Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 869-875
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

Classification Of Problems Of Determining The Maximum Common Fragments For Two Structures Of a Temporal Digraph

Ali Rashid Ibrahim[a]

[a] Department of Applied Mathematics, College of Science, University of Anbar, Ramadi, Iraq

(Communicated by Madjid Eshaghi Gordji)

Abstract

A new approach is proposed for classifying the problems of determining the maximum common fragments (MCF) for two connected structures included in the T -digraph, based on the type of the maximum common fragment. A tree of classification the problems of determining the maximum common fragments (MCF) for two structures $t_iG, t_jG\,(MCF\,(t_iG, t_jG))$ included in the T -digraph is proposed. Examples are given for a digraph tG with three types of its fragments (parts), and for five connectivity types of digraphs. The formulation of six basic problems of determining the maximum common fragments (MCF) for two connected structures included in the T -digraph is given. A classification is proposed for an isomorphic embedding of a digraph into another.

Keywords: temporal digraph, maximum common fragment, maximum common subgraph, spanning subgraph, induced subgraph, classification of maximum common fragments, Isomorphic embedding.

1. Introduction

Currently, topical research in theoretical and applied graph theory is the study of dynamic digraphs, that is, digraphs whose structure changes with time [1] − [16]. In such digraphs, called temporal digraphs (T -digraphs), the analyzed set of vertices, or arcs, or both vertices and arcs, can change at different times. The structure of the T -digraph helps to study, predict and optimize the behavior of dynamical systems.

*Corresponding author
Email address: sc.alirashed@uoanbar.edu.iq (Ali Rashid Ibrahim)

Received: November 2020 *Revised:* March 2021

As one of the central theoretical problems in [1], the problem of determining the maximum common fragment (MCF) of the structures of a T -digraph, which does not change with time, is singled out. The MCF can be a subgraph of a digraph, or a spanning subgraph of a digraph, or an induced of a digraph [17]. Below we propose a classification of the types of maximum common fragments that can be the result of solving a problem, and which determine the variety of types of problems for determining the maximum common fragment of a T -digraph. The highlighted problem has not only theoretical, but also broad applied interest, for example, in a new area of management related to the management of social networks, its structures [10] . Currently, work on the structural analysis of corporate social networks is relevant, in particular on the analysis of networks of intracorporate communications of the company's employees [18]. The determination and analysis of stable, non-changing with time, communities of employees of the company on formal and informal connections helps the head of the company to create a single team of employees united by a common aim. Solving the problem of determining the maximum subsets (basic, most significant) of the company's employees who are in constant working interaction throughout the entire period of execution of one or more important projects (tasks) will allow the head of the company to make informed management decisions.

2. Basic definitions

$G = \left\langle V^{(t)}, E^{(t)}, T \right\rangle$ we call a temporal digraph (T -digraph), where $V^{(t)}$ is the set of vertices of the graph at time t and $\left|V^{(t)}\right| = p^{(t)}$ (number of vertices), $T = 1 \ldots N$ is the set of natural numbers defining (discrete) time, $E^{(t)}$ is the set of arcs (the family of correspondences or mappings) $\Gamma_t \in E^{(t)}$ of the set of vertices $V^{(t)}$ into itself at time $t \in T$[4]. We denote by tG the T - digraph at time t.

The vertex v_i is reachable from the vertex v_j in the digraph tG, if there is a path from v_j to v_i. A digraph tG is called strongly connected or strong if any two of its different vertices are mutually reachable [19][20][21]. A digraph tG is called unilaterally connected or unilateral, if for any two different vertices at least one is reachable from the other [22] . A connected digraph tG is called weakly connected, if it contains pairs of different vertices with unilateral connection [23]. A digraph tG is not connected, if it contains at least two different unreachable vertices.

A digraph tG is called strongly connected or strong, if for any two different vertices there is at least one path connecting these vertices. This also means that any two vertices of a strong connected graph are mutually reachable.
As follows from the above introduced definitions, each strong digraph tG is unilateral, and every unilateral digraph tG is weak, but the converse statements are not true.

