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Maximal Ideal Graph of Commutative Semirings

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Abstract

In this paper a new kind of graph on a commutative semiring is introduced and investigated. The maximal ideal graph of S, denoted by $\mathrm{MG}(S)$, is a graph with all nontrivial ideals of S as vertices and two distinct vertices I and J are adjacent if and only if $I+J$ is a maximal ideal of S. In this article, some interrelation between the graph theoretic properties of this graph and some algebraic properties of semirings are studied. We investigated the basic properties of the maximal ideal graph such as diameter, girth, clique number, cut vertex, planar property.

Keywords: Semiring; Maximal ideal; The maximal ideal graph; Connectedness; Diameter; Girth; Planar property.

1. Introduction

There has been a lot of activity over the past several years in associating a graph to an algebraic system such as a ring or semiring [1,5,8,9,10,14]

In 1988, Beck [5] introduced the concept of the zero-divisor graph. Since then, others have introduced and studied many researches in this area. Gupta et al. [14] in 2015 defined a variation of zero-divisor graphs. Recently, the study of such graphs of rings are extended to include semirings as in [8,9,10]

In 2020, Abdulqadr [1] introduced the maximal ideal graph of a commutative ring R denoted by $\mathrm{MG}(R)$, is the undirected graph with all non-trivial ideals of R as vertices and two distinct vertices I and J are adjacent if and only if $I+J$ is a maximal ideal of R. In this paper, we introduce maximal ideal graph of a commutative semiring, as a generalization of this notion. Throughout this paper S will be a commutative semiring with identity, also, $\mathbb{N}$ be the semiring of all non-negative integers.

A commutative semiring S is defined as an algebraic system $(S,+,\cdot)$ such that $(S,+)$ and $(S,\cdot)$ are commutative semigroups, connected by $a(b+c)=ab+ac$ for all $a,b,c\in S$, and there exists

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$0, 1 \in S$ such that $s + 0 = s$ and $s0 = 0s = 0$. A commutative semiring is a semifield if each non-zero element in S has multiplicative inverse. Clearly, any ring is a semiring. A nonempty subset I of a semiring S is defined to be an ideal of S if $a, b \in I$ and $s \in S$ implies that $a + b, sa \in I$. An ideal I of a semiring S is called subtractive if $a+b \in I$ and $a \in I$ imply $b \in I$ for all $a, b \in S$. We say a semiring is subtractive if each of its ideals is subtractive see [2,16] . An ideal of a semiring S is maximal if and only if it is not properly contained in any other ideal of S. A semiring is said to be local if it has a unique maximal ideal M and we denote it by (S, M). A semiring is said to be semi-local if the set of its maximal ideals is finite. An ideal $I \neq \{0\}$ of a semiring S is minimal if and only if it does not contain any ideal of S other than itself and 0. The set of non-trivial ideals is denoted by $\mathrm{Id}(S)$. The set of maximal ideals of S is denoted by $\max(S)$, and the intersection of all maximal ideals of S is called the Jacobson radical of S and is denoted by $J(S)$. The set of minimal ideals of S is denoted by $\min(S)$. A semiring S is Noetherian (respectively, Artinian) if any non-empty set of ideals of S has a maximal member (respectively, minimal member) with respect to set inclusion.

The maximal ideal graph helps us to consider the algebraic properties of semirings using graph theoretical tools. In our investigation of $\mathrm{MG}(S)$, maximal ideals play an important role to find some connections between the graph theoretic properties of this graph and some algebraic properties of semirings. In section 2, we show that $\mathrm{MG}(S)$ cannot be a complete graph if S has more than one maximal ideal. Fire explore some of the properties and characterizations of these graphs. For instance, the semirings S, for which the graph $\mathrm{MG}(S)$ is star or complete bipartite, are characterized.

In Section 3, the planarity is investigated. At the first of this section, one of the important properties of $\mathrm{MG}(S)$ is introduced, which help us to gain interesting results about the girth of $\mathrm{MG}(S)$. Also, then number of maximal ideals of S.

In Section 4, under one condition it is shown that $\mathrm{MG}(S)$ is a connected graph and $\mathrm{diam}(\mathrm{MG}(S)) \leq 3$.

