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New subclass of analytic functions defined by subordination

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Abstract

By using the subordination relation " $\prec$ ", we introduce an interesting subclass of analytic functions as follows:

$$\mathcal{S}^*_\alpha := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1}{(1-z)^\alpha}, \quad |z| < 1 \right\},$$

where $0 < \alpha \leq 1$ and $\mathcal{A}$ denotes the class of analytic and normalized functions in the unit disk $|z| < 1$. In the present paper, by the class $\mathcal{S}^*_\alpha$ and by the Nunokawa lemma we generalize a famous result connected to starlike functions of order $1/2$. Also, coefficients inequality and logarithmic coefficients inequality for functions of the class $\mathcal{S}^*_\alpha$ are obtained.

Keywords: Nunokawa's lemma; univalent; subordination; starlike functions; coefficients estimates; logarithmic coefficients.
2010 MSC: 30C45

1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk on the complex plane $\mathbb{C}$ and let $\mathcal{H}$ be the class of all analytic functions f in Δ. We denote by $\mathcal{A}$ the class of all functions that are analytic and normalized by $f(0) = 0$ and $f'(0) = 1$ in Δ. Moreover for each $f \in \mathcal{A}$, we have the following representation:

$$f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots \qquad (z \in \Delta). \tag{1.1}$$

Also, we denote by $\mathcal{S}$, the class of all univalent (one–to–one) functions in Δ. The well–known class of analytic functions p with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ $(z \in \Delta)$ is denoted by $\mathcal{P}$. Let f and g belong

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to the class $\mathcal{A}$. Then we say that f is subordinate to g, written by $f(z) \prec g(z)$, if there exists a Schwarz function w such that $f(z) = g(w(z))$ for all $z \in \Delta$. In particular, if $g \in \mathcal{S}$, then the following equivalence relationship holds true

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$

A function $f \in \mathcal{A}$ is said to be starlike of order γ if f satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in \Delta),$$

for some $0 \leq \gamma < 1$, [16]. We denote by $\mathcal{S}^*(\gamma)$ the class of starlike functions of order γ and we denote by $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ the class of starlike functions. Next, we say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{Q}(\gamma)$ if it satisfies the following condition

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \gamma \qquad (0 \leq \gamma < 1, z \in \Delta).$$

Also, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\gamma)$ if f satisfies

$$\mathrm{Re}\left\{f'(z)\right\} > \gamma \qquad (0 \leq \gamma < 1, z \in \Delta).$$

We note that $f(z) \in \mathcal{R}(\gamma)$ if and only if $zf'(z) \in \mathcal{Q}(\gamma)$. It is well–known that any $f(z) \in \mathcal{R}(\gamma)$ is univalent in Δ (see [12] or [21]). Let φ be analytic function with the positive real part mapping the unit disk Δ onto a domain symmetric with respect to real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Ma and Minda [8] introduced and studied the class $\mathcal{S}^*(\varphi)$ including of all functions $f \in \mathcal{A}$ so that

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (z \in \Delta). \tag{1.2}$$

They obtained distortion, covering, and growth theorems. The class $\mathcal{S}^*(\varphi)$ is generalization of many well–known classes. For example, by selecting $\varphi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ we have the class $\mathcal{S}^*[A, B]$ of Janowski starlike functions [3]. Also, the class $\mathcal{S}^*[-1, 1]$ is the well–known class of starlike functions. Recently, several authors have defined many interesting subclasses of $\mathcal{S}^*$ by restricting the value of $zf'(z)/f(z)$ to lie in a certain precise domain in the right–half plane. The Table 1 shows more details about some subclasses of starlike functions with different choices for φ.

Motivated by the above classes we define.