When removing one or several vertices and incident arcs in tG, we obtain an induced subgraph of the digraph tG, an example of an induced subgraph of tG is shown in fig. $1-A$. When removing one or several arcs in tG, we obtain a spanning subgraph of the digraph tG, thus, the spanning subgraph has the same set of vertices as tG, but the set of its arcs is a subset of the set of arcs of tG, an example of a spanning subgraph of tG is shown in fig. $1-B$, and when removing vertices and incident arcs in tG, and then removing arcs from the resulting part, leads to the selection of a subgraph of the digraph tG, an example of a subgraph of tG is shown in fig. $1-C$. In fig. 1, shows an example of a digraph tG and three types of its fragments (parts).

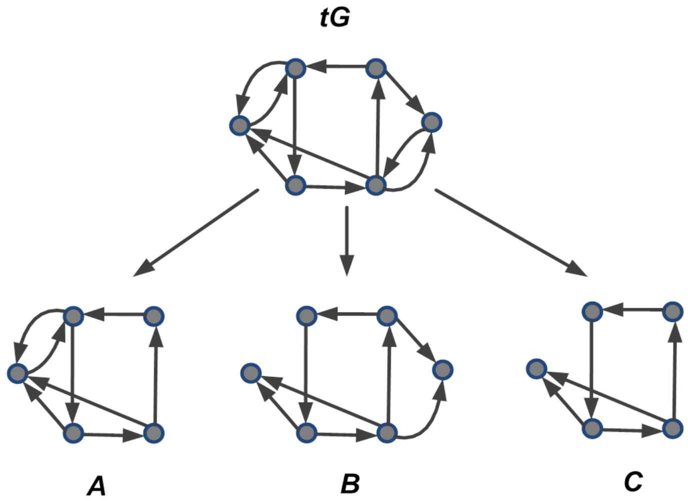

Figure 1: An example of a digraph tG and three types of its fragments (parts) ($A-$ is an induced subgraph of a tG -digraph, B - is a spanning subgraph of a tG -digraph, C - is a subgraph of a tG digraph)

In fig. 2 examples are shown to demonstrate the five connectivity types of digraphs.

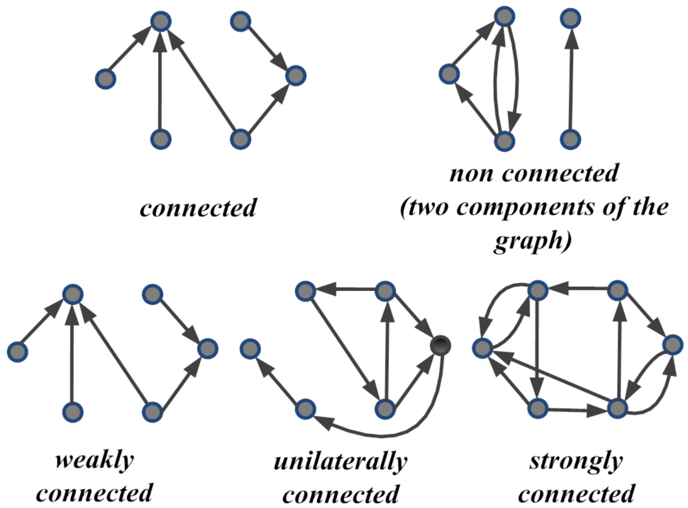

Figure 2: Examples of five connectivity types of digraphs

A digraph $t_1G = \left\langle V^{(t_1)}, E^{(t_1)}, T \right\rangle$ is isomorphic to the digraph $t_2G = \left\langle V^{(t_2)}, E^{(t_2)}, T \right\rangle$ (denoted by $t_1G \approx t_2G$), if there exists a mapping $\varphi : V^{(t_1)} \to V^{(t_2)}$, such that

$$\left(\forall v_i, v_j \in V^{(t_1)}\right) \left(\langle v_i, v_j \rangle \in E^{(t_1)} \leftrightarrow \langle \varphi\left(v_i\right), \varphi\left(v_j\right) \rangle \in E^{(t_2)}\right)$$

where $\varphi\left(v_i\right), \varphi\left(v_j\right) \in V^{(t_2)}$. The set of all isomorphisms of the digraph tG onto itself forms a group by multiplication of permutations φ and is denoted by $Aut(tG)$. The order of the group is denoted by $|Aut(tG)|$. The order of this group is called the symmetry number of the digraph. A digraph tG is called identical, if $|Aut(tG)| = 1$.

