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On existence of solutions for some nonlinear fractional differential equations via Wardowski–Mizoguchi–Takahashi type contractions

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Abstract

Using the concept of extended Wardowski-Mizoguchi-Takahashi contractions, we investigate the existence of solutions for three type of nonlinear fractional differential equations. To patronage our main results, some examples of nonlinear fractional differential equations are given.

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1. Introduction

Fixed point theory is a powerful mixture of several branches of mathematics such as analysis, topology, and geometry. This theory has been applied as a very vigorous and substantial instrumentation in the scrutiny of nonlinear phenomena. This theory has been applied in biology, chemistry, economics, engineering, game theory, physics, logic programming etc particularly. After stating the Banach contraction principle many authors have tried to develop this interesting field of mathematics. For more circumstance in this direction we refer the reader to [7]-[9].

Let Π be the family of all maps $\pi : [0, \infty) \longrightarrow [0, \infty)$ so that

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1. $\pi(s) = 0 \quad \Leftrightarrow \quad s = 0$;
2. π is nondecreasing and lower semi-continuous;
3. $\limsup_{\kappa \longrightarrow 0^+} \frac{\kappa}{\pi(\kappa)} < \infty$.

Consider,
(H): $\sigma_n \preceq \sigma$ for each $n \geq 0$ where $\{\sigma_n\} \subseteq X$ is an increasing sequence with $\sigma_n \to \sigma$ as $n \to \infty$.

Gordji and Ramezani [10] propounded an alternative of Mizoguchi-Takahashi Theorem [14] for single-valued mappings.

Theorem 1.1. *[10] Assume that $(X, d, \preceq)$ be a complete partially ordered metric space and let $\mu : X \longrightarrow X$ be an increasing mapping so that there is $\sigma_0 \in X$ with $\sigma_0 \preceq \mu(\sigma_0)$. Assume that there is $\pi \in \Pi$ so that*

$$\pi(d(\mu(\iota), \mu(\kappa))) \leq \alpha(\pi(d(\iota, \kappa)))\pi(d(\iota, \kappa)) \tag{1.1}$$

for all comparable elements $\iota, \kappa \in X$, where $\alpha : [0, \infty) \longrightarrow [0, 1)$ avers the reservation that $\limsup_{s \longrightarrow x^+} \alpha(s) < 1$, for all $x > 0$. If either μ is continuous, or, (H) holds, then μ admits a fixed point.

Definition 1.2. *[11] A self-mapping μ on X is triangular α-admissible if*

(T1) $\alpha(\mu(\iota), \mu(\kappa)) \geq 1$ *provided that* $\alpha(\iota, \kappa) \geq 1$ *where* $\iota, \kappa \in X$,

(T2) $\alpha(\iota, \kappa) \geq 1$ *provided that* $\alpha(\iota, \zeta) \geq 1$ *and* $\alpha(\zeta, \kappa) \geq 1$ *where* $\iota, \kappa, \zeta \in X$.

Lemma 1.3. *[11] Let there is $\sigma_0 \in X$ so that $\alpha(\sigma_0, \mu(\sigma_0)) \geq 1$, where μ is a triangular α-admissible mapping. Let $\sigma_n = \mu^n \sigma_0$. So,*

$$\alpha(\sigma_k, \sigma_l) \geq 1 \text{ for all } k, l \in \mathbb{N} \text{ with } k < l.$$

Denote by Σ the set of all functions $\sigma : (0, \infty) \longrightarrow [0, 1)$ such that

$$\limsup_{\iota \longrightarrow x^+} \sigma(\iota) < 1,$$

for any $x > 0$.

Denote by Ξ the set of all functions $\Upsilon : (0, \infty) \to \mathbb{R}$ so that:

(δ_1) Υ is strictly increasing and continuous.

(δ_2) $\Upsilon(s) = 0 \leftrightarrow s = 1$.

As examples of elements of Ξ:

(i) $\Upsilon_1(x) = x \ln(x)$,

(ii) $\Upsilon_2(x) = \ln(x)$,

(iii) $\Upsilon_3(x) = -\frac{1}{\sqrt{x}} + 1$

(iv) $\Upsilon_4(x) = -\frac{1}{x} + 1$

Denote by Π' the family of all maps $\pi : [0, \infty) \longrightarrow [0, \infty)$ so that

1. $\pi(s) = 0 \quad \Leftrightarrow \quad s = 0$;
2. π is nondecreasing and continuous.

