + \frac{\partial}{\partial x}\left(W^4\right) \tag{1.2}$$

A spatially uniform solution $F = \text{const}$ of (1.1) and the corresponding solution $W \equiv 0$ of (1.2) are stable under small perturbations because the linearised equation, $\partial F/\partial t = \partial^6 F/\partial x^6$, is purely dissipative.

Further below, for a more compact presentation, we will also use primes/Roman numerals to denote derivatives on $x$.

To explain the mechanism behind (1.1), denote the typical amplitude of the variation of the $F$ -function by $\Delta F > 0$ and the typical spatial scale of the variation by $\Delta x > 0$. If the initial amplitude $\Delta F$ is sufficiently large, it grows because of the pumping effect of the (3rdorder) nonlinear excitation, $-\left(F^{03}\right)^0 = \left(-3F^{02}\right)F^{00}$. Effectively, this is an anti-diffusion term with the nonlinear anti-diffusion coefficient $\left(-3F^{02}\right)$. Evaluating the terms by the order of magnitude in absolute value, we have

$$\left(F'^3\right)' \sim (\Delta F)^3/(\Delta x)^4$$

As $\Delta F$ grows, the higher-order nonlinearity comes into play,

$$\left(F^0\right)^4 \sim (\Delta F)^4/(\Delta x)^4$$

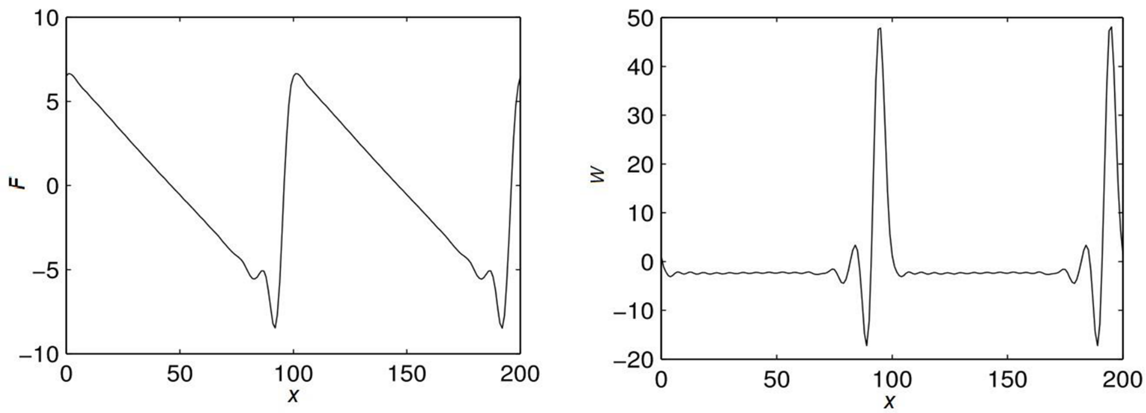


Figure 1: A train of kink-shaped (F) and pulse-shaped (W) auto-waves. The F-wave moves upwards and to the left; the W-wave moves horizontally to the left.[6]

It acts so that the sides of the $F$ -profile get steeper (the $W$ -profile locally surges in amplitude), which makes $\Delta x$ small. On the sides the dissipation prevails because it is of higher order in $\Delta x$

$$F^{VI} \sim \Delta F/(\Delta x)^6$$

As a result, the steep sections are smoothed out and the profile, instead of turning into a singularity, becomes a smooth self-sustained dissipation structure. It typically assumes the form of one or more kinks (steps) in terms of $F$ or pulses in terms of $W$; they are shown in Fig. 1. Each individual pulse is essentially a stable auto-solution with the amplitude and velocity dictated by the dynamic equation rather than an initial condition.

The 6 th-order dissipation in Eq. (1.1) hints an analogy with the dissipation in a viscous flow between elastic walls. It is well-established that the latter is too represented by the 6 th-order spatial derivative [11, 12]. This leads us to an idea to extend the model [11] by extra excitation-like terms that could represent active motion of the elastic walls in order to make the fluid-wall model structurally similar to (1.2) .