In order to make this paper easier to follow, we recall in this section various notions which will be used in the sequel [11,12] . Let G be a graph. Then V(G) and E(G) denote the set of vertices and edges of G, respectively. The set of vertices adjacent to vertex v of the graph G is called the neighborhood of v and denoted by N(v). In addition, for two distinct vertices u and v in G, the notation $\{\mathrm{u}, \mathrm{v}\} \in \mathrm{E(G)}$ means that u and v are adjacent. The degree of a vertex v of any graph G is denoted by deg(v) and defined as the number of edges incident on v. A vertex of degree 0 is called an isolated vertex. The complete graph of order n, denoted by $\mathrm{K_n}$, is a graph with n vertices in which every two distinct vertices are adjacent.

For a positive integer r, an r-partite graph is one whose vertex set V(G) can be partitioned into r subsets $\mathrm{V_1},\ \mathrm{V_2}, \ldots, \mathrm{V_r}$(called partite sets) such that every element of E(G) joins a vertex of V_i to a vertex of $V_j, i \neq j$. The complete bipartite graph (2-partite graph) with exactly two partitions of size m and n is denoted by $\mathrm{K_{n,m}}$. A graph G is said to be star if $\mathrm{G} = \mathrm{K_{n,1}}$. Two vertices u and v of a graph G are said to be connected in G if there exists a path between them. A graph G is called connected if there exists a path between any two distinct vertices. Otherwise, G is called disconnected. A graph G is said to be totally disconnected if it has no edges. Let G be a connected graph. The distance between two distinct vertices u and v of G, denoted by $\mathrm{d(u, v)}$, is the length of the shortest path connecting u and v, if such a path exists; otherwise, we set $\mathrm{d(u, v)} = \infty$. The diameter, eccentricity, and radius of a connected graph G are defined by $\mathrm{diam(G)} = \mathrm{Max}\{\mathrm{d(u, v)} : \mathrm{u, v} \in \mathrm{V(G)}\}$ $\mathrm{e(v)} = \mathrm{Max}\{\mathrm{d(u, v)} : \text{for all } \mathrm{u} \in \mathrm{V(G)}\}$ and $\mathrm{rad(G)} = \mathrm{Min}\{\mathrm{e(v)} : \mathrm{v} \in \mathrm{V(G)}\}$, respectively. A vertex v of a connected graph G is a cut-vertex if the components of $\mathrm{G} - \mathrm{v}$ are more than the components of G. The girth of a graph G, denoted by gr(G), is the length of a shortest cycle in G, provided G contains a cycle; otherwise; $\mathrm{gr(G)} = \infty$. A k-coloring of a graph G is a function $\mathrm{C} : \mathrm{V(G)} \to \{1, 2, \ldots, \mathrm{k}\}$ such that $\mathrm{C(u)} \neq \mathrm{C(v)}$ whenever u is adjacent to v. If a k-coloring of G exists, then G is k-colorable. The chromatic number of G is defined by $\chi(\mathrm{G}) = \min\{\mathrm{k} : \mathrm{G} \text{ is k-colorable}\}$. A complete subgraph $\mathrm{K_n}$ of a graph G is called a clique, and $\omega(G)$ is the clique number of G, which is the greatest integer $r \geq 1$ such that $K_r \subseteq G$. A graph G is called a planar graph if it can be drawn on a plane in such a way that any two of its edges either meet only at their end vertices or do not meet at all. A graph G is perfect if every induced subgraph H of G satisfies $\chi(\mathrm{H}) = \omega(\mathrm{H})$. A graph is a split graph if it can be partition in an independent set and a clique.

2. The Maximal Ideal Graph of a Semiring

In this section, we introduce the concept of the maximal ideal graph of a commutative semiring with identity. We illustrate this concept by examples and remarks and give some of its properties and characterizations. We begin with the key definition of this paper.

Definition 2.1. *Let S be a commutative semiring with identity. The maximal ideal graph of S, denoted by $\mathrm{MG}(S)$, is the undirected graph with all non-trivial ideals of S as vertices and two distinct vertices I and J are adjacent if and only if $I + J$ is a maximal ideal of S.*

Proposition 2.2. *[13, Proposition 6.59] Every ideal of a semiring S is contained in a maximal ideal of S.*

The proof of the next result is the same as in [1, Lemma 2.2], for the sake of completeness, a proof will be given.