For $\iota, \kappa \in X$, set

$$\Delta(\iota, \kappa) = \max\{d(\iota, \kappa), d(\iota, \mu(\iota)), d(\kappa, \mu(\iota))\}.$$

Consider,

(K): $\alpha(\sigma_n, \sigma) \geq 1$ for each $n \geq 0$ provided that $\{\sigma_n\}$ is a sequence in X so that $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ for each integer $n \geq 0$ and $\sigma_n \to \sigma$ as $n \to +\infty$.

Now, we recall the main results of [15]. In fact, in [15], the authors have extended the Mizoguchi-Takahashi fixed point result motivated by Wardowski's approach [16].

Theorem 1.4. *[15] Let μ be a self-mapping on the complete metric space (X, d) and assume that there is a function $\alpha : X^2 \to [0, \infty)$ satisfying*

$$\Upsilon(\alpha(\iota, \kappa)\pi(d(\mu(\iota), \mu(\kappa)))) \leq \Upsilon(\sigma(\pi(d(\iota, \kappa)))) + \Upsilon(\pi(\Delta(\iota, \kappa))) \tag{1.2}$$

for all $\iota, \kappa \in X$ with $\mu(\iota) \neq \mu(\kappa)$, where $\Upsilon \in \Xi$, $\sigma \in \Sigma$ and $\pi \in \Pi'$. Suppose that μ is a triangular α-admissible mapping and assume that there is $\sigma_0 \in X$ for which $\alpha(\sigma_0, \mu(\sigma_0)) \geq 1$. Then μ admits one fixed point if,

(I) either μ is continuous, or;

(II) (K) holds.

Moreover, such fixed point is unique if for any two fixed points ι, κ of μ, we have $\alpha(\iota, \kappa) \geq 1$.

Note that if π is considered as the identity function, Theorem 1.4 can be expressed as the following result.

Theorem 1.5. *[15] Let f be a self-mapping on the complete metric space (X, d) and let there is a function $\alpha : X^2 \to [0, \infty)$ such that*

$$\Upsilon(d(\mu(\iota), \mu(\kappa))) \leq \Upsilon(\sigma(d(\iota, \kappa))) + \Upsilon(\Delta(\iota, \kappa)) \tag{1.3}$$

for all $\iota, \kappa \in X$ with $\alpha(\iota, \kappa) \geq 1$ and $\mu(\iota) \neq \mu(\kappa)$, where $\Upsilon \in \Xi$ and $\sigma \in \Sigma$. Suppose that μ is a triangular α-admissible mapping and there is $\sigma_0 \in X$ for which $\alpha(\sigma_0, \mu(\sigma_0)) \geq 1$. Then μ admits one fixed point if,

(I) either μ is continuous, or;

(II) (K) holds.

Moreover, such fixed point is unique if $\alpha(\iota, \kappa) \geq 1$ for any two fixed points ι, κ of μ.

Fractional calculus is an important field for research in mathematics considering the properties of derivatives and integrals of non-integer orders and has played an important role in the study of nonlinear fractional differential equations that arise from the modeling of nonlinear phenomena. In particular, this discipline contains the study of methods for solving fractional equations.

The theory of fractional calculus includes even complex orders so that fractional calculus becomes very darling and has many applications.

Now, let us recall some introductive definitions of fractional differential equations ([1], [2] and [6]).

