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Infinitesimal generators of Lie symmetry group of parametric ordinary differential equations

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Abstract

Lie's theory of symmetry groups plays an important role in analyzing and solving differential equations; for instance, by decreasing the order of equation. Moreover, there are some analytic methods to find the infinitesimal generators that span the Lie algebra of symmetries. In this paper, we first converted the problem of finding infinitesimal generators in to the problem of solving a system of polynomial equations in the context of computational algebraic geometry. Then, we used Gröbner basis a novel computational tool to solve this problem. As far as we know, when a differential equation contains some parameters, there is no linear algebraic algorithm up to our knowledge to deal with these parameters; so, we must apply the algorithms, which are based on Gröbner basis.

Keywords: Point symmetry of ODEs, Infinitesimal generators, Gröbner basis.
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1. Introduction

To gain a better understanding of the formulation of the basic laws of nature and many technological problems, it is useful to consider them as differential equations. This leads to finding solutions such equations are useful. Norwegian mathematician, Sophos Lie, spent most of his life on the Lie groups theory to find solutions to differential equations through the systematic use of symmetries [13]. Today, Lie group analysis is an essential tool in many sources such as analysis, geometry, number theory, differential equations, physics, and atomic structures, and so on. Symmetric methods are important, especially when it comes to finding solutions to nonlinear differential equations because most standard methods are inadequate for these cases. Texts are suggested for better and more

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complete learning of symmetric methods includes [2], [10], [15], [16] and [18]. When working with Li groups, the most difficult step in symmetric methods is to find the symmetries (infinitesimal generators) of the differential equations. In this paper, we presented the infinitesimal generators of Lie symmetry group of ordinary differential equations of the form

$$y^{(n)} = w(x, y, y^{'}, ..., y^{(n-1)}), \qquad n \geq 2. \tag{1.1}$$

Infinitesimal generators $X = \xi(x, y)\partial x + \eta(x, y)\partial y$ of differential equations, are determined by solving the linearized symmetry condition. Infinitesimal generators are useful to determine the general solution of ODE or enable us to reduce the order of the ODE. So far, no general scheme has been developed for solving partial differential equations whose solution gives the infinitesimals of the symmetry group. Peter Haydon stated that finding such solutions using some ansatz is easier than trying to directly solve determining equations directly [10]. ODE tools package in Maple to find the solutions of determining equations with an ansatz, collected by E. S. Cheb-Terrab and coworkers [4]. This package is based on six algorithms to finds Lie point symmetries. In this paper, using an algorithm, we found a specific type of symmetry that had polynomial ansatz. In fact, the bridge between Lie's theory and computational algebraic geometry is that we substitute ξ and η by their Taylor series at the origin up to the degree equal to the order of equation, with unknown coefficients. By substituting this taylor series into a given determining equations, we obtained a system of polynomial equations. The Gröbner basis is one of the strongest tools for solving system of equations in computer algebra. These bases were introduced by Bruno Buchberger [3] in 1965. He also produced the fundamental algorithms to compute them in his Ph.D. thesis. There are many applications of Gröbner bases such as graph coloring problems [8], robotics [7], coding theory [17], solving Diophantine equations (Pell) [5], solving fuzzy systems [1], and so on. In fact, we found infinitesimal generators by dividing ξ and η on Gröbner basis.
We also illustrated this method on the partial differential equations as well as the parametric differential equations. Indeed, we intended to find a solution for a parametric polynomial system to describe all the different behaviors of such system. The main difficulty with this process was to analyze the parametric system to obtain infinitesimal generators. Kapur [11, 12] computed a parametric Gröbner basis G from prametric ideal basis in K[U, X]. Furthermore, we found infinitestimal generators for each of the cases through applying the parametric Gröbner- based algorithms.

This paper has been structured as follows: Subsection 2.1 recalls some basic facts about point symmetries of ODEs. Subsection 2.2 presents basic concepts of Gröbner basis and comprehensive Gröbner systems. Finally, section 3 offers a new approch for computing infinitestimal generators for ODE, PDE, and parametric differential equations.

2. Preliminaries

2.1. Basic facts on the point symmetries of ODEs

We begin by recalling some basic facts about point symmetries of ODEs, more details have been given in [10]. A point symmetry of the ODE (1.1) is a smooth invertible transformation of the (x, y) plane that maps the set of solutions of (1.1) in to itself. To facilitate this expression, we consider a one-parameter Lie group of transformatin in the form

$$\begin{aligned} \widehat{x} &= x + \epsilon\xi(x, y) + O(\epsilon^2), \\ \widehat{y} &= y + \epsilon\eta(x, y) + O(\epsilon^2), \\ \widehat{y}^{(k)} &= y^{(k)} + \epsilon\eta^{(k)} + O(\epsilon^2), k \geq 1. \end{aligned} \tag{2.1}$$

for each ε in some neighbourhood of zero. Specifically, a point symmetry is a diffeomorphism

$$\Gamma : (x, y) \mapsto (\widehat{x}(x, y), \widehat{y}(x, y)). \tag{2.2}$$

This map induces the n-th prolongation map

$$\Gamma^* : (x, y, \ldots, y^{(n)}) \to (\widehat{x}, \widehat{y}, \ldots, \widehat{y}^{(n)}),$$

where $\widehat{y}(0) = \widehat{y}$ and for each $k = 1, \ldots, n$

$$\widehat{y}^{(k)} = \frac{D_x \widehat{y}^{(k-1)}}{D_x \widehat{x}}.$$

Here, D_x is the total derivative whit respect to x,

$$D_x = \partial_x + y^{'} \partial_y + y^{''} \partial_{y'} + \cdots . \tag{2.3}$$