Lemma 2.3. *1. Every non-maximal ideal is adjacent to at least one maximal ideal in $\mathrm{MG}(S)$.*

2. If $M_1, M_2, \ldots, M_n \in \max(S)$ such that $\bigcap_{i=1}^{n} M_i \notin \max(S) \cup \{(0)\}$, then the ideal $\bigcap_{i=1}^{n} M_i$ is adjacent to every $M_i \in \max(S)$ in $\mathrm{MG}(S)$, for $1 \leq i \leq n$.

Proof . For(1): Let $I \in V(\mathrm{MG}(S)) \setminus \max(S)$. So IM, for some $M \in \max(S)$. Then clearly, $I + M = M$. Thus I is adjacent to M.

For(2): Clearly $\bigcap_{i=1}^{n} M_i \subseteq M_t$, for $1 \leq t \leq n$. So the proof is a direct consequence of (1). □

We recall that for a graph G, a subset E of the vertex-set of G is called a dominating set if every vertex not in E is adjacent to a vertex in E. The domination number, $\gamma(\mathrm{G})$, of G is the minimum cardinality of a dominating set of G (see [15], [19]).

Theorem 2.4. *Let S be a semiring. Then $\{\max(S)\}$ is a dominating set of $\mathrm{MG}(S)$.*

Proof . This is an immediate consequence of Lemma 2.3(1). □

Example 2.5. *Let $S = \mathbb{N}$ be a semiring of non-negative integers. The set $\mathbb{N}\setminus\{1\}$ is a unique maximal ideal of the semiring S which contains all ideals of S see [13, Example 6.60]$.Theset\{N \setminus\{1\}\}$ is a dominating set for $\mathrm{MG}(S)$. Hence $\gamma(\mathrm{MG}(S))$ the domination number of $\mathrm{MG}(S)$ is equal to 1.*

Recall [13, p. 118] that an ideals I and H of a semiring S are comaximal if and only if $I + H = S$.

Remark 2.6. *The comaximal ideals of S are not adjacent in $\mathrm{MG}(S)$*

. The next main result shows the adjacency between ideal vertices of $\mathrm{MG}(S)$

Theorem 2.7. *Let I, J and M be three distinct vertices of $\mathrm{MG}(S)$ with $M \in \max(S)$. Then:*

1. $M \in N(I) \cap N(J)$ if and only if $M \in N(I + J)$, where $I + J \neq M, S$.

2. *If $I \subset J(S)$, then $M \in N(I)$.*
3. *If $I \subset J$ and $J \notin \max(S)$, then $I \notin N(J)$*
4. *If $I \in N(JL)$, then $I \in N(J \cap L) \cap N(J)$, for every vertices L in which $JL \neq (0)$.*

Proof . For (1): Let $I + J \neq M$. If $M \in N(I) \cap N(J)$, then by Lemma 2.3, $I, J \subset M$. This means that $I + J \subset M$. Thus, $M \in N(I + J)$. Similarly, If $M \in N(I + J)$, then $M \in N(I) \cap N(J)$.

For (2): Let $I \subset J(S)$. Then $I \subset I + J(S) = J(S) \subseteq M$. By Lemma 2.3, $M \in N(I)$.

For (3): Let $I \subset J$ and $J \notin \max(S)$. Clearly $I + J = J \notin \max(S)$, this completes the proof.

For (4): Let I, J, and L be an ideals of S such that $I \in$ N(JL). In semiring theory, it is clear that $JL \subseteq J \cap L$, but, in general, we do not have equality. Thus, $N(JL) \subseteq N(J \cap L)$ and so $I \in N(J \cap L)$. Similarly, we can show, $I \in N(J)$. This completes the proof. □

An ideal I of S is called small if $I + K = S$, for some ideal K of S, implies $K = S$ [16].

Proposition 2.8. *Let S be a semiring. If I and J are two vertices of $MG(S)$ such that $I \subseteq J$ and J is small ideal of S. Then* $\deg(I) \leq \deg(J)$.