## 2. The Previous model [11]

Let us follow [11], consider the flow between infinite elastic walls, assuming symmetry with respect to the middle plane, $z = 0$; hence it will suffice to analyse only half of the flow, $0 < z < H(t)$. All the variables are assumed uniform in $y$ -direction. Adopting the lubrication theory [2] equate the pressure gradient to the viscous friction,

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{\eta}\frac{\partial p}{\partial x} \tag{2.1}$$

where $x$ and $z$ are the coordinates along and across the flow respectively, $v(x, z, t)$ the flow velocity in the $x$ -direction, $p(x, t)$ the pressure, and $\eta$ the viscosity. The pressure is assumed $z$ -independent, so that integrating (2.1) on $z$ gives

$$v = \frac{1}{2\eta}\frac{\partial p}{\partial x}\left(z^2 - H^2\right) + v(x, H, t) \tag{2.2}$$

The mass flux is

$$Q = \int_0^H v dz = -\frac{H^3}{3\eta}\frac{\partial p}{\partial x} + v(x, H, t)H \tag{2.3}$$

Define the displacement, $w(x,t)$, of the wall in the $z$ -direction from the neutral position, $w = H_0$, by

$$H = H_0 + w \tag{2.4}$$

Then the continuity equation becomes

$$\frac{\partial w}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{2.5}$$

Substituting (2.3) into (2.5) gives

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\frac{H^3}{3\eta}\frac{\partial p}{\partial x} - v(x, H, t)H\right]. \tag{2.6}$$

Equation (2.6) links the displacement of the flow boundary, coinciding with the wall's position, to the flow pressure. The elasticity theory [5, 14] provides the reverse link from the pressure to the displacement,

$$p = D\frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right) \tag{2.7}$$

where

$$N = \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{2.8}$$

In (2.7) and (2.8) $u(x,t)$ is the wall's displacement along the flow, $D$ the flexural rigidity of the wall, $E$ Young's modulus, $h$ the thickness of the wall, $v$ Poisson's ratio, and $N$ the force caused by the displacements. Substituting (2.8) and (2.7) into (2.6), and using the no-slip boundary condition,

$$v(x, H, t) = \frac{\partial u}{\partial t}$$

we obtain

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{D}{3\eta}\left(H^3 w^V\right)' - \frac{Eh}{6\eta\left(1-\nu^2\right)}\left[H^3\left(w'^3\right)''\right]' \\ &- \frac{Eh}{3\eta\left(1-\nu^2\right)}\left[H^3\left(u'w'\right)''\right]' - \left(\frac{\partial u}{\partial t}H\right)'\end{aligned} \tag{2.9}$$

The shear stress in the fluid is represented as usual by $T = \eta\partial v/\partial z$, therefore on the boundary, $z = H$, using (2.2)

$$T = p^0 H \tag{2.10}$$

This shear stress must be equal to the shear stress produced by the wall,

$$T = N^0 \tag{2.11}$$

Equating (2.10) and (2.11) with the use of (2.8), we have

$$\frac{E}{1-\nu^2}\left[u'' + \frac{1}{2}\left(w'^2\right)'\right] = p'H \tag{2.12}$$

The three equations (2.9),(2.12) and (2.7) form a closed system with respect to the three functions of interest, $w(x,t), u(x,t)$ and $p(x,t)$. As a straightforward exercise, let us solve the system for small-amplitude perturbations of the neutral state. They satisfy the linearised equations

$$\begin{aligned} &\frac{E}{1-\nu^2}u'' = DH_0 w^V \\ &\frac{\partial w}{\partial t} = \frac{H_0^3}{3\eta}Dw^{VI} - \frac{\partial u'}{\partial t}H_0 \end{aligned} \tag{2.13}$$

Looking for spatially periodic solutions, relevant, for example, to a closed-loop configuration,

$$w = A(t)\sin(kx), \quad u = B(t)\cos(kx),$$

we obtain

$$\begin{aligned} &\frac{E}{1-\nu^2}B = -DH_0k^3A \\ &\frac{dA}{dt} = -\frac{H_0^3}{3\eta}Dk^6A + H_0k\frac{dB}{dt} \end{aligned}$$

It is easy to deduce the solution

$$A(t) \sim B(t) \sim \exp\left[-\frac{H_0^3DEk^6}{3\eta E + 3\eta DH_0^2\left(1-\nu^2\right)k^4}t\right]$$

The expression under the exponent is strictly negative ($1 - v^2 > 0$). Thus, as expected for a closed passive system, the classical theory gives a decaying wave. Now suppose that, when deflecting from the neutral position, the elastic walls exert extra pressure, in addition to (2.7),

$$p = Dw^{IV} - \left(Nw^0\right)^0 + p_1 \tag{2.14}$$

where $p_1$ depends on $w$. We postulate that $p_1$ is proportional to the 4 th power of the vertical displacement,

$$p_1 = -\alpha w^4, \quad \alpha > 0 \tag{2.15}$$

The motivation for this assumption is to achieve analogy with the term $W^3$ in (1.2). Further, we suppose that the walls actively move along the flow, thereby producing an extra shear stress relative to (2.12). We postulate that the wall's motion along the flow, represented by the displacement $u$ and velocity $\partial u/\partial t$, is coupled with $w$. Specifically, the $H$ -weighted velocity along the flow, $H\partial u/\partial t$, combined with the other $u$ -containing term in (2.9) , is proportional to the 5 th power of $w$. This assumption will result in analogy with the term $W^4$ in (1.2),

$$-\frac{Eh}{3\eta\left(1-\nu^2\right)}H^3\left(u'w'\right)'' - \frac{\partial u}{\partial t}H = \beta w^5, \quad \beta > 0 \tag{2.16}$$