Definition 1.1. *Let $f \in \mathcal{A}$ and $0 < \alpha \leq 1$. Then we say that a function f belongs to the class $\mathcal{S}^*_\alpha$ if it satisfies the following subordination relation*

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{(1-z)^\alpha} \quad (z \in \Delta). \tag{1.3}$$

Remark 1.2. *Since the function*

$$q_\alpha(z) := \frac{1}{(1-z)^\alpha} = 1 + \sum_{n=1}^{\infty} Q_n z^n \quad (0 < \alpha \leq 1), \tag{1.4}$$

Table 1: Some subclass of $\mathcal{S}^*$

Authors	$\varphi(z)$	Year	Ref.
Ma and Minda	$1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$	1992	[9]
Sokół and Stankiewicz	$\sqrt{1+z}$	1996	[20]
Sokół	$\frac{3}{3+(\alpha-3)z-\alpha z^2}$, $(-3<\alpha\le 1)$	2011	[19]
Kuroki and Owa	$1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right)$, $(0\le\alpha<1,\beta>1)$	2011	[7]
Mendiratta *et al.*	$\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$	2014	[10]
Mendiratta *et al.*	e^z	2015	[11]
Raina and Sokół	$z+\sqrt{1+z^2}$	2015	[14]
Sharma *et al.*	$1+\frac{4z}{3}+\frac{2z^2}{3}$	2016	[18]
Kumar and Ravichandran	$1+(z/k)\frac{k+z}{k-z}$, $(k=1+\sqrt{2})$	2016	[6]
Kargar *et al.*	$1+\frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$, $(\pi/2\le\alpha<\pi)$	2017	[5]
Kargar *et al.*	$1+\frac{z}{1-\alpha z^2}$, $(0\le\alpha\le 1)$	2019	[4]

where

$$Q_n=\prod_{k=2}^{n+1}\frac{k-2+\alpha}{k-1}\quad (n=1,2,3,\ldots), \tag{1.5}$$

is univalent and $\mathrm{Re}\{q_\alpha(z)\}>2^{-\alpha}$*, by the subordination principle, if* $f\in\mathcal{S}^*_\alpha$*, then*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>2^{-\alpha}\quad (0<\alpha\le 1).$$

Indeed, the set $q_\alpha(\Delta)$ *lies in the right–hand half plane (see Fig. 1 for* $\alpha=1/3$*) and thus* $\mathcal{S}^*_\alpha\subset\mathcal{S}^*$*. On the other hand* $1/2\le 2^{-\alpha}<1$*, and thus by [15] the members of the class* $\mathcal{S}^*_\alpha$ *are univalent. Also,* $\mathcal{S}^*_1$ *becomes the class of starlike functions of order* $1/2$*.*

The following lemmas will be useful.

Lemma 1.3. (Nunokawa [13]) Let $p(z)$ be an analytic function in $|z|<1$ of the form

$$p(z)=1+\sum_{n=m}^{\infty}c_nz^n\quad (c_m\neq 0),$$

with $p(z)\neq 0$ in $|z|<1$. If there exists a point z_0, $|z_0|<1$, such that

$$\mathrm{Re}\{p(z)\}>0\quad\text{for}\quad |z|<|z_0|$$

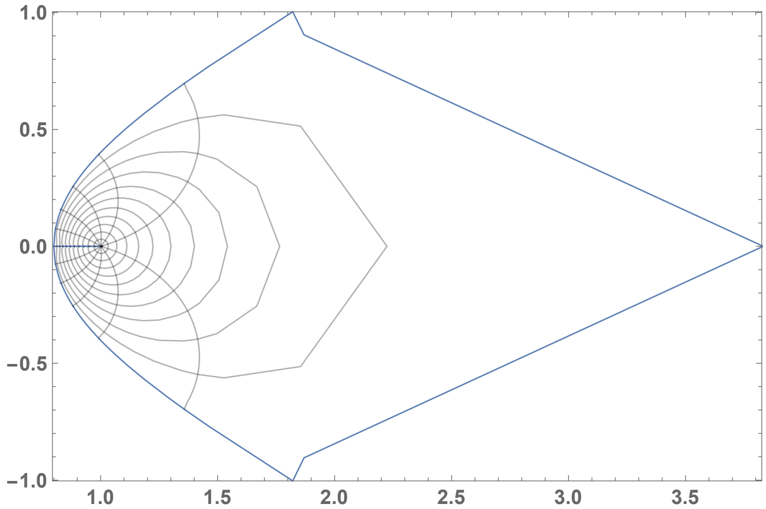

Figure 1: The boundary curve of $q_{1/3}(\Delta)$

and

$$\mathrm{Re}\{p(z_0)\} = 0, \quad a = |p(z_0)| \neq 0,$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where k is real number and

$$k \geq \frac{m}{2}\left(a + \frac{1}{a}\right) \geq m \geq 1 \quad \text{when} \quad p(z_0) = ia \tag{1.6}$$

and

$$k \leq -\frac{m}{2}\left(a + \frac{1}{a}\right) \leq -m \leq -1 \quad \text{when} \quad p(z_0) = -ia. \tag{1.7}$$