Proof . Let I and J be two vertices of $\mathrm{MG}(S)$ such that $I \subseteq J$ and J is small ideal of S. Let K be a vertex adjacent to I. So $I + K = M$, for some $M \in \max(S)$. Now, $I + K = M \subseteq J + K$. If $J + K = S$, and J small ideal, then $K = S$. Hence, $M = S$, which is a contradiction. Then $J + S \neq S$ and so $M = J + K$. Thus K is adjacent to J. Hence $\deg(I) \leq \deg(J)$. □

Theorem 2.9. *Let S be a semiring and* $\mathrm{n} > 1$, *if* $|\max(S)| = \mathrm{n} < \infty$. *Then the following hold:*

(a) There is no vertex in $\mathrm{MG}(S)$ *which is adjacent to every other vertex.*
(b) $\mathrm{MG}(S)$ *cannot be a complete graph.*
(c) If $J(S) \neq \{0\}$ then it is a cut vertex of $MG(S)$.

Proof . For (a): Since the comaximal ideals are not adjacent in $\mathrm{MG}(S)$, this proves (a).
For (b): This is a direct consequence of (a).
For (c): By Remark 2.6, the comaximal ideals of S are not adjacent in $\mathrm{MG}(S)$. By Lemma 2.3 (2), an ideal $J(S) = \bigcap_{i=1}^{n} M_i$ is adjacent to every $M_i \in \max(S)$. It can be easily seen that $J(S)$ is a cut vertex. □

Proposition 2.10. *If $\{I, J\} \in E(\mathrm{MG}(S))$ with $I, J \notin \max(S)$, then there exists a unique $M \in \max(S)$ such that $M \in N(I) \cap N(J)$.*

Proof . Suppose that $M_1, M_2 \in \max(S)$ and each of I and J are adjacent to both M_1 and M_2 in $\mathrm{MG}(S)$. Then by Lemma 2.3, $I, J \subset M_1 \cap M_2$. Since $I + J \in \max(S)$, then $M_1 = I + J = M_2$. □

Corollary 2.11. *Suppose I_1 and I_2 are two are adjacent non-maximal ideals of a semiring S, then the set $\{I_1, I_2, I_1 + I_2\}$ forms a triangle in* $\mathrm{MG}(S)$.

Proof . This is a direct consequence of Proposition 2.10. □

The next result shows that the degree of maximal ideals determines the finiteness of $\mathrm{MG}(S)$.

Proposition 2.12. *Let $S \cong S_1 \times \cdots \times S_n$, where (S_i, M_i) is a local Artinian semiring. If* $\deg(I) < \infty$, *for every $I \in \max(S)$, then* $\mathrm{MG}(S)$ *is a finite graph and S is Artinian.*

Proof . Assume that $S \cong S_1 \times \cdots \times S_n$, where (S_i, M_i) is a local Artinian semiring. So, the maximally of I gives that $I = S_1 \times S_2 \times \cdots \times S_{i-1} \times M_i \times S_{i+1} \times \cdots \times S_n$, where $1 \leq i \leq n$. Since $\deg(I)$ is finite, then $\mathrm{Id}(S_i)$ is finite. Thus, $\mathrm{MG}(S)$ is a finite graph and so S is Artinian. □

The next result gives the conditions on $\mathrm{MG}(S)$ for which S is a local semiring.

Theorem 2.13. *If $MG(S) \cong K_n$ or $MG(S) \cong K_{n,1}$, where $n \in \mathbb{Z}^+$, then S is a local semiring.*

Proof . If $MG(S) \cong K_n$, then by Theorem 2.7, S is local. Suppose that $MG(S)$ is a star with center I. If $\mathrm{MG}(S)$ consists of only one edge, then it refers to completeness case. Assume that $|\mathrm{MG}(S)| \geq 3$. If $I \notin \max(S)$, then by Lemma 2.3, $\mathrm{V}(\mathrm{MG}(S)) \backslash \{I\} = \max(S)$. Thus $I = J(S) \neq (0)$. Now, suppose that $M, T \in \max(S)$ with $M \neq T$. Obviously $(0) \neq MT \notin \max(S)$. Thus $MT = I = J(S)$. This contradicts that $|\mathrm{MG}(S)| \geq 3$. Therefore, $I \in \max(S)$. By Lemma 2.3, $\max(S) = \{I\}$. This completes the proof. □

The converse of Theorem 2.13 will be true if V(MG(S)) is a totally ordered set. We illustrate it in the following result.