This relation implies that an extra (active) shear stress takes place; we denote it $T_1$. The total stress satisfies the continuity condition on the boundary

$$T_1 + \frac{E}{1-\nu^2}\left[u'' + \frac{1}{2}\left(w'^2\right)'\right] = p'H \tag{2.17}$$

where the pressure $p$ is represented by (2.14). Although we are not able to justify the concrete values of the powers of $w$ used in (2.15) and (2.16), at least these relations reflect the natural hypothesis that we adopt about the larger active response from the walls to the larger deformation $w$. Under the assumptions (2.14),(2.15) and (2.16), the equation (2.9) governing the dynamics of the vertical displacement becomes $u$ -independent:

$$\begin{aligned}\frac{\partial w}{\partial t} = \frac{D}{3\eta}\left(H^3 w^V\right)' - \frac{Eh}{6\eta\left(1-\nu^2\right)}\left[H^3\left(w'^3\right)''\right]' \\ -\frac{1}{3\eta}\alpha\left[H^3\left(w^4\right)'\right]' + \beta\left(w^5\right)'\end{aligned} \tag{2.18}$$

We arrived at a closed system. The procedure of finding solution is as follows. The function $w(x,t)$ is obtained from (2.18) under, say, periodic boundary conditions. Then $u(x,t)$ is found from (2.16) and the pressure from (2.14),(2.15). The extra shear stress $T_1(x,t)$ and total stress are obtained from (2.17). It is convenient to introduce the function $f$ by $w = f^0$ and present (2.18) in equivalent form that allows to draw direct analogy and make comparison to the combustion model (1.1),

$$\frac{\partial f}{\partial t} = \frac{D}{3\eta}H^3 f^{VI} - \frac{Eh}{6\eta\left(1-\nu^2\right)}H^3\left[\left(f''\right)^3\right]'' - \frac{1}{3\eta}\alpha H^3\left[\left(f'\right)^4\right]' + \beta\left(f'\right)^5 + K \tag{2.19}$$

where $K$ is the constant of integration. Note that the first two terms in the right-hand side here originate from the classical theory and, therefore, are dissipative. Comparing with (1.1), take notice of the higher order of nonlinearity of the excitation term (the 4 th instead of the 3rd); it is necessary to overpower the 3 rd-order classical term in (2.19) . Accordingly, the last term in ( 2.19 ) has even higher order of nonlinearity (5th) required to counterbalance the excitation as the function grows. Now we evaluate the terms by the order of magnitude in absolute value. The excitation is

$$\frac{1}{3\eta}\alpha H^3\left[\left(f'\right)^4\right]' \sim \left(f'\right)^3 f'' \sim (\Delta f)^4/(\Delta x)^5$$

(omitting the coefficients). As the amplitude of the function variation increases, the 5 thorder nonlinearity comes into play,

$$\beta\left(f^0\right)^5 \sim (\Delta f)^5/(\Delta x)^5$$

It steepens the profile of $f$ on its side segments, where eventually the dissipation prevails due to the higher order in $\Delta x$,

$$\frac{D}{3\eta}H^3 f^{VI} \sim \Delta f/(\Delta x)^6$$

## 3. Our Model

Our model has the same nature as (2.19). We follow the same procedure as in section (2) with the require changes in different places. We obtain the following forms:

$$\frac{\partial w}{\partial t} = \frac{D}{3\eta}H^3\left(w^V\right)' - \frac{Eh}{6\eta\left(1-\nu^2\right)}H^3\left[\left(w'\right)^3\right]'' - \frac{1}{3\eta}\alpha H^3\left[\left((w)^3\right)' + \left(w'\right)^2\right] + \beta(w)^4 \tag{3.1}$$

$$\frac{\partial f}{\partial t}=\frac{D}{3\eta}H^3f^{VI}-\frac{Eh}{6\eta\left(1-\nu^2\right)}H^3\left[\left(f''\right)^3\right]''-\frac{1}{3\eta}\alpha H^3\left[\left(\left(f'\right)^3\right)'+\left(f''\right)^2\right]+\beta\left(f'\right)^4+K \tag{3.2}$$