The symmetry condition for the ODE (1.1) is

$$\widehat{y}^{(n)} = \omega(\widehat{x}, \widehat{y}, \widehat{y}^{'}, \ldots, \widehat{y}^{(n-1)}), \tag{2.4}$$

provided that (1.1) holds. We are going now to state the general form of infinitesimal generators of Lie symmetry group, where the infinitesimal generators $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ are determined by solving the linearized condition (2.4). We substitute (2.1) into the symmetry condition (2.4) and expand the result in powers of ϵ; by equating the the $\mathrm{O}(\epsilon)$ terms yield the linearized symmetry condition:

$$\eta^{(n)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + ... + \eta^{(n-1)}\omega_{y^{(n-1)}}, \tag{2.5}$$

provided that (1.1) holds. This makes it easy to split the linearized symmetry condition using all terms that are multiplied by the highest power of $y^{(n-1)}$, and so on, which are determining equations for the Lie point symmetries. Lie point symmetries of PDEs are also calculated by the same procedure as for ODEs. However, PDEs involve several independent variables.

2.2. Basic concepts of Gröbner basis and comprehensive Gröbner systems

In this section, we present briefly the basic concepts of Gröbner basis and comprehensive Gröbner systems. For a more detailed discussion, we refer the reader to [3, 7, 19]. Using the method of Gröbner bases, we can solve systems of polynomial equations in a very nice fashion.

Let the ring of all polynomials in $x_1, x_2, \cdots, x_n$, with coefficients in field K is denoted by $R = K[x_1, x_2, \cdots, x_n]$. An expression of the form $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in R$ with non-negative exponents is called a term. In order to define Gröbner basis, we need to define the term order.

One the most important of a term order will be *lexicographical order* (or *Lex order* for short). We now introduce that as follows:

Definition 2.1. *[7] Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geqslant 0}$, We say $\alpha >_{lex} \beta$ if, in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the leftmost nonzero entry is positive. We will write $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} >_{lex} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ if $\alpha >_{lex} \beta$.*

Suppose $>$ be an arbitrary term order on R. For any non-zero polynomial f, the maximum term appearing in f with respect to $>$ is denoted by $LT(f)$, and is called the *leading term* of f. The coefficient of $LT(f)$ is the *leading coefficient* of f is denoted by $LC(f)$.

Now, we can define a Gröbner basis for an ideal in R as follows:

Definition 2.2. *[7] Fix a monomial order $>$. A* Gröbner basis *of an ideal I in R with respect to $>$ is a finite set of polynomials $G = \{g_1, \dots, g_m\} \subset I$ with the property that for every nonzero $f \in I$, $LT(f)$ is divisible by $LT(g_i)$ for some i. A Gröbner basis G is called a* reduced Gröbner basis *for I if for any $g_i \in G$, $LC(g_i) = 1$ and no term of g_i lies in the ideal generated by $\{LT(g_j) | 1 \leq j \neq i \leq m\}$.*

We introduce the following theorem which are used in next section.

Theorem 2.3. *Let I be an ideal in $K[\boldsymbol{x}]$, $G = \{g_1, \dots, g_t\}$ be a Gröbner basis for I and $f \in K[\boldsymbol{x}]$. Then there is a unique $r \in K[\boldsymbol{x}]$ with the following two properties:*

1. *No monomial of r is divisible by any of $lm(g_1), \dots, lm(g_t)$.*
2. *There is $g \in I$ such that $f = g + r$.*

The remainder on division of f by G, r, is sometimes called the normal form of f.

Proof . See [7] □

Now, we describe the concept Gröbner bases for a polynomial ideal with parametric coefficients, in this case, these are called comprehensive Gröbner bases. In general, comprehensive Gröbner bases and comprehensive Gröbner systems are called "parametric Gröbner bases". Let K be a field, R be the polynomial ring $K[U]$ in the parameters $U = u_1, ..., u_m$, and $R[X]$ be the polynomial ring over the parameter ring R in the variables $X = x_1, ..., x_n$ and $X \cap U = \emptyset$, i.e., X and U disjoint sets. $K[U][X]$ denotes parametric polynomial ring over K, where consisted of the set of all parametric polynomial as

$$\sum_{i=1}^{t} p_i X^{\alpha_i}$$

where $p_i \in K[U]$ is a polynomial on U with a coefficients in K, for each i. Assume that $\overline{K}$ is an algebraically closed filed containing K. A specialization σ_a corresponding to a where $a \in \overline{K}^m$ of $K[U][X]$ is a ring-homomorphism

$$\begin{aligned} \sigma_a : K[U][X] &\to \overline{K}[X] \\ f &\to f(a) \end{aligned}$$

where $f \in K[U][X]$. The comprehensive Gröbner systems has been defined by Weispfenning [19] as follows; if we take the parameter space $\mathbf{P}$ and its set of parametric polynomials G from a comprehensive Gröbner systems for a parametric polynomial ideal I, then $\sigma(G)$ constitutes a Gröbner basis of the ideal generated by $\sigma(I)$ under the specialization σ with respect to the parameter space $\mathbf{P}$ of the parameters. Comprehensive Gröbner systems are also important parts for solving problem of parametric polynomials.