For a continuous function $\mu : [0,\infty) \to \mathbb{R}$, the Caputo-derivative of fractional order α is defined by

$$^{c}D^{\alpha}\mu(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\sigma)^{n-\alpha-1}\mu^n(\sigma)d\sigma \ \ (n-1<\alpha<n, n=[\alpha]+1),$$

whereas the Reimann-Liouville fractional derivative of order α is defined by

$$D^{\alpha}\mu(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_0^x (x-\sigma)^{n-\alpha-1}\mu(\sigma)d\sigma \ \ (n-1<\alpha<n, n=[\alpha]+1).$$

2. Main results

Now, we are able to state and prove the main result of this work. We give our results in the following three forms:

Form 1: In this form, we consider the following nonlinear fractional differential equation

$$^{c}D^{\alpha}\iota(x) = \mu(x,\iota(x)) \ (x \in \mathcal{J}, 1<\alpha\leq 2), \tag{2.1}$$

with the following boundary value conditions:

$$\iota(0)=0, \iota(1)=\int_0^{\eta}\iota(\sigma)d\sigma \ (0<\eta<1),$$

where $\mu : [0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Here, $X = C([0,1],\mathbb{R})$ is the Banach space of continuous functions from $[0,1]$ to $\mathbb{R}$ with the supremum morm $\|\iota\|_\infty = \sup\{|\iota(x)| : x\in[0,1]\}$.

Theorem 2.1. *Suppose that*

(i) there are functions $\Upsilon \in \Xi$, $\sigma\in\Sigma$ and $\xi : \mathbb{R}^2\to\mathbb{R}$ satisfying

$$\Upsilon(\frac{5}{\Gamma(\alpha+1)}|\mu(x,a)-\mu(x,b)|) \leq \Upsilon(\sigma(|a-b|)) + \Upsilon(|a-b|) \tag{2.2}$$

for all $x\in\mathcal{J}$ and $a,b\in\mathbb{R}$ with $\xi(a,b)\geq 0$.

(ii) there is $\iota_0\in C(\mathcal{J})$ such that $\xi(\iota_0(x),\Lambda\iota_0(x))\geq 0$ for all $x\in\mathcal{J}$, where the operator $\Lambda : C(\mathcal{J})\to C(\mathcal{J})$ is defined by

$$\begin{aligned}\Lambda\iota(x) &= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^x(x-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma - \tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^{\eta}(1-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma \\ &+\tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\textstyle\int_0^{\eta}(\int_0^s(s-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma)ds \ (0\leq x\leq 1);\end{aligned} \tag{2.3}$$

(iii) $\xi(\iota(x),\kappa(x))\geq 0$ implies $\xi(\Lambda\iota(x),\Lambda\kappa(x))\geq 0$ for each $x\in\mathcal{J}$ and for each $\iota,\kappa\in C(\mathcal{J})$;

(iv) if $\{\iota_n\}$ is a sequence in $C(\mathcal{J})$ such that $\iota_n\to\iota$ in $C(\mathcal{J})$ and $\xi(\iota_n,\iota_{n+1})\geq 0$ for all n, then $\xi(\iota_n,\iota)\geq 0$ for all n.

Then, the problem (2.1) *has at least one solution.*

Proof . It is well known that $\iota(x)$ is a solution of the problem (2.1) if and only if it is a solution of the fractional integral equation

$$\begin{aligned}\iota(x) &= \tfrac{1}{\Gamma(\alpha)}\int_0^x (x-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma - \tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^1(1-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma \\ &+ \tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta(\int_0^s(s-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma)ds\ (x\in\mathcal{J});\end{aligned} \tag{2.4}$$

Thus, the existence of a solution for (2.1) is equivalent to this fact that Λ admits one fixed point. Now, let $\iota,\kappa\in C(\mathcal{J})$ be such that $\xi(\iota(x),\kappa(x))\geq 0$ for all $x\in\mathcal{J}$ and $\Lambda\iota\neq\Lambda\kappa$. Then, for all $x\in\mathcal{J}$ with $\Lambda\iota(x)\neq\Lambda\kappa(x)$, we have $\iota(x)\neq\kappa(x)$. From (i), we have

$$|\mu(x,\iota(x))-\mu(x,\kappa(x))|\leq\frac{\Gamma(\alpha+1)}{5}\Upsilon^{-1}[\Upsilon(\sigma(|\iota(x)-\kappa(x)|))+\Upsilon(|\iota(x)-\kappa(x)|)].$$