where $f, w, D, \eta, H, E, h, \eta, v, \alpha, K$ are the same as in section 2 . Note that the first two terms in the right-hand side here originate from the classical theory and, therefore, are dissipative. Take notice of the excitation term, it has two terms $\left(\left(\left(f^0\right)^3\right)^0 \text{ and } \left(f^{00}\right)^2\right)$ ) and the higher order of nonlinearity is 3) ; it is necessary to overpower the 3rd-order classical term in (3.2). Accordingly, the last term in (3.2) has even higher order of nonlinearity (4th) required to counterbalance the excitation as the function grows. Now we evaluate the terms by the order of magnitude in absolute value. The excitation is

$$\frac{1}{3\eta}\alpha H^3\left[\left(f'\right)^3\right]'\sim\left(f'\right)^2f''\sim(\Delta f)^3/(\Delta x)^4$$
$$\frac{1}{3\eta}\alpha H^3\left[\left(f''\right)^2\right]\sim(\Delta f)^2/(\Delta x)^2$$

(omitting the coefficients). As the amplitude of the function variation increases, the 4 th- order nonlinearity comes into play,

$$\beta\left(f^0\right)^4\sim(\Delta f)^4/(\Delta x)^4$$

It steepens the profile of $f$ on its side segments, where eventually the dissipation prevails due to the higher order in $\Delta x$,

$$\frac{D}{3\eta}H^3f^{VI}\sim\Delta f/(\Delta x)^6$$

This mechanism facilitates the energy flow from low to high wavenumbers in the way similar to the Kuramoto-Sivashinsky (KS) equation [9], however the excitation in our model and in the models (1.1) and (2.19) as well is nonlinear, while in the KS equation it is linear. Because of the analogy between (3.2), (2.19) and (1.1) we expect pulse-shaped waves to exist in the hydroelastic model as well. Our model meets the two key requirements: (1.1) be autonomous and (1.2) have the capacity to produce auto-pulses. The pulses are a corollary of the assumption that the walls actively push the blood against the viscous forces.

Trying to maintain parallel with the actual arteries we also require that the overall displacement of the artery's material over one period, $\tau$, is zero,

$$\int_\tau\frac{\partial u}{\partial t}dt=0,\quad\int_\tau\frac{\partial w}{\partial t}dt=0 \tag{3.3}$$

The second condition in (3.3) is guaranteed provided the boundary conditions are periodic. Indeed, the wave solution depends on $x/\lambda+t$, where $\lambda$ is the wave velocity, so integrating w. r. $tt$ is equivalent to integrating on $x/\lambda$. Integrating equation (3.1) over one or several periods gives zero because all the terms in the equation's right-hand side are full derivatives and the expressions under them are periodic. The first condition of (3.3) can be transformed using

$$-\frac{Eh}{3\eta\left(1-\nu^2\right)}H^3\left(u'w'\right)''-\frac{\partial u}{\partial t}H=\beta w^4,\quad\beta>0 \tag{3.4}$$

to the form

$$\int_\tau\left[\frac{Eh}{3\eta\left(1-\nu^2\right)}H^2\left(u'w'\right)''+\frac{\beta}{H}w^4\right]dt=0 \tag{3.5}$$

Transferring from time- to space-integral and replacing $H$ by $H_0$ (assuming that the dis- placement $w$ is small compared to $H_0$ so that $H^2 u'w' = (H_0 + w)^2 u'w' \approx H_0^2 u'w'$ and also $(1/H)w^4 = 1/(H_0 + w)\, w^4 \approx (1/H_0)\, w^4)$, Eq. (3.5) is reduced to

$$\int_L^Z w^4 dx = 0 \tag{3.6}$$

where $L$ is the period in space. Now, integrate the main equation (3.2) over the spatial period using $\partial/\partial t = \lambda \partial/\partial x$. Using periodicity of all the terms, and remembering that $w = f^0$ we see that condition (3.6) translates into

$$\lambda \int_0^L f' dx = \beta \int_0^L (f')^4 dx + KL = \beta \int_0^L w^4 dx + KL = KL \tag{3.7}$$