Definition 2.4. *[12] Let $I \subset K[U][X]$ be a parametric ideal and $\prec$ be a monomial ordering on X. Then the set*

$$\zeta(I) = \{(E_i, N_i, G_i) | i = 1, ..., l\} \subset K[U] \times K[U] \times K[U][X]$$

is said a comprehensive Gröbner system for I if for each $(\lambda_1, ..., \lambda_t) \in \overline{K}^m$ and each specialization

$$\begin{aligned} \sigma_{(\lambda_1,...,\lambda_m)} : K[U][X] &\to \overline{K}[X] \\ \sum_{i=1}^{t} p_i X^{\alpha_i} &\mapsto \sum_{i=1}^{t} p_i(\lambda_1, ..., \lambda_m) X^{\alpha_i} \end{aligned}$$

there exists an $1 \leqslant i \leqslant l$ such that $(\lambda_1, ..., \lambda_m) \in V(E_i) \setminus V(N_i)$ and $\sigma_{(\lambda_1,...,\lambda_m)(G_i)}$ is a Gröbner basis for $\sigma_{(\lambda_1,...,\lambda_m)(I)}$ with respect to $\prec$. Because of simplicity, we call E_i and N_i the null and non-null conditions respectively.

We refer the reader to [12] for more details. For instance, let $F = \{f_1 = (a-1)x + y^2, f_2 = ay + a\} \subset Q[a][x, y]$, x, y variables, a parameter. Choosing the lexicographic ordering $x \succ y$, we have the following Gröbner systems:

G_i	N_i	E_i
$\{-x+y^2\}$	$\{\}$	$\{a\}$
$\{1\}$	$\{a\}$	$\{a-1\}$
$\{(a-1)x+y^2, ay+a\}$	$\{a, a-1\}$	$\{\}$

For instanse, $\sigma_{a=1}(G_i) = \{1\}$ is also a Gröbner basis of $\sigma_{a=1}(f_1, f_2) = \{y^2, y+1\}$.

3. Computing infinitesimal generators

When the Lie groups studied infinitesimal generators seen. In this section, we look for infinitesimal generator of a given ordinary differential equation of order two or greater. Roughly speaking, X is infinitesimal generator of form $X = \xi(x, y)\partial x + \eta(x, y)\partial y$ for point symmetries from ODEs. Here, an algorithm is presented to find all the infinitesimal generator that has polynomial ansatz. The main bridge between Lie theory and computational algebraic geometry in the topic of this paper is to substitute the functions $\xi(x, y)$ and $\eta(x, y)$ by their Taylor series at the origin up to the degree equal to the order of equation, with unknown coefficients. The main algorithm used to help us achieve our aim is presented below:

Algorithm 1 (Main Algorithm)

Input: ODEs of order $n \geq 2$.
Output: Computing infinitesimal generators.

- 1. Apply linearized symmetry condition on given ODE.
- 2. Extract coefficients of all terms that are multiplied by the highest power of $y^{(n-1)}$, which are the system of PDEs.
- 3. Substitute the functions $\xi(x, y)$ and $\eta(x, y)$ by their Taylor series at the origin up to the degree equal to the order of equation, with unknown coefficients.
- 4. Substitute $\xi(x, y)$ and $\eta(x, y)$ into determining equations gained from step 2.
- 5. Compute Gröbner basis for the ideal generated by these systems of linear equations from previous step with respect to lex order.
- 6. Find all the infinitesimal generator by dividing $\xi(x, y)$ and $\eta(x, y)$ on Gröbner basis.

Below, we illustrate the algorithm with an example:

Example 3.1. *[10] Consider the simplest second-order ODE,*

$$y'' = 0. \tag{3.1}$$

The linearized symmetry condition for this ODE is

$$\eta^{(2)} = 0,$$

proided that (3.1) holds. From (3.21) [10],

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 = 0.$$

By reding off all terms that are multiplied by a particular power of y', the linearized symmetry condition splits into the following system of determining equations:

$$\eta_{xx} = 0, \quad 2\eta_{xy} - \xi_{xx} = 0, \quad \eta_{yy} - 2\xi_{xy} = 0, \quad \xi_{yy} = 0. \tag{3.2}$$

Now, let's consider the functions $\xi(x, y)$ and $\eta(x, y)$ by their Taylor series at the origin up to a degree equal to the order of differential equation that is two. Then, general solution of the (3.2) *as*

$$\eta(x, y) := c_0 + c_1x + c_2y + c_3x^2 + c_4y^2 + c_5xy;$$
$$\xi(x, y) := d_0 + d_1x + d_2y + d_3x^2 + d_4y^2 + d_5xy;$$

for arbitrary constants c_i and d_j, i,j=0,...,5. After substitution these ξ and η into (3.2) *and doing some simplifications, we receive to the following system of equations:*

$$\begin{cases} f_1 = 2d_3 = 0 \\ f_2 = 2d_5 - 2c_3 = 0 \\ f_3 = 2d_4 - 2c_5 = 0 \\ f_4 = 2c_4 = 0. \end{cases} \tag{3.3}$$

Let I be the ideal generated by f_1, f_2, f_3, f_4. By computing a Gröbner basis for I with respect to lex order we receive to

$$G := \{c_4, d_5 - c_3, d_4 - c_5, d_3\}.$$

To find all the infinitesimal generator, we divide ξ and η on Gröbner basis.

```
a := NormalForm(xi(x,y), G, T),
b := NormalForm(eta(x,y), G, T),
X := a*Rx+b*Ry.
```