Lemma 1.4. (Rogosinski [17]) Let $q(z) = \sum_{n=1}^{\infty} q_n z^n$ be analytic and univalent in Δ, and suppose that $q(z)$ maps Δ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} p_n z^n$ is analytic in Δ and satisfies the following subordination

$$p(z) \prec q(z) \quad (z \in \Delta),$$

then

$$|p_n| \leq |q_1| \quad (n = 1, 2, \ldots).$$

In the present paper, we first find a lower bound for $\mathrm{Re}\{f(z)/z\}$ when $f \in \mathcal{S}^*_\alpha$ and as a corollary we improve the well–known result concerning starlike functions of order 1/2. Also, we estimate the coefficients and logarithmic coefficients of functions f which belong to the class $\mathcal{S}^*_\alpha$.

2. Main Results

The first result is the following.

Theorem 2.1. Let $f \in \mathcal{S}^*_\alpha$. Then

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2^\alpha} \quad (0 < \alpha \leq 1). \tag{2.1}$$

This means that $\mathcal{S}^*_\alpha \subset \mathcal{Q}(2^{-\alpha})$. The result is sharp.

Proof . Let the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\alpha$. We define

$$\left(1 - 2^{-\alpha}\right) p(z) + 2^{-\alpha} = \frac{f(z)}{z} \quad (z \in \Delta). \tag{2.2}$$

Then p is analytic in Δ and $p(0) = 1$. By an easy calculation, (2.2) yields that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{(1 - 2^{-\alpha})\, zp'(z)}{(1 - 2^{-\alpha})\, p(z) + 2^{-\alpha}} \quad (z \in \Delta). \tag{2.3}$$

Suppose that there exists a point $z_0 \in \Delta$ so that

$$\operatorname{Re}\{p(z)\} > 0 \quad \text{for} \quad |z| < |z_0|$$

and

$$\operatorname{Re}\{p(z_0)\} = 0 \quad a = |p(z_0)| \neq 0.$$

Applying Nunokawa's lemma, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik, \tag{2.4}$$

where

$$k \geq \frac{1 + a^2}{2a} \quad \text{when} \quad p(z_0) = ia \tag{2.5}$$

and

$$k \leq -\frac{1 + a^2}{2a} \quad \text{when} \quad p(z_0) = -ia. \tag{2.6}$$

We consider the case $p(z_0) = ia$. The proof of the case $p(z_0) = -ia$ is similar and therefore we omit the details. From (2.3), we have

$$\begin{aligned}
\operatorname{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} &= \operatorname{Re}\left\{1 + \frac{(1 - 2^{-\alpha})\, z_0 p'(z_0)}{p(z_0)} \times \frac{p(z_0)}{(1 - 2^{-\alpha})\, p(z_0) + 2^{-\alpha}}\right\} \\
&= \operatorname{Re}\left\{1 + \left(1 - 2^{-\alpha}\right) ik \frac{ia}{(1 - 2^{-\alpha})\, ia + 2^{-\alpha}}\right\} \\
&= 1 - \operatorname{Re}\left\{\frac{\beta ak}{(1 - \beta) + ia\beta}\right\} \quad (\beta := 1 - 2^{-\alpha}) \\
&= 1 - \frac{\beta(1 - \beta)ka}{(1 - \beta)^2 + a^2\beta^2} \\
&\leq 1 - \frac{\beta(1 - \beta)}{2} \frac{1 + a^2}{(1 - \beta)^2 + a^2\beta^2} \\
&< 1 - \frac{1}{2}\frac{\beta}{1 - \beta} \qquad (2.7) \\
&\leq 1 - \beta = 2^{-\alpha} \quad (\text{when} \;\; \beta \to 0^+ \;\; \text{or} \;\; \beta \to (1/2)^-).
\end{aligned}$$

To justify the inequality (2.7), we have left to show that for any $a \in (0, \infty)$ the following inequality hlods

$$h(a) := \frac{1+a^2}{(1-\beta)^2 + a^2\beta^2} > \frac{1}{(1-\beta)^2} = h(0).$$