Now, we have

$$\begin{aligned}\Big|\Lambda\iota(x)-\Lambda\kappa(x)\Big| &= \Big|\tfrac{1}{\Gamma(\alpha)}\int_0^x(x-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma - \tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^1(1-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma \\ &+\tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta(\int_0^s(s-\sigma)^{\alpha-1}\mu(\sigma,\iota(\sigma))d\sigma)ds \\ &-\tfrac{1}{\Gamma(\alpha)}\int_0^x(x-\sigma)^{\alpha-1}\mu(\sigma,\kappa(\sigma))d\sigma+\tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^1(1-\sigma)^{\alpha-1}\mu(\sigma,\kappa(\sigma))d\sigma \\ &-\tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta(\int_0^s(s-\sigma)^{\alpha-1}\mu(\sigma,\kappa(\sigma))d\sigma)ds\Big| \\ &\leq\tfrac{1}{\Gamma(\alpha)}\int_0^x(x-\sigma)^{\alpha-1}|\mu(\sigma,\iota(\sigma))-\mu(\sigma,\kappa(\sigma))|d\sigma \\ &+\tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^1(1-\sigma)^{\alpha-1}|\mu(\sigma,\iota(\sigma))-\mu(\sigma,\kappa(\sigma))|d\sigma \\ &+\tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta(\int_0^s(s-\sigma)^{\alpha-1}|\mu(\sigma,\iota(\sigma))-\mu(\sigma,\kappa(\sigma))|d\sigma)ds \\ &\leq\tfrac{1}{\Gamma(\alpha)}\int_0^x(x-\sigma)^{\alpha-1}\tfrac{\Gamma(\alpha+1)}{5}\Upsilon^{-1}[\Upsilon(\sigma(|\iota(x)-\kappa(x)|))+\Upsilon(|\iota(x)-\kappa(x)|)]d\sigma \\ &+\tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^1(1-\sigma)^{\alpha-1}\tfrac{\Gamma(\alpha+1)}{5}\Upsilon^{-1}[\Upsilon(\sigma(|\iota(x)-\kappa(x)|))+\Upsilon(|\iota(x)-\kappa(x)|)]d\sigma \\ &+\tfrac{2x}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta(\int_0^s(s-\sigma)^{\alpha-1}\tfrac{\Gamma(\alpha+1)}{5}\Upsilon^{-1}[\Upsilon(\sigma(|\iota(x)-\kappa(x)|))+\Upsilon(|\iota(x)-\kappa(x)|)]d\sigma)ds \\ &\leq\tfrac{\Gamma(\alpha+1)}{5}\Upsilon^{-1}[\Upsilon(\sigma(\|\iota-\kappa\|)+\Upsilon(\|\iota-\kappa\|)](\tfrac{1}{\Gamma(\alpha+1)}+\tfrac{2}{\Gamma(\alpha+1)}+\tfrac{2}{\Gamma(\alpha+1)}) \\ &=\Upsilon^{-1}[\Upsilon(\sigma(\|\iota-\kappa\|)+\Upsilon(\|\iota-\kappa\|)].\end{aligned} \tag{2.5}$$

Therefore

$$\|\Lambda\iota-\Lambda\kappa\|\leq\Upsilon^{-1}[\Upsilon(\sigma(\|\iota-\kappa\|)+\Upsilon(\|\iota-\kappa\|)],$$

and so

$$\Upsilon(\|\Lambda\iota-\Lambda\kappa\|)\leq\Upsilon(\sigma(\|\iota-\kappa\|)+\Upsilon(\|\iota-\kappa\|).$$

Therefore, by Theorem (2.1) Λ admits one fixed point and so the problem ((2.1)) admits one solution in $C(\mathcal{J})$. □

Remark 2.2. *Note that if we take $\Upsilon(x)=-\frac{1}{x}+1$ and $\sigma(x)=\frac{2}{3}$, for all $x\in(0,\infty)$, then contraction (2.9) is equal with*

$$|\mu(x,a)-\mu(x,b)|\leq\frac{\Gamma(\alpha+1)}{5}\frac{|a-b|}{1+\frac{1}{2}|a-b|} \tag{2.6}$$

for all $x\in\mathcal{J}$ and for all $a,b\in\mathbb{R}$ with $\xi(a,b)\geq 0$.