3. Isomorphic embedding of a digraph into another digraph

A digraph $t_1G = \left\langle V^{(t_1)}, E^{(t_1)}, T \right\rangle$ is isomorphically embedded into a digraph $t_2G = \left\langle V^{(t_2)}, E^{(t_2)}, T \right\rangle$ as a fragment, if t_2G contains a fragment f, such that $f \approx t_1G$ (denoted by $t_1G \subseteq^f t_2G\right)$. As we explained above that the fragment of any directed graph is classified into three types (subgraph, spanning, induced subgraph), and this classification is according to the process of removing the vertices and arcs. Therefore, we can classify the isomorphic embedding of a digraph $t_1G = \left\langle V^{(t_1)}, E^{(t_1)}, T \right\rangle$ into a digraph $t_2G = \left\langle V^{(t_2)}, E^{(t_2)}, T \right\rangle$ as follows:

1- If a fragment f is a subgraph in t_2G, then a digraph $t_1G = \left\langle V^{(t_1)}, E^{(t_1)}, T \right\rangle$ is isomorphically embedded into a digraph $t_2G = \left\langle V^{(t_2)}, E^{(t_2)}, T \right\rangle$ as a subgraph and denoted by $t_1G \subseteq^S t_2G$.
2- If f is a spanning subgraph in t_2G, then a digraph $t_1G = \left\langle V^{(t_1)}, E^{(t_1)}, T \right\rangle$ is isomorphically embedded into a digraph $t_2G = \left\langle V^{(t_2)}, E^{(t_2)}, T \right\rangle$ as a spanning subgraph, and denoted by $t_1G \subseteq^{SS} t_2G$.
3- If f is an induced subgraph in t_2G, then a digraph $t_1G = \left\langle V^{(t_1)}, E^{(t_1)}, T \right\rangle$ is isomorphically embedded into a digraph $t_2G = \left\langle V^{(t_2)}, E^{(t_2)}, T \right\rangle$ as induced subgraph, and denoted by $t_1G \subseteq^{IS} t_2G$.

4. The maximum common fragment for two structures of *T* -digraph

By the maximum common fragment (MCF) for two structures t_1G, t_2G of T -digraph (MCF (t_1G, t_2G)) , we mean a fragment $f1 = t_{1,2}G^* = \left\langle V^{(t_{1,2})}, E^{(t_{1,2})}, T \right\rangle$, such that $t_{1,2}G^* \subseteq^f t_1G$ and $t_{1,2}G^* \subseteq^f t_2G$.

Since each fragment is one of three types (subgraph, spanning subgraph, induced subgraph), it is possible to determine the maximum common fragment (MCF) for two structures t_1G, t_2G of T -digraph according to its type, as follows:

1. 1- If a fragment $f1$ is a subgraph, then the maximum common subgraph (MCS) for two structures t_1G, t_2G of T -digraph $(MCS\,(t_1G, t_2G))$ is denoted by $t_{1,2}G^*$ and $t_{1,2}G^* = \left\langle V^{(t_{1,2})}, E^{(t_{1,2})}, T \right\rangle$, such that $t_{1,2}G^* \subseteq^S t_1G$ and $t_{1,2}G^* \subseteq^S t_2G$.
2- If a fragment $f1$ is a spanning subgraph, then the maximum common spanning subgraph $(MCSS)$ for two structures t_1G, t_2G of T -digraph $(MCSS\,(t_1G, t_2G))$ is denoted by $t_{1,2}G^*$ and $t_{1,2}G^* = \left\langle V^{(t_{1,2})}, E^{(t_{1,2})}, T \right\rangle$, such that $t_{1,2}G^* \subseteq^{SS} t_1G$ and $t_{1,2}G^* \subseteq^{SS} t_2G$.
3- If a fragment $f1$ is an induced subgraph, then the maximum common induced subgraph $(MCIS)$ for two structures t_1G, t_2G of T-digraph $(MCIS\,(t_1G, t_2G))$ is denoted by $t_{1,2}G^*$ and $t_{1,2}G^* = \left\langle V^{(t_{1,2})}, E^{(t_{1,2})}, T \right\rangle$, such that $t_{1,2}G^* \subseteq^{IS} t_1G$ and $t_{1,2}G^* \subseteq^{IS} t_2G$.