Since h is increasing function, this is clear. But by Lemma 1.3 this is contradictory and we have

$$\mathrm{Re}\,\{p(z)\} > 0 \quad (z \in \Delta).$$

Therefore the inequality (2.1) holds. For the sharpness consider the function

$$\begin{aligned} \ell_\alpha(z) &:= \frac{z}{(1-z)^\alpha} \quad (0 < \alpha \le 1) \\ &= z + \alpha z^2 + \frac{1}{2}\alpha(\alpha+1)z^3 + \frac{1}{6}\alpha(\alpha+1)(\alpha+2)z^4 + \mathrm{O}(z^5). \end{aligned} \tag{2.8}$$

It is clear that ℓ_α is holomorphic and

$$\frac{z\ell'_\alpha(z)}{\ell_\alpha(z)} = 1 + \frac{\alpha z}{1-z} =: H_\alpha(z) \quad (z \in \Delta).$$

Since $q_\alpha(z) = 1/(1-z)^\alpha$ is univalent, $H_\alpha(0) = 1 = q_\alpha(0)$ and $H_\alpha(\Delta) \subset q_\alpha(\Delta)$ (because $\mathrm{Re}\{H_\alpha(z)\} = 1 - \alpha/2 \ge 2^{-\alpha} = \mathrm{Re}\{q_\alpha(z)\}$ when $0 < \alpha \le 1$), we get $H_\alpha(z) \prec q_\alpha(z)$. This means that $\ell_\alpha(z) \in \mathcal{S}^*_\alpha$. On the other hand

$$\mathrm{Re}\left\{\frac{\ell_\alpha(z)}{z}\right\} = \mathrm{Re}\left\{\frac{1}{(1-z)^\alpha}\right\} > \frac{1}{2^\alpha} \quad (0 < \alpha \le 1).$$

This completes the proof. $\square$

The Figure 2 shows the image of the unit disk under the function $\ell_\alpha(z)$ for $\alpha = 1/2$.

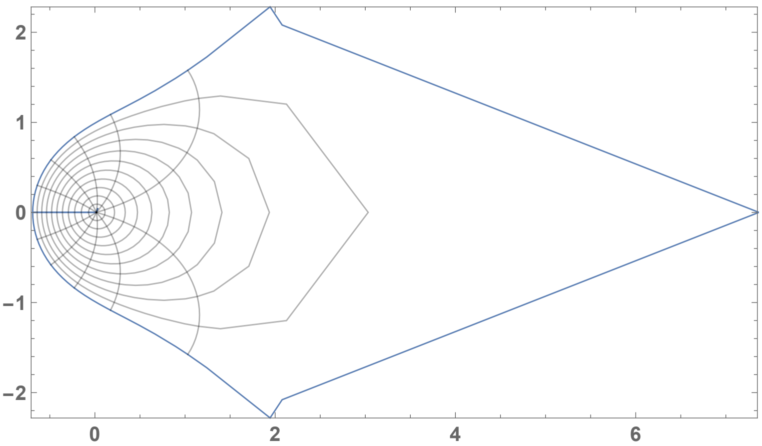

Figure 2: The boundary curve of $\ell_{1/2}(\Delta)$

Taking $\alpha = 1$ in the above Theorem 2.1, we get the following well–known result.

Corollary 2.2. *If*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \Delta),$$

then

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2} \quad (z \in \Delta).$$

Since $1/2 \leq 2^{-\alpha} < 1$ when $0 < \alpha \leq 1$, we have the following.

Corollary 2.3. *Let f be starlike function of order $\gamma(1/2 \leq \gamma < 1)$. Then we have*

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \gamma \quad (1/2 \leq \gamma < 1,\, z \in \Delta).$$

In other words, we have $\mathcal{S}^*(\gamma) \subset \mathcal{Q}(\gamma)$ when $1/2 \leq \gamma < 1$.

We remark that the Corollary 2.3 generalizes the well–known result given in the Corollary 2.2.

Next, we obtain the sharp estimates of coefficients of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_\alpha^*$.

Theorem 2.4. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_\alpha^*$, then

$$|a_n| \leq \prod_{k=2}^{n} \frac{k-2+\alpha}{k-1} \quad (n = 2, 3, \ldots). \tag{2.9}$$

The inequality is sharp.