Example 2.3. *Consider the differential equation of fractional order*

$${}^cD^\alpha\iota(x)=\frac{\Gamma(\alpha+1)}{5}\frac{e^{-x}sin(x^2+1)|\iota(x)|}{1+|\iota(x)|}(x\in\mathcal{J},1<\alpha\leq 2), \tag{2.7}$$

with the following boundary value conditions:

$$\iota(0) = 0, \iota(1) = \int_0^\eta \iota(\sigma)d\sigma \ (0 < \eta < 1).$$

Here,

$$\mu(x,a) = \frac{\Gamma(\alpha+1)}{5}\frac{e^{-x}sin(x^2+1)|a|}{1+|a|}.$$

We have

$$\begin{aligned}|\mu(x,a)-\mu(x,b)| &= \frac{\Gamma(\alpha+1)}{5}e^{-x}|sin(x^2+1)||\frac{|a|}{1+|a|}-\frac{|b|}{1+|b|}| \\ &\le \frac{\Gamma(\alpha+1)}{5}\frac{\Big||a|-|b|\Big|}{(1+|a|)(1+|b|)} \le \frac{\Gamma(\alpha+1)}{5}\frac{\Big||a|-|b|\Big|}{1+\frac{1}{2}(\Big||a|-|b|\Big|)} \\ &\le \frac{\Gamma(\alpha+1)}{5}\frac{|a-b|}{1+\frac{1}{2}|a-b|},\end{aligned}$$

which is (2.6)*. Thus, by the Remark 2.2, the problem* (2.7) *admits one solution in* $C(\mathcal{J})$*.*

Form 2: In this form, we consider the following nonlinear fractional differential equation:

$$^cD^\alpha\iota(x) + \mu(x,\iota(x)) = 0 \ (x \in \mathcal{J}, 1 < \alpha), \tag{2.8}$$

with the following boundary value conditions:

$$\iota(0) = \iota(1) = 0,$$

where $\mu : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\mathcal{J} = [0,1]$.

It is well known that the Green function of the above problem is as follows:

$$G(x,\sigma) = \frac{1}{\Gamma(\alpha)}\begin{cases} x(1-\sigma)^{\alpha-1} - (x-\sigma)^{\alpha-1}, & 0 \le \sigma \le x \le 1, \\ x(1-\sigma)^{\alpha-1}, & 0 \le x \le \sigma \le 1,\end{cases}$$

that is, the problem (2.8) is equivalent to the fractional integral equation

$$\iota(x) = \int_0^1 G(x,\sigma)\mu(\sigma,\iota(\sigma))d\sigma.$$

Theorem 2.4. *Suppose that*

(i) there are functions $\Upsilon \in \Xi$, $\sigma \in \Sigma$ *and* $\xi : \mathbb{R}^2 \to \mathbb{R}$ *satisfying*

$$\Upsilon(|\mu(x,a)-\mu(x,b)|) \le \Upsilon(\sigma(|a-b|)) + \Upsilon(|a-b|) \tag{2.9}$$

for all $x \in \mathcal{J}$ *and for all* $a, b \in \mathbb{R}$ *with* $\xi(a,b) \ge 0$ *and* $\mu(x,a) \ne \mu(x,b)$*.*

(ii) there is $\iota_0 \in C(\mathcal{J})$ *such that* $\xi(\iota_0(x), \int_0^1 G(x,\sigma)\mu(\sigma,\iota(\sigma))d\sigma) \ge 0$ *for all* $x \in \mathcal{J}$*.*

(iii) for each $x \in \mathcal{J}$ *and for each* $\iota, \kappa \in C(\mathcal{J})$*,*

$$\xi(\iota(x),\kappa(x)) \ge 0 \Longrightarrow \xi(\int_0^1 G(x,\sigma)\mu(\sigma,\iota(\sigma))d\sigma, \int_0^1 G(x,\sigma)\mu(\sigma,\kappa(\sigma))d\sigma) \ge 0;$$

(iv) if $\{\iota_n\}$ is a sequence in $C(\mathcal{J})$ such that $\iota_n \to \iota$ in $C(\mathcal{J})$ and $\xi(\iota_n(x), \iota_{n+1}(x)) \geq 0$ for all n, then $\xi(\iota_n(x), \iota(x)) \geq 0$ for all n.