5. Classification of problems of determining the maximum common fragments (parts) for two structures of one T-digraph

In fig. 3 , a classification tree is proposed for the problems of determining the maximum common fragments (parts) for two structures $t_iG, t_jG\,(MCF\,(t_iG, t_jG))$ included in the T -digraph.
The classification is based on four parameters:

A) The type of the maximum common part (subgraph, spanning subgraph, induced subgraph);
B) The type of connection of the maximum common part (connected, disconnected, weakly connected, strongly connected, unilaterally connected);
C) The number of determined maximum common parts (one; all);
D) The number of connected components of the maximum common part $(1, 2, \ldots, n)$.

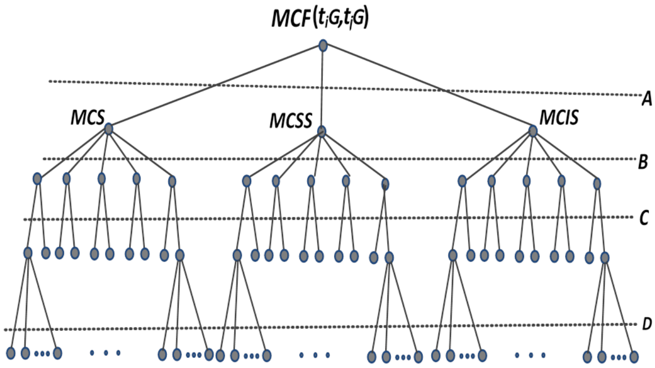

Figure 3: A tree of classification the problems of determining the maximum common fragments (parts) for two structures $t_iG, t_jG\,(MCF\,(t_iG, t_jG))$ included in the T -digraph.

Classification sections ($A-$ the type of $MCF, B-$ the type of connection of $MCF, C-$ The number of determined $MCF, D-$ The number of connected components of MCF.)

Now, on the basis of the type of MCF, we can give the formulation of the six basic problems of determining the maximum common fragments of two connected structures of the T -digraph.

Problem 1.1. Given a temporal digraph $G = \left\langle V^{(t)}, E^{(t)}, T\right\rangle, T = \{t_1, \ldots, t_n\}$, consisting of non-isomorphic structures $(t_1G, t_2G, \ldots, t_nG)$. It is necessary to determine only one maximum common connected subgraph of the T -digraph, that is, find a connected $MCS\,(t_1G, t_2G, \ldots, t_nG)$, such that $MCS = t_{1,2,\ldots,n}G^* = \left\langle V^{(t_{1,2,\cdots,n}}\right), E^{(t_{1,2,\ldots,n})}, T\rangle$, and $t_{1,2,\ldots,n}G^* \subseteq^S t_1G, t_{1,2,\ldots,n}G^* \subseteq^S t_2G$ $\ldots, t_{1,2,\ldots,n}G^* \subseteq^S t_nG$.

Problem 1.2. Given a temporal digraph $G = \left\langle V^{(t)}, E^{(t)}, T\right\rangle, T = \{t_1, \ldots, t_n\}$, consisting of non-isomorphic structures $(t_1G, t_2G, \ldots, t_nG)$. It is necessary to determine all non-isomorphic maximum common connected subgraphs of the T -digraph, that is, find all connected non- isomorphic $MCS\,(t_1G, t_2G, \ldots, t_nG)$.

Problem 2.1. Given a temporal digraph $G = \left\langle V^{(t)}, E^{(t)}, T\right\rangle, T = \{t_1, \ldots, t_n\}$, consisting of non-isomorphic structures $(t_1G, t_2G, \ldots, t_nG)$. It is necessary to determine only one maximum common connected spanning subgraph of the T -digraph, that is, find a connected $MCSS\,(t_1G, t_2G, \ldots, t_nG)$, such that $MCSS = t_{1,2,\ldots,n}G^* = \left\langle V^{(t_{1,2,\ldots,n})}, E^{(t_{1,2,\ldots,n})}, T\right\rangle$, and $t_{1,2,\ldots,n}G^* \subseteq^{SS} t_1G, t_{1,2,\ldots,n}G^* \subseteq^{SS}$ $t_2G, \ldots, t_{1,2,\ldots,n}G^* \subseteq^{SS} t_nG$.
Problem 2.2. Given a temporal digraph $G = \left\langle V^{(t)}, E^{(t)}, T\right\rangle, T = \{t_1, \ldots, t_n\}$, consisting of non-isomorphic structures $(t_1G, t_2G, \ldots, t_nG)$. It is necessary to determine all non-isomorphic maximum common connected spanning subgraph of the T -digraph, that is, find all connected non-isomorphic $MCSS\,(t_1G, t_2G, \ldots, t_nG)$.