Proof . Let f of the form (1.1) belongs to $\mathcal{S}_\alpha^*$. Then by Definition 1.1 we have $\phi(z) \prec q_\alpha(z)$ where

$$\phi(z) := \frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} \lambda_n z^n \quad (z \in \Delta) \tag{2.10}$$

and q_α is given by (1.4). Since q_α is convex in the unit disk Δ, we get

$$|\lambda_n| \leq |Q_1| = \alpha, \tag{2.11}$$

where α is the coefficient of z in the Taylor series of q_α. Now, from the relation (2.10), we get

$$zf'(z) = \phi(z)f(z). \tag{2.12}$$

Equating the coefficients of z^n in both sides of the last equation (2.12) we have

$$(n-1)a_n = \lambda_{n-1} + \lambda_{n-2}a_2 + \cdots + \lambda_1 a_{n-1} \quad (n = 2, 3, \ldots). \tag{2.13}$$

A simple calculation and using (2.11), give us

$$(n-1)|a_n| \leq \alpha \sum_{k=2}^{n} |a_{k-1}| \quad (|a_1| = 1), \tag{2.14}$$

or

$$|a_n| \leq \frac{\alpha}{n-1} \sum_{k=2}^{n} |a_{k-1}|.$$

It is a simple exercise (by using the mathematical induction) that

$$\frac{\alpha}{n-1} \sum_{k=2}^{n} |a_{k-1}| \leq \prod_{k=2}^{n} \frac{k-2+\alpha}{k-1}.$$

Therefore, we have the inequality (2.9). Also, it is easy to see that equality occurs for the coefficients of the function $\ell_\alpha(z)$ given by (2.8). This completes the proof. □

Putting $\alpha = 1$, in the Theorem 2.4, we have the following well–known result

Corollary 2.5. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be starlike function of order* $1/2$. *Then* $|a_n| \leq 1$.

We note that the above Corollary 2.5 is trivial, because $\mathcal{K} \subset \mathcal{S}^*(1/2) \subset \mathcal{Q}(1/2)$, where $\mathcal{K}$ is the class of convex functions and for each function $f \in \mathcal{K}$, we have $|a_n| \leq 1$. Also, Dvorák [2] proved that if f of the form (1.1) is odd and belongs to $\mathcal{Q}(1/2)$, then $|a_n| \leq 1$.

The logarithmic coefficients $\gamma_n := \gamma_n(f)$ of $f \in \mathcal{A}$ are defined by

$$\log \left\{ \frac{f(z)}{z} \right\} = \sum_{n=1}^{\infty} 2\gamma_n z^n \quad (z \in \Delta). \tag{2.15}$$

The logarithmic coefficients have had great impact in the development of the theory of univalent functions. For example, de Branges by using of this concept, was able to prove the famous Bieberbach's conjecture [1]. Thus, the next theorem is related to logarithmic coefficients.

Theorem 2.6. Let $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*_\alpha$, $0 < \alpha \leq 1$ and γ_n be the logarithmic coefficients of f. Then

$$|\gamma_n| \leq \frac{\alpha}{2n} \quad (n \geq 1).$$

The inequality is sharp.

Proof . Let $f \in \mathcal{A}$ and $0 < \alpha \leq 1$. If f belongs to the class $\mathcal{S}^*_\alpha$, then by using the Definition 1.1 we have

$$\frac{zf'(z)}{f(z)} - 1 = z \left(\log \left\{ \frac{f(z)}{z} \right\} \right)' = \sum_{n=1}^{\infty} 2n\gamma_n z^n \prec q_\alpha(z) - 1,$$

where q_α given by (1.4). Moreover if $f \in \mathcal{S}^*_\alpha$, then

$$\sum_{n=1}^{\infty} 2n\gamma_n z^n \prec \sum_{n=1}^{\infty} Q_n z^n \quad (0 < \alpha \leq 1),$$

where Q_n defined in (1.5). Thus by Lemma 1.4 we obtain that $|\gamma_n| \leq \alpha/2n$ for $n \geq 1$. It is easy to see that equality occurs for the logarithmic coefficients of the function $\ell_\alpha(z)$ given by (2.8) concluding the proof. $\square$

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