Then, the problem (2.8) has at least one solution.

Proof . Define $\Lambda : C(\mathcal{J}) \to C(\mathcal{J})$ by

$$\Lambda\iota(x) = \int_0^1 G(x,\sigma)\mu(\sigma, \iota(\sigma))d\sigma.$$

Thus, the existence of a solution for (2.8) is equivalent to this fact that Λ admits one fixed point. Now, let $\iota, \kappa \in C(\mathcal{J})$ be such that $\xi(\iota(x), \kappa(x)) \geq 0$ for all $x \in \mathcal{J}$ and $\Lambda\iota \neq \Lambda\kappa$. Then, for those $x \in \mathcal{J}$ with $\Lambda\iota(x) \neq \Lambda\kappa(x)$, we have also $\iota(x) \neq \kappa(x)$. From (i), we have

$$|\mu(x, \iota(x)) - \mu(x, \kappa(x))|) \leq \Upsilon^{-1}[\Upsilon(\sigma(|\iota(x) - \kappa(x)|)) + \Upsilon(|\iota(x) - \kappa(x)|)].$$

Now, we have

$$\begin{aligned} |\Lambda\iota(x) - \Lambda\kappa(x)| &= |\int_0^1 G(x,\sigma)\mu(\sigma, \iota(\sigma))d\sigma - \int_0^1 G(x,\sigma)\mu(\sigma, \kappa(\sigma))d\sigma| \\ &\leq \int_0^1 G(x,\sigma)|\mu(\sigma, \iota(\sigma)) - \mu(\sigma, \kappa(\sigma))|d\sigma \\ &\leq \int_0^1 G(x,\sigma)\Upsilon^{-1}[\Upsilon(\sigma(|\iota(x) - \kappa(x)|) + \Upsilon(|\iota(x) - \kappa(x)|)]d\sigma \\ &\leq \Upsilon^{-1}[\Upsilon(\sigma(\|\iota - \kappa\|) + \Upsilon(\|\iota - \kappa\|)]. \end{aligned}$$

Therefore,

$$\|\Lambda\iota - \Lambda\kappa\| \leq \Upsilon^{-1}[\Upsilon(\sigma(\|\iota - \kappa\|) + \Upsilon(\|\iota - \kappa\|)],$$

and so,

$$\Upsilon(\|\Lambda\iota - \Lambda\kappa\|) \leq \Upsilon(\sigma(\|\iota - \kappa\|) + \Upsilon(\|\iota - \kappa\|).$$

Therefore, Theorem (1.5) yields that problem (2.8) admits one solution in $C(\mathcal{J})$. $\square$

Note that if we take $\Upsilon(x) = -\frac{2}{x} + 2$ and $\sigma(x) = \frac{4}{5}$, for all $x \in (0, \infty)$, then contraction (2.9) is equall with

$$|\mu(x,a) - \mu(x,b)| \leq \frac{|a-b|}{1 + \frac{1}{4}|a-b|}. \tag{2.10}$$

for all $x \in \mathcal{J}$ and for all $a, b \in \mathbb{R}$ with $\xi(a,b) \geq 0$.

Example 2.5. *Consider the differential equation of fractional order*

$${}^cD^\alpha\iota(x) = -\frac{e^{-x}cos(x^2+1)|\iota(x)|}{1 + \frac{1}{2}|\iota(x)|} \ (x \in \mathcal{J}, 1 < \alpha), \tag{2.11}$$

with the following boundary value conditions:

$$\iota(0) = \iota(1) = 0.$$

Here,

$$\mu(x,a) = -\frac{e^{-x}cos(x^2+1)|a|}{1 + \frac{1}{2}|a|}.$$

We have

$$
\begin{aligned}
|\mu(x,a)-\mu(x,b)| &= e^{-x}|cos(x^2+1)||\frac{|a|}{1+\frac{1}{2}|a|}-\frac{|b|}{1+\frac{1}{2}|b|}| \\
&\leq \frac{\Big||a|-|b|\Big|}{(1+\frac{1}{2}|a|)(1+\frac{1}{2}|b|)} \leq \frac{\Big||a|-|b|\Big|}{1+\frac{1}{4}(\Big||a|-|b|\Big|)} \\
&\leq \frac{|a-b|}{1+\frac{1}{4}|a-b|},
\end{aligned}
$$

therefore (2.10) *holds. Thus, Theorem 1.5 yields that the problem* (2.11) *admits one solution in* $C(\mathcal{J})$.