Problem 3.1. Given a temporal digraph $G = \left\langle V^{(t)}, E^{(t)}, T\right\rangle, T = \{t_1, \ldots, t_n\}$, consisting of non-isomorphic structures $(t_1G, t_2G, \ldots, t_nG)$. It is necessary to determine only one maximum common connected induced subgraph of the T -digraph, that is, find a connected $MCSS\left(t_1G, t_2G, \ldots, t_nG\right)$, such that $MCIS = t_{1,2,\ldots,n}G^* = \left\langle V^{(t_{1,2,\ldots n})}, E^{(t_{1,2,\ldots,n})}, T\right\rangle$, and $t_{1,2,\ldots,n}G^* \subseteq^{IS} t_1G, t_{1,2,\ldots,n}G^* \subseteq^{IS} t_2G, \ldots, t_{1,2,\ldots,n}G^* \subseteq^{IS} t_nG$.

Problem 3.2. Given a temporal digraph $G = \left\langle V^{(t)}, E^{(t)}, T\right\rangle, T = \{t_1, \ldots, t_n\}$, consisting of non-isomorphic structures $\left(t_1G, t_2G, \ldots, t_nG\right)$. It is necessary to determine all non-isomorphic maximum common connected induced subgraphs of the T -digraph, that is, find all connected non-isomorphic $MCIS\left(t_1G, t_2G, \ldots, t_nG\right)$.
In fig. 4, shows an example of a T -digraph G, consisting of all non-isomorphic connected digraphs tG with the number of vertices $p = 3$.

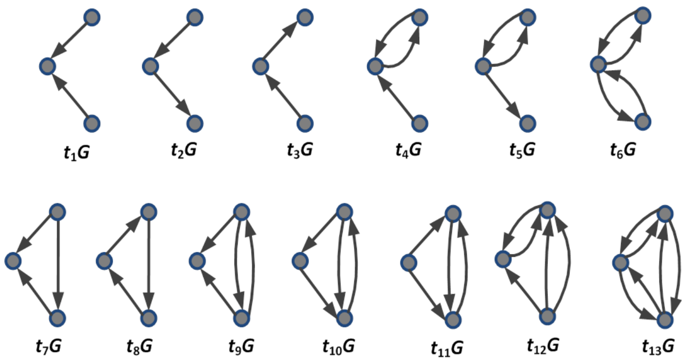

Figure 4: An example of a T -digraph G, consisting of all non-isomorphic connected digraphs tG with the number of vertices $p = 3$.

6. Conclusions and Future Work

The paper considers the actual problem of analyzing temporal digraphs associated with determining its maximum common fragment that does not change with time. Four parameters are distinguished, on the basis of which a classification of problems of determining the maximum common fragments for two structures t_iG, t_jG included in the T -digraph is proposed. A classification tree is given, which includes $N = 30 \times nCC$, where nCC is the number of connectivity components in the required maximum common fragment of two analyzed tG. The existence of all three types of connectivity of the maximum common fragment (MCF) of the result is substantiated when analyzing any combination of pairs of connected t_iG, t_jG out of three types of their connectivity. Six basic classes of problems of determining the maximum common fragments of structures included in the T -digraph, which do not change with time and characterize its resistance to external influences, have been identified. The formulation of basic problems is presented. Also, based on the type of the maximum common fragment, a classification of an isomorphic embedding of a digraph into another is given, so any digraph can be isomorphically embedded into another digraph as a subgraph, an induced subgraph, and as a spanning subgraph.
Further research will be devoted to the actual analysis of the T -digraph and the solution of two problems associated with the determination of the maximum common connected induced subgraph $(MCIS)$ of two structures of the T -digraph.