From here

$$f(L,t) - f(0,t) = \frac{KL}{\lambda} \tag{3.8}$$

Therefore, in order to guarantee the zero displacement of an element of the wall after one or several periods, the boundary condition has to have the special form (3.8), where the wave velocity $\lambda$ is part of solution. Condition (3.8) can be achieved, for example, via an iteration process. However, for simplicity in this paper we limit our attention to the case $K = 0$, in which the boundary condition (3.8) becomes the condition of periodicity,

$$f(L,t) - f(0,t) = 0$$

In distinction to the combustion model (1.2), the hydro-elastic model (3.1) generates only pulses travelling to the left. This is caused by the asymmetry: the term $-(f^0)^2 f^{00}$ acts as excitation only on sections with positive slopes, $f^0 > 0$, because on those sections the term is effectively an anti-diffusion with the negative nonlinear anti-diffusion coefficient, $-(f^0)^2 < 0$, whereas on sections with negative slopes, $f^0 < 0$, it acts as normal nonlinear diffusion. We are satisfied with this property as it implies that our simulated artery "knows" the direction to "heart", which would be to the right in Fig. 1 . Accordingly the artery only supports pulses travelling in the opposite direction that is to the left.

## 4. Conclusion

We presented the autonomous model of a hypothetical flow between active elastic walls in order to mimic arterial flows. Lubrication theory is used for the flow, and the walls are supposed to actively exert pressure and shear stress. The analogy with the combustion front equations analyzed earlier, led us to an anticipation that the model would be capable of generating auto-wave solutions in the form of pulses - this main point is proved in the current paper.

The expressions adopted for the active components of the pressure (2.15) and shear stress (2.16)-(2.17) are empirical; they contain the empirical coefficients $\alpha$ and $\beta$. Note though that for such a complex system as biological, the more details and processes are attempted to be taken into account the more coefficients are inevitably involved into consideration. As always, the values of such coefficients may not be easy to determine. From this point of view the fact that our model has only two such coefficients can be regarded as a positive feature. Finally we remark that the capability of our model to exhibit self-organised behaviour is new in arterial pulse simulation. Yet, it remains to be seen whether or not the model can lead to any direct applications.

## References


[1] F. Ahmed, D. Strunin, M. Mohammed and R. Bhanot, *Numerical solution for the fluid flow between active elastic walls*, ANZIAM Journal (E) (former J. of Australian Math. Soc. Ser. B) 57 (2016) 163–177.

[2] R. Huang and Z. Suo, *Wrinkling of a compressed elastic film on a viscous layer*, J. Appl. Phys. 91 (2002) 1135–1142.

[3] J.R. King, *The isolation oxidation of silicon: the reaction-controlled case*, SIAM J. Appl. Math. 49 (1989) 1064–1080.

[4] C. Kleinstreuer, *Biofluid Dynamics*, Taylor and Francis, Boca Raton, 2006.

[5] L.D. Landau and E.M. Lifshitz, *Theory of elasticity*, Pergamon. London, (1959) 57-60.

[6] K.S. Matthys, J. Alastruey, J. Peiro, A.W. Khir, P. Segers, P.R. Verdonck, K.H. Parker and S.J. Sherwin, *Pulse wave propagation in a model human arterial network: Assessment of 1-D numerical simulations against in vitro measurements*, J. Biomech. 40 (2007) 3476–3486.

[7] M.S. Olufsen, C.S. Peskin, W.Y. Kim, E.M. Pedersen, A. Nadim and J. Larsen, *Numerical simulation and experimental validation of blood flow in arteries with structured-tree outflow conditions*, Ann. Biom. Engin. 28 (2000) 1281–1299.

[8] A.J. Roberts, *A One-Dimensional Introduction to Continuum Mechanics*, World Scientific, Singapore, 1994.

[9] S.J. Sherwin, L. Formaggia, J. Peiro and V. Franke, *Computational modelling of 1D blood flow with variable mechanical properties and its application to the simulation of wave propagation in the human arterial system*, Int. J. Numer. Meth. Fluids 43 (2003) 673–700.

[10] D.V. Strunin, *Autosoliton model of the spinning fronts of reaction*, IMA J. Appl. Math. 63 (1999) 163-177.

[11] D.V. Strunin, *Fluid flow between active elastic plates*, ANZIAM Journal (E) (former J. of Australian Math. Soc., Ser. B) 50 (2009) C871–C883.

[12] D.V. Strunin, *Phase equation with nonlinear excitation for nonlocally coupled oscillators*, Physica D: Nonlin. Phenomena 238 (2009) 1909–1916.

[13] D.V. Strunin and M.G. Mohammed, *Range of validity and intermittent dynamics of the phase of oscillators with nonlinear self-excitation*, Communic. Nonlin. Sci. Numer. Simul. 29 (2015) 128–147.

[14] S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill. New York, 1987.