Form 3: In this form, we consider the following nonlinear fractional differential equation

$$
{}^cD^{\alpha}\iota(x) +{}^c D^{\sigma}\iota(x) = \mu(x,\iota(x)) \ (x\in\mathcal{J}, 0<\sigma<\alpha<1), \tag{2.12}
$$

with the following boundary value conditions:

$$
\iota(0)=\iota(1)=0,
$$

where $\mu:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $\mathcal{J}=[0,1]$.

Recall that the Green function of the above problem is

$$
G(x)=x^{\alpha-1}E_{\alpha-\sigma,\alpha}(-x^{\alpha-\sigma}),
$$

where $E_{\alpha,\sigma}$ is the Mittag-Leffler function (see [6]), that is, the problem (2.12) is equivalent to the fractional integral equation

$$
\iota(x)=\int_0^1 G(x-\sigma)\mu(\sigma,\iota(\sigma))d\sigma.
$$

Theorem 2.6. *Suppose that*

(i) there are functions $\Upsilon\in\Xi$, $\sigma\in\Sigma$ *and* $\xi:\mathbb{R}^2\to\mathbb{R}$ *satisfying*

$$
\Upsilon(\frac{1}{\Gamma(\alpha+1)}|\mu(x,a)-\mu(x,b)|)\leq\Upsilon(\sigma(|a-b|))+\Upsilon(|a-b|) \tag{2.13}
$$

for all $x\in\mathcal{J}$ *and for all* $a,b\in\mathbb{R}$ *with* $\xi(a,b)\geq 0$ *and* $\mu(x,a)\neq\mu(x,b)$.

(ii) there is $\iota_0\in C(\mathcal{J})$ *such that* $\xi(\iota_0(x),\int_0^x G(x-\sigma)\mu(\sigma,\iota(\sigma))d\sigma)\geq 0$ *for all* $x\in\mathcal{J}$.

(iii) for each $x\in\mathcal{J}$ *and for each* $\iota,\kappa\in C(\mathcal{J})$,

$$
\xi(\iota(x),\kappa(x))\geq 0 \Longrightarrow \xi(\int_0^x G(x-\sigma)\mu(\sigma,\iota(\sigma))d\sigma, \int_0^x G(x-\sigma)\mu(\sigma,\kappa(\sigma))d\sigma)\geq 0;
$$

(iv) if $\{\iota_n\}$ *is a sequence in* $C(\mathcal{J})$ *such that* $\iota_n\to\iota$ *in* $C(\mathcal{J})$ *and* $\xi(\iota_n(x),\iota_{n+1}(x))\geq 0$ *for all* n, *then* $\xi(\iota_n(x),\iota(x))\geq 0$ *for all* n.

Then, the problem (2.12) *has at least one solution.*

Proof . Define $\Lambda : C(\mathcal{J}) \to C(\mathcal{J})$ by

$$\Lambda\iota(x) = \int_0^x G(x-\sigma)\mu(\sigma, \iota(\sigma))d\sigma.$$

Thus, the existence of a solution for (2.12) is equivalent to this fact that Λ admits one fixed point. Now, let $\iota, \kappa \in C(\mathcal{J})$ be such that $\xi(\iota(x), \kappa(x)) \geq 0$ for all $x \in \mathcal{J}$ and $\Lambda\iota \neq \Lambda\kappa$. Then, for those $x \in \mathcal{J}$ with $\Lambda\iota(x) \neq \Lambda\kappa(x)$, we have also $\iota(x) \neq \kappa(x)$ and from (i), we have

$$|\mu(x, \iota(x)) - \mu(x, \kappa(x))|) \leq \Gamma(\alpha+1)\Upsilon^{-1}[\Upsilon(\sigma(|\iota(x) - \kappa(x)|)) + \Upsilon(|\iota(x) - \kappa(x)|)].$$