References

[1] M. A. Aizerman, L. A. Gusev, S. V. Petrov and I. M. & Smirnova, *A dynamic approach t analysis of structures represented as graphs (fundamentals of graph dynamics)*, I. Avtomatika i Telemekhanika, 7 (1977) 135-151.
[2] F. Kuhn and R. & Oshman, *Dynamic networks: models and algorithms*, ACM SIGACT News, 42(1)(2011) 82-96.
[3] V. A. Kokhov, *Two approaches to determining similarity of two digraphs*, Journal of Computer and Systems Sciences International, 51(5)(2012) 695-714.
[4] A. R. Ibrahim, *A method for analyzing the problem of determining the maximum common fragments of temporal directed tree, that do not change with time*, International Journal of Nonlinear Analysis and Applications, 12(1)(2021) 111-118.
[5] P. Holme and J. & Saramki, *Temporal networks*, Physics reports, 519(3)(2012) 97-125.
[6] A. Casteigts, P. Flocchini, W. Quattrociocchi and N. & Santoro, *Time-varying graphs and dynamic networks*, International Journal of Parallel, Emergent and Distributed Systems, 27(5)(2012) 387-408.
[7] P. J. Bramsen, *Doing time: inducing temporal graphs Doctoral dissertation*, Massachusetts Institute of Technology, (2006).
[8] D. Braha and Y. & Bar-Yam, *Time-dependent complex networks: Dynamic centrality, dynamic motifs, and cycles of social interactions*, In Adaptive Networks, Springer, Berlin, Heidelberg, (2009) 39-50.
[9] N. Santoro, W. Quattrociocchi, P. Flocchini, A. Casteigts and F. & Amblard, *Time-varying graphs and social network analysis: Temporal indicators and metrics*, arXiv preprint arXiv:1102.0629 (2011).
[10] A. M. J. Hass, *Configuration management principles and practice*, Addison-Wesley Professional, (2003).
[11] V. Kostakos, *Temporal graphs*, Physica A: Statistical Mechanics and its Applications, 388(6)(2009) 1007-1023.
[12] J. Tang, M. Musolesi, C. Mascolo and V. & Latora, *Temporal distance metrics for social network analysis*, In Proceedings of the 2nd ACM workshop on Online social networks, (2009) 31-36.
[13] P. Basu, A. Barnoy, M. P. Johnson and R. & Ramanathan, *Modeling and analysis of timevarying graphs*, arXiv Prepr arXiv10120260, (2010).
[14] A. Paranjape, A. R. Benson and J. & Leskovec, *Proceedings of the Tenth ACM International Conference on Web Search and Data Mining*, WSDM17 (2017).
[15] D. Kempe, J. Kleinberg and A. & Kumar, *Connectivity and inference problems for temporal networks*, Journal of Computer and System Sciences, 64(4)(2002) 820-842.
[16] V. Campos, R. Lopes, A. Marino and A. & Silva, *Edge-Disjoint Branchings in Temporal Graphs*, In International Workshop on Combinatorial Algorithms, Springer, Cham, (2020) 112-125.
[17] V. Lozin, *Graph theory notes*, (2015).
[18] T. V. Shinkarenko, R. G. Smirnov and A. V. & Beloshitskiy, *Studying the organizational structure effectiveness on the basis of internal communication analysis*, Upravlenets, 11(2)(2020) 27-40.
[19] K. Cai and H. & Ishii, *Average consensus on general strongly connected digraphs*, Automatica, 48(11)(2012) 2750-2761.
[20] A. Frank, *How to make a digraph strongly connected*, Combinatorica, 1(2)(1981) 145-153.
[21] F. Harary and R. Z. & Norman, *Some properties of line digraphs*, Rendiconti del Circolo Matematico di Palermo, 9(2)(1960) 161-168.
[22] R. A. Brualdi, F. Harary and Z. & Miller, *Bigraphs versus digraphs via matrices*, Journal of Graph Theory, 4(1)(1980) 51-73.
[23] J. J. P. Veerman and E. & Kummel, *Diffusion and consensus on weakly connected directed graphs*, Linear Algebra and its Applications, 578 (2019) 184-206.