For this example,

$$X := (c_0 + c_1x + c_2y + d_5x^2 + d_4xy)\partial_x + (d_0 + d_1x + d_2y + d_4y^2 + d_5xy)\partial_y. \tag{3.4}$$

If in (3.4)*, we put one of the coefficients c_i or d_j, i,j=0,...,5 one respectively and the rest coefficients zero. We obtained infinitesimal generator*

$$\partial_x, x\partial_y, y\partial_y, \partial_x, x\partial_x, y\partial_x, xy\partial_y + x^2\partial_x, y^2\partial_y + xy\partial_x.$$

Theorem 3.2. *The main algorithm finds the infinitesimal generators from the given differential equation.*

Proof . Suppose that I is the ideal generated by these determining equations replaced by Taylor series instead of ξ and η. Variety of the mentioned ideal equal variety of Gröbner basis. The coefficients c_i or d_j which are zero in ξ and η. Now the remainder of the division of ξ and η on Gröbner basis is the coefficients c_i or d_j which are not zero. Infact, which are the same as infinitesimal generators. □

3.1. ODE examples

In such subsection, three ODE examples are given to illustrate the performance of computinging infinitesimal generators by the proposed method. PDE and parametric examples are given in further subsections.

Example 3.3. *[10] In studing the ODE*

$$y^{''}=\frac{y^{'2}}{y}-y^2, \tag{3.5}$$

the linearized symmetry condition is

$$\eta^{(2)}=\eta(-\frac{y^{'2}}{y^2}-2y)+\eta^{(1)}(\frac{2y^{'}}{y}), \quad \textit{when (3.5) holds.}$$

That is from (3.20) and (3.21) [10],

$$\eta_{xx}+(2\eta_{xy}-\xi_{xx})y^{'}+(\eta_{yy}-2\xi_{xy})y^{'2}-\xi_{yy}y^{'3}+(\eta_y-2\xi_x-3\xi_y y^{'})(\frac{y^{'2}}{y}-y^2)=$$
$$\eta(-\frac{y^{'2}}{y^2}-2y)+(\eta_x+(\eta_y-\xi_x)y^{'}-\xi_y y^{'2})(\frac{2y^{'}}{y}).$$

By comparing powers of $y^{'}$, we obtain the determining equations of the form:

$$\begin{aligned}\xi_{yy}+\frac{1}{y}\xi_y&=0,\\ \eta_{yy}-2\xi_{xy}-\frac{1}{y}\eta_y+\frac{1}{y^2}\eta&=0,\\ 2\eta_{xy}-\xi_{xx}+3y^2\xi_y-\frac{2}{y}\eta_x&=0,\\ \eta_{xx}-y^2(\eta_y-2\xi_x)+2y\eta&=0.\end{aligned} \tag{3.6}$$

We consider the general solution of the (3.6) *is*

$$\xi(x,y):=c_0+c_1x+c_2y+c_3x^2+c_4y^2+c_5xy;$$
$$\eta(x,y):=d_0+d_1x+d_2y+d_3x^2+d_4y^2+d_5xy;$$

for arbitrary constants c_i and d_j, i,j=0,...,5. We substitute this ξ and η into (3.6). *Let I be the ideal generated by this substitution and simplification*

$$\begin{cases} f_1=4c_4y+c_2+c_5x\\ f_2=d_4y^2-2c_5y^2+d_0+d_1x+d_3x^2\\ f_3=-2c_3y+3c_2y^3+6c_4y^4+3c_5y^3x-2d_1-4d_3x\\ f_4=2d_3+d_2y^2+d_5y^2x+2c_1y^2+4c_3y^2x+2c_5y^3+2d_0y+2d_1yx+2d_3yx^2.\end{cases} \tag{3.7}$$

Then by computing Gröbner basis w.r.t. $c_i \prec_{lex} d_j$ for I, we have the following:

$$G:=\{c_5,c_4,c_3,c_2,d_5,d_4,d_3,d_2+2c_1,d_1,d_0\}.$$

Hence by dividing ξ and η on Gröbner basis, the general solution of the linearized symmetry condition is

$$\begin{aligned} \xi(x,y) &:= c_0 + c_1 x, \\ \eta(x,y) &:= -2c_1 y, \end{aligned}$$

where (as usual) c_0, c_1 are constants. We obtained infinitesimal generator

$$\partial_x, x\partial_x - 2y\partial_y.$$

Example 3.4. *[10] Consider the following ODE:*

$$y''' = y^{-3}. \tag{3.8}$$

The linearized symmetry condition for this ODE is

$$\eta^{(3)} = -3y^{-4}\eta$$

proided that (3.8) *holds. That is from (3.22) [10],*

$$\begin{aligned} &\eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy}y'^4 \\ +3(\eta_{xy} - \xi_{xx} &+ (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2)y'' - 3\xi_y y''^2 + (\eta_y - 3\xi_x - 4\xi_y y')y^{-3} = -3y^{-4}\eta. \end{aligned} \tag{3.9}$$