Now, we have

$$\begin{aligned}
|\Lambda\iota(x) - \Lambda\kappa(x)| &= |\int_0^x G(x-\sigma)\mu(\sigma, \iota(\sigma))d\sigma - \int_0^x G(x-\sigma)\mu(\sigma, \kappa(\sigma))d\sigma| \\
&\leq \int_0^x G(x-\sigma)|\mu(\sigma, \iota(\sigma)) - \mu(\sigma, \kappa(\sigma))|d\sigma \\
&\leq \Gamma(\alpha+1)\int_0^x G(x-\sigma)\Upsilon^{-1}[\Upsilon(\sigma(|\iota(x) - \kappa(x)|) + \Upsilon(|\iota(x) - \kappa(x)|)]d\sigma \\
&\leq \Upsilon^{-1}[\Upsilon(\sigma(\|\iota - \kappa\|)) + \Upsilon(\|\iota - \kappa\|)].
\end{aligned}$$

Note that

$$G(x) = x^{\alpha-1}E_{\alpha-\sigma,\alpha}(-x^{\alpha-\sigma}) \leq \frac{1}{\Gamma(\alpha)}\frac{x^{\alpha-1}}{1+x^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}x^{\alpha-1}.$$

Therefore, $\sup_{x\in\mathcal{J}} \int_0^x G(x-s)ds \leq \frac{1}{\alpha\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha+1)}$. Thus,

$$\|\Lambda\iota - \Lambda\kappa\| \leq \Upsilon^{-1}[\Upsilon(\sigma(\|\iota - \kappa\|)) + \Upsilon(\|\iota - \kappa\|)],$$

and so

$$\Upsilon(\|\Lambda\iota - \Lambda\kappa\|) \leq \Upsilon(\sigma(\|\iota - \kappa\|) + \Upsilon(\|\iota - \kappa\|).$$

Therefore, Theorem 1.5 implies that Λ admits one fixed point and so the problem (2.12) admits one solution in $C(\mathcal{J})$. □

Note that if we take $\Upsilon(x) = -\frac{1}{x} + 1$ and $\sigma(x) = \frac{2}{3}$, for all $x \in (0, \infty)$, then contraction (2.13) is equall with

$$|\mu(x, a) - \mu(x, b)| \leq \Gamma(\alpha+1)\frac{|a-b|}{1+\frac{1}{2}|a-b|}. \tag{2.14}$$

for all $x \in \mathcal{J}$ and for all $a, b \in \mathbb{R}$ with $\xi(a, b) \geq 0$.

Example 2.7. *Consider the differential equation of fractional order*

$$^cD^\alpha\iota(x) +^c D^\sigma\iota(x) = \Gamma(\alpha+1)\frac{e^{-x}[cos(x^2+1) + sin(e^x)]|\iota(x)|}{2(1+|\iota(x)|)} \; (x \in \mathcal{J}, 0 < \sigma < \alpha < 1), \tag{2.15}$$

with the following boundary value conditions $\iota(0) = \iota(1) = 0$. *Here,*

$$\mu(x, a) = \Gamma(\alpha+1)\frac{e^{-x}[cos(x^2+1) + sin(e^x)]|a|}{2(1+|a|)}.$$

We have

$$\begin{aligned}|\mu(x,a)-\mu(x,b)| &= \Gamma(\alpha+1)e^{-x}|[cos(x^2+1)+sin(e^x)]||\frac{|a|}{2(1+|a|)}-\frac{|b|}{2(1+|b|)}| \\ &\le \Gamma(\alpha+1)\frac{\Big||a|-|b|\Big|}{(1+|a|)(1+|b|)} \le \Gamma(\alpha+1)\frac{\Big||a|-|b|\Big|}{1+\frac{1}{2}(\Big||a|-|b|\Big|)} \\ &\le \Gamma(\alpha+1)\frac{|a-b|}{1+\frac{1}{2}|a-b|}.\end{aligned}$$

So, the inequality (2.14) *holds. Thus, by Theorem 1.5 the problem* (2.15) *admits one solution in* $C(\mathcal{J})$.

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