By comparing powers of $y'', y'y'', y'$, we obtain the determining equations:

$$\begin{aligned} y^4\eta_{xxx} + 3\eta + \eta_y y - 3\xi_x y &= 0, \\ y^4(3\eta_{xxy} - \xi_{xxx}) - 4\xi_y y &= 0, \\ y^4(\eta_{xyy} - \xi_{xxy}) &= 0, \\ y^4(\eta_{yyy} - 3\xi_{xyy}) &= 0, \\ y^4\xi_{yyy} &= 0, \\ y^4(\eta_{xy} - 3\xi_{xx}) &= 0, \\ y^4(\eta_{yy} - 3\xi_{xy}) &= 0, \\ y^4\xi_{yy} &= 0, \\ y^4\xi_y &= 0. \end{aligned} \tag{3.10}$$

Our assumption for the solution of the (3.10) *is*

$$\begin{aligned} \xi(x,y) &:= c_0 + c_1x + c_2y + c_3x^2 + c_4y^2 + c_5xy + c_6x^3 + c_7y^3 + c_8x^2y + c_9xy^2; \\ \eta(x,y) &:= d_0 + d_1x + d_2y + d_3x^2 + d_4y^2 + d_5xy + d_6x^3 + d_7y^3 + d_8x^2y + d_9xy^2; \end{aligned}$$

for arbitrary constants c_i and d_j, i,j=0,...,9. [In this example, if we put $\eta(x,y) := c_0 + c_1x + c_2y$ and $\xi(x,y) := d_0 + d_1x + d_2y$, we get the same results]. We substitute this ξ and η into (3.10)*. Let I be* *the ideal generated by the product this placement and simplification*

$$\begin{cases} f_1 = 6d_6y^4 + 3d_0 + 3d_1x + 4d_2y + 3d_3x^2 + 5d_4y^2 + 4d_5xy + 3d_6x^3 + \\ 6d_7y^3 + 4d_8x^2y + 5d_9xy^2 - 3c_1y - 6c_3yx - 3c_5y^2 - 9c_6yx^2 - 6c_8xy^2 - 3c_9y^3 \\ f_2 = 6d_8y^4 - 6c_6y^4 - 4c_2y - 8c_4y^2 - 4c_5xy - 12c_7y^3 - 4c_8x^2y - 8c_9xy^2 \\ f_3 = 2(d_9 - c_8)y^4 \\ f_4 = 6(d_7 - c_9)y^4 \\ f_5 = 6c_7y^4 \\ f_6 = (d_5 + 2d_8x + 2d_9y - 2c_3 - 6c_6x - 2c_8y)y^4 \\ f_7 = (2d_4 + 6d_7y + 2d_9x - 3c_5 - 6c_8x - 6c_9y)y^4 \\ f_8 = 2(c_4 + 3c_7y + c_9x)y^4 \\ f_9 = (c_2 + 2c_4y + c_5x + 3c_7y^2 + c_8x^2 + 2c_9xy)y^4. \end{cases} \tag{3.11}$$

Then by computing Gröbner basis w.r.t. $c_i \prec_{lex} d_j$ *for* I*, we have the following:*

$$G := \{c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, d_9, d_8, d_7, d_6, d_5, d_4, d_3, 4d_2 - 3c_1, d_1, d_0\}.$$

From by dividing the ξ *and* η *on Gröbner basis, the tangent vector field is*

$$\xi(x, y) := c_0 + c_1x,$$
$$\eta(x, y) := \frac{3}{4}c_1y,$$

where (as usual) c_0, c_1 *are constants. Like before*

$$X := (c_0 + c_1x)\partial_x + (\frac{3}{4}c_1y)\partial_y. \tag{3.12}$$

By placement one of the coefficients x *and* y *one and the rest coefficients zero in* (3.12)*, respectively. We obtained infinitesimal generator:*

$$\partial_x,\ x\partial x + \frac{3}{4}y\partial y.$$

Example 3.5. *[10]Consider the Blasius equation,*

$$y^{'''} = -yy^{''}. \tag{3.13}$$

The linearized symmetry condition for this ODE is

$$\eta^{(3)} = -\eta y^{''} - \eta^2 y \qquad\qquad \text{when } (3.13) \text{ holds.}$$

That is from (3.21) and (3.22) [10],

$$\begin{aligned} \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y^{'} + 3(\eta_{xyy} - \xi_{xxy})y^{'2} + (\eta_{yyy} - 3\xi_{xyy})y^{'3} - \xi_{yyy}y^{'4} \\ +3(\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y^{'} - 2\xi_{yy}y^{'2})y^{''} - 3\xi_y y^{''2} + (\eta_y - 3\xi_x - 4\xi_y y^{'})(-yy^{''}) \\ = -\eta y^{''} - y[\eta_{xx} + (2\eta_{xy} - \xi_{xx})y^{'} + (\eta_{yy} - 2\xi_{xy})y^{'2} - \xi_{yy}y^{'3} + (\eta_y - 2\xi_x - 3\xi_y y^{'})y^{''}]. \end{aligned} \tag{3.14}$$

By comparing powers of $y^{''}, y^{'}y^{''}, y^{'}$*, we obtain the determining equations:*

$$
\begin{aligned}
\eta_{xxx} + y\eta_{xx} &= 0,\\
3\eta_{xxy} - \xi_{xxx} + y(2\eta_{xy} - \xi_{xx}) &= 0,\\
3(\eta_{xyy} - \xi_{xxy}) + y(\eta_{yy} - 2\xi_{xy}) &= 0,\\
\eta_{yyy} - 3\xi_{xyy} - y\xi_{yy} &= 0,\\
\xi_{yyy} &= 0,\\
3(\eta_{xy} - \xi_{xx}) - y(\eta_y - 3\xi_x) + \eta + y(\eta_y - 2\xi_x) &= 0,\\
3(\eta_{yy} - 3\xi_{xy}) + y\xi_y &= 0,\\
\xi_{yy} &= 0,\\
\xi_y &= 0.
\end{aligned}
\tag{3.15}
$$

We consider the tangent vector field of the form

$$
\begin{aligned}
\xi(x,y) &:= c_0 + c_1x + c_2y + c_3x^2 + c_4y^2 + c_5xy + c_6x^3 + c_7y^3 + c_8x^2y + c_9xy^2;\\
\eta(x,y) &:= d_0 + d_1x + d_2y + d_3x^2 + d_4y^2 + d_5xy + d_6x^3 + d_7y^3 + d_8x^2y + d_9xy^2;
\end{aligned}
$$

for arbitrary constants c_i and d_j, i,j=0,...,9. We substitute this ξ and η into (3.15). *Let I be the ideal generated by*

$$
\begin{cases}
f_1 = 6d_6 + 2d_3y + 6d_6yx + 2d_8y^2\\
f_2 = 6d_8 - 6c_6 + 2d_5y + 4d_8xy + 4d_9y^2 - 2c_3y - 6c_6yx - 2c_8y^2\\
f_3 = 6d_9 - 6c_8 + 2d_4y + 6d_7y^2 + 2d_9xy - 2c_5y - 4c_8xy - 4c_9y^2\\
f_4 = 6d_7 - 6c_9 - 2c_4y - 6c_7y^2 - 2c_9xy\\
f_5 = 6c_7\\
f_6 = d_0 + 3d_5 - 6c_3 + d_1x + d_2y + yc_1 + d_5xy + d_8x^2y + d_9xy^2\\
+3c_6yx^2 + 2c_8xy^2 + 2c_3yx + c_5y^2 + c_9y^3 - 18c_6x + 6d_9y + d_3x^2 + d_4y^2 + d_6x^3\\
+d_7y^3 + 6d_8x - 6c_8y\\
f_7 = 6d_4 + 18d_7y + 6d_9x - 9c_5 - 18c_8x - 18c_9y + c_2y + 2c_4y^2 + c_5xy\\
+3c_7y^3 + c_8x^2y + 2c_9xy^2\\
f_8 = 2c_4 + 6c_7y + 2c_9x\\
f_9 = c_2 + 2c_4y + c_5x + 3c_7y^2 + c_8x^2 + 2c_9xy.
\end{cases}
\tag{3.16}
$$

Then by computing Gröbner basis w.r.t. $c_i \prec_{lex} d_j$ for I, we have the following:

$$
G := \{c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, d_9, d_8, d_7, d_6, d_5, d_4, d_3, d_2 + c_1, d_1, d_0\}.
$$

By dividing the ξ and η on Gröbner basis like before

$$
X := (c_0 + c_1x)\partial_x + (-c_1y)\partial_y. \tag{3.17}
$$

By placement one of the coefficients x and y one and the rest coefficients zero in (3.17), *respectively. We obtained infinitesimal generator:*

$$
\partial_x, -y\partial y + x\partial x.
$$

3.2. PDE example

The technique taken to carry out Lie point symmetry for PDEs is essentially the same as for ODEs, however, there are more calculations for PDEs because they have several independent variables. This algorithm works to calculate the infinitesimal generators of the PDEs with finite Lie algebras with some changes, which we explain with an example.

Example 3.6. *[10] To better understand the implementation of the algorithm, consider the following PDE:*

$$u_t + uu_x = u_{xx}. \tag{3.18}$$

The linearized symmetry condition is

$$\eta^t + u\eta^x + u_x\eta = \eta^{xx} \qquad \text{when } (3.18) \text{ holds.} \tag{3.19}$$

That is from (8.29) to (8.31) [10], using (3.18) *to eliminate u_{xx}, we see the highest-order derivative terms in* (3.19) *have a factor u_{xt} in the following:*

$$0 = -2\tau_x u_{xt} - 2\tau_u u_x u_{xt}.$$

This leads to

$$\tau_x = \tau_u = 0.$$

By removing this terms from the linearized symmetry condition; the remaining terms are

$$\begin{aligned} \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t + u(\eta_x + (\eta_u - \xi_x)u_x - \xi_u u_x^2) + u_x\eta = \\ \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x + (\eta_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 + (\eta_u - 2\xi_x - 3\xi_u u_x)(u_t + uu_x). \end{aligned}$$

By reding off all terms that are multiplied by a particular power of u_x and $u_x u_t$, the linearized symmetry condition splits into the following system of determining equations:

$$\begin{aligned} \eta_{xx} - \eta_t - u\eta_x &= 0 \\ 2\eta_{xu} - \xi_{xx} - \xi_x u - \eta + \xi_t &= 0 \\ \eta_{uu} - 2\xi_{xu} - 2u\xi_u &= 0 \\ \xi_{uu} &= 0 \\ -2\xi_x + \tau_t &= 0 \\ \xi_u &= 0 \\ \tau_x &= 0 \\ \tau_u &= 0. \end{aligned}$$

We consider the functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ by their Taylor series at the origin up to the degree equal to the order of partial differential equation that is two,

$$\begin{aligned} \xi(x,t,u) &:= c_0 + c_1x + c_2t + c_3u + c_4x^2 + c_5t^2 + c_6u^2 + c_7xt + c_8xu + c_9tu; \\ \tau(x,t,u) &:= d_0 + d_1x + d_2t + d_3u + d_4x^2 + d_5t^2 + d_6u^2 + d_7xt + d_8xu + d_9tu; \\ \eta(x,t,u) &:= e_0 + e_1x + e_2t + e_3u + e_4x^2 + e_5t^2 + e_6u^2 + e_7xt + e_8xu + e_9tu, \end{aligned}$$

for arbitrary constants c_i, d_j and e_k, i,j,k=0,...,9. After replacing the above Taylor series in determining equations and simplifying, we have a system of linear equations in the following:

$$\begin{cases} f_1 = -2e_4ux - e_7tu - e_8u^2 - e_1u - 2e_5t - e_7x - e_9u - e_2 + 2e_4 = 0 \\ f_2 = -2c_4ux - c_7tu - c_8u^2 - e_4x^2 - e_5t^2 - e_6u^2 - e_7tx - e_8ux - e_9tu - c_1u \\ +2c_5t + c_7x + c_9u - e_1x - e_2t - e_3u + c_2 - 2c_4 - e_0 + 2e_8 = 0 \\ f_3 = -4c_6u^2 - 2c_8ux - 2c_9tu - 2c_3u - 2c_8 + 2e_6 = 0 \\ f_4 = 2c_6 = 0 \\ f_5 = 4c_4x + 2c_7t + 2c_8u - 2d_5t - d_7x - d_9u + 2c_1 - d_2 = 0 \\ f_6 = 2c_6u + c_8x + c_9t + c_3 = 0 \\ f_7 = 2d_4x + d_7t + d_8u + d_1 = 0 \\ f_8 = 2d_6u + d_8x + d_9t + d_3 = 0. \end{cases} \tag{3.20}$$

Let I be the ideal generated by $f_1, \dots, f_8$. Then by computing Gröbner basis w.r.t. $c_i \prec_{lex} d_j \prec_{lex} e_k$ for I, we have the following:

$$G := \{c_9, c_8, c_6, c_5, c_4, c_3, d_9, d_8, d_7, d_6, -c_7 + d_5, d_4, d_3, -2c_1 + d_2,$$
$$d_1, c_7 + e_9, e_8, e_7, e_6, e_5, e_4, c_1 + e_3, e_2, -c_7 + e_1, -c_2 + e_0\}.$$

By dividing the ξ, τ and η on Gröbner basis, we have

$$\xi(x,t,u) = c_7tx + c_1x + c_2t + c_0,$$
$$\tau(x,t,u) = c_7t^2 + 2c_1t + d_0,$$
$$\eta(x,t,u) = c_7(x - ut) - c_1u + c_2.$$

For this example,

$$X := (c_7tx + c_1x + c_2t + c_0)\partial_x + (c_7t^2 + 2c_1t + d_0)\partial_t + (c_7(-tu + x) - c_1u + c_2)\partial_u. \tag{3.21}$$

In (3.21) the coefficients c_i and d_j once one and we set the rest to zero. Therefore, the infinitesimal generator is as follows:

$$\partial_x,\ \partial_t,\ t\partial_x + \partial_u,\ x\partial_x + 2t\partial_t - u\partial_u,\ xt\partial_x + t^2\partial_t + (x - ut)\partial_u,$$

3.3. parametric example

Many engineering problems are parameterized and then solved for different parameter values. The person is also interested for what parameters to find the structure of the solution space. In the following, we present a new approch for computing infinitestimal generators in parametric ODE. For the sake of simplicity, we consider parametric ODE of the form

$$y^n = w(x, y, y', ..., y^{n-1}, u_1, ..., u_m), \tag{3.22}$$

that u_i are parameters. Similar to the previous algorithm, the linearized symmetry condition on given parametric ODE splits into the system of determining equations. By taking the appropriate ansatz for ξ and η and placement in determining equations, we arise systems of parametric polynomials ideals. Kapur (2010) computed a parametric Gröbner basis G from prametric ideal basis in K[U, X] [11, 12]. We found infinitestimal generators by dividing on parametric Gröbner basis for each of the cases. We give now an example to illustrate what described above.

Example 3.7. *Consider the parametric equation [14]*

$$y''' = 2yy'' - \beta y'^2, \tag{3.23}$$

where β is arbitrary parameter. The linearized symmetry condition is

$$\eta^{(3)} = 2y''\eta - 2\beta y'\eta^{(1)} + 2y\eta^{(2)} \qquad \text{when (3.23) holds.}$$

By reding off all terms that are multiplied by a particular power of y'', $y'y''$, the linearized symmetry condition splits into the following system of determining equations:

$$\begin{aligned} \eta_{xxx} - 2y\eta_{xx} &= 0,\\ 3(\eta_{xxy} - \xi_{xxx}) - 2y(2\eta_{xy} - \xi_{xx}) + 2\beta\eta_x &= 0,\\ 3(\eta_{xyy} - \xi_{xxy}) - 2y(\eta_{yy} - 2\xi_{xy}) + \beta\eta_y + \beta\xi_x &= 0,\\ \eta_{yyy} - 3\xi_{xyy} + 2y\xi_{yy} + 2\beta\xi_y &= 0,\\ \xi_{yyy} &= 0,\\ 3(\eta_{xy} - \xi_{xx}) + 2y(\eta_y - 3\xi_x) - 2\eta - 2y(\eta_y - 2\xi_x) &= 0,\\ \eta_{yy} - 3\xi_{xy} - 8y\xi_y &= 0,\\ \xi_{yy} &= 0,\\ \xi_y &= 0 \end{aligned} \tag{3.24}$$

We consider the functions $\xi(x,y)$ and $\eta(x,y)$ by their Taylor series at the origin up to the degree equal to the order of differential equation that is three,

$$\begin{aligned} \xi(x,y) &:= c_0 + c_1x + c_2y + c_3x^2 + c_4y^2 + c_5xy + c_6x^3 + c_7y^3 + c_8x^2y + c_9xy^2;\\ \eta(x,y) &:= d_0 + d_1x + d_2y + d_3x^2 + d_4y^2 + d_5xy + d_6x^3 + d_7y^3 + d_8x^2y + d_9xy^2; \end{aligned}$$

for arbitrary constants c_i and d_j, i,j=0,...,9. After replacing the above Taylor series in determining equations and simplifying, we have a system of linear parametric equations in the following:

$$\left\{\begin{array}{l} f_1 = -12d_6xy - 4d_8y^2 - 4d_3y + 6d_6\\ f_2 = 6\beta d_6x^2 + 4\beta d_8xy + 2\beta d_9y^2 + 4\beta d_3x + 2\beta d_5y - 8d_8xy - 8d_9y^2 + 12c_6xy\\ +4c_8y^2 + 2\beta d_1 - 4d_5y + 4c_3y + 6d_8 - 18c_6\\ f_3 = 3\beta d_7y^2 + \beta d_8x^2 + 2\beta d_9xy + 3\beta c_6x^2 + 2\beta c_8xy + \beta c_9y^2 + 2\beta d_4y + \beta d_5x\\ +2\beta c_3x + \beta c_5y - 12d_7y^2 - 4d_9xy + 8c_8xy + 8c_9y^2 + \beta d_2 + \beta c_1 - 4d_4y + 4c_5y + 6d_9 - 6c_8\\ f_4 = 6\beta c_7y^2 + 2\beta c_8x^2 + 4\beta c_9xy + 4\beta c_4y + 2\beta c_5x + 12c_7y^2 + 4c_9xy + 2\beta c_2\\ +4c_4y + 6d_7 - 6c_9\\ f_5 = 6c_7\\ f_6 = -2d_6x^3 - 2d_7y^3 - 2d_8x^2y - 2d_9xy^2 - 6c_6x^2y - 4c_8xy^2 - 2c_9y^3 - 2d_3x^2\\ -2d_4y^2 - 2d_5xy - 4c_3xy - 2c_5y^2 - 2d_1x - 2d_2y + 6d_8x + 6d_9y - 2d_1y - 18c_6x\\ -6c_8y - 2d_0 + 3d_5 - 6c_3\\ f_7 = -24c_7y^3 - 8c_8x^2y - 16c_9xy^2 - 16c_4y^2 - 8c_5xy + 6d_7y + 2d_9x - 8c_2y - 6c_8x\\ -6c_9y + 2d_4 - 3c_5\\ f_8 = 6c_5y + 2c_9x + 2c_4\\ f_9 = 3c_7y^2 + c_8x^2 + 2c_9xy + 2c_4y + c_5x + c_2. \end{array}\right. \tag{3.25}$$

We consider $F = \{f_1, \ldots, f_9\} \subset K[\beta][x,y]$, where β parameter, x,y variables and $\succ$ the lexicographic order such that $x \succ y$. Then a comprehensive Gröbner system for $< F >$ with respect to $\succ$ is

Like the previous, for obtaining infinitestimal generators for every line of the table, we divide ξ and η on parametric Gröbner basis.

G_i	N_i	E_i
$\{c_9, c_8, c_7, c_6, c_5, c_4, c_2, d_9, d_8, d_7, d_6, d_5 + 2 * c_3, d_4, d_3, c_1 + d_2, d_1, 6 * c_3 + d_0\}$	$\{\}$	$\{\beta - 3\}$
$\{c_9, c_8, c_7, c_6, c_5, c_4, c_3 * (\beta - 3), c_2, d_9, d_8, d_7, d_6, d_5, d_4, d_3, c_1 + d_2, d_1, d_0\}$	$\{\beta - 3\}$	$\{\}$

When is $\beta = 3$, this equation is known as the Chazy equation, more complete discussion of the Chazy equation and its symmetries can be found in [6]

$$\xi(x, y) = c_3 x^2 + c_1 x + c_0,$$
$$\eta(x, y) = (-2xy - 6)c_3 - yc_1.$$

So, infinitestimal generators are

$$\partial_x,\ x\partial_x - y\partial_y,\ x^2\partial_x - (2xy + 6)\partial_y.$$

When is $\beta \neq 3$

$$\xi(x, y) = c_1 x + c_0,$$
$$\eta(x, y) = -yc_1.$$

We have the infinitestimal generators in the following:

$$\partial_x,\ x\partial_x - y\partial_y.$$

4. Conclusion

As we saw in the above examples, all polynomials in the appeared system are linear and so one can solve it by linear algebra techniques. However, there are two important reasons that we prefer to use Gröbner basis for this purpose:

- This is shown in some scientific papers (see [9]) that Gröbner basis has better algorithmic performance in comparison by linear algebra techniques to solve systems of linear equations.
- When the differential equation contains some parameters, the system of equations will also contain these parameters and so, we must analyse a parametric system. In this case, there is no linear algebraic algorithm up to our knowledge to deal with parameters and so we must apply the algorithms which are based on Gröbner basis.

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