Proposition 2.14. *If* V(MG(S)) *is a totally ordered set, then* MG(S) *is a star.*

Proof . Since V(MG(S)) is a totally ordered set, then MG(S) contains a vertex I which is adjacent to each other vertex. If J and H are two distinct vertices of MG(S) such that J $\neq$ I and H $\neq$ I, then either H $\subset$ J or J $\subset$ H. For both cases, J and K are not adjacent vertices. Thus MG(S) is a star with center I. □

Corollary 2.15. *For any prime number p, the graph* $\mathrm{MG}(\mathbb{Z}_{p^n})$ *is star.*

Proof . It follows from Proposition2.14. □

Now, we give the condition for which MG(S) be a complete bipartite, as follows.

Theorem 2.16. *Let $J(S) \notin \max(S) \cup \{(0)\}$. Then* $\mathrm{MG}(S) \cong \mathrm{K_{m,n}}; \mathrm{m}, \mathrm{n} \in \mathbb{Z}^+$ *if and only if* $\mathrm{Id}(S) - \max(S) \subseteq \mathrm{J(S)}$.

Proof . Suppose that $\mathrm{Id}(S) - \max(S) \subseteq \mathrm{J}(S)$.
Choose $\mathrm{V_1} = \max(\mathrm{S})$ and $\mathrm{V_2} = \{\mathrm{I} \in \mathrm{V(MG(S))} : \mathrm{I} \subseteq \mathrm{J(S)}\}$. From Lemma 2.3 , every two vertices in $\mathrm{V_1}$ are independent with respect to the graph MG(S). Since $|\mathrm{MG(S)}| \neq 1$, then $\mathrm{J(S)} \notin \max(\mathrm{S})$. Thus $\mathrm{I} + \mathrm{J} \notin \max(\mathrm{S})$ for every $\mathrm{I}, \mathrm{J} \in \mathrm{V_2}$. This means that every two vertices in $\mathrm{V_2}$ are independent with respect to the graph MG(S). On the other hand, Theorem 2.7 mentions that every $\mathrm{I} \in \mathrm{V_1}$ is adjacent to each $\mathrm{J} \in \mathrm{V_2}$, this completes the proof.

Conversely, if MG(S) is a complete bipartite with partite sets $\mathrm{W_1}$ and $\mathrm{W_2}$, we can prove that $\mathrm{W_i} = \max(\mathrm{S})$ and $\mathrm{W_j} = \{\mathrm{I} \in \mathrm{V(MG(S))} : \mathrm{I} \subseteq \mathrm{J(S)}\}$, for $\mathrm{i}, \mathrm{j} = 1, 2$ with $\mathrm{i} \neq \mathrm{j}$. This completes the proof. □

Corollary 2.17. *Let* $\mathrm{J(S)} \notin \max(\mathrm{S}) \cup \{(0)\}$. *If* MG(S) *is not a complete bipartite, then* MG(S) *is a 3-partite graph.*

Proof . Since MG(S) is not a complete bipartite, then by Theorem 2.16, $\mathrm{I} \nsubseteq \mathrm{J(S)}$, for some $I \in MG(S) \backslash \max(S) \cdot We$ set $V_1 = \max(S)$, $V_2 = \{I \in \mathrm{V}(MG(S)) : I \subseteq J(S)\}$ and $\mathrm{V_3} = \mathrm{V(MG(S))} \backslash (\mathrm{V_1 U V_2})$. It is not difficult to show that every two vertices in $\mathrm{V_i}$ are independent, for $i = 1, 2, 3$. Thus MG(S) is a 3-partite graph. □

Example 2.18. *The graph* $\mathrm{MG}(\mathbb{Z}_{24})$ *is a 3-partite graph, as the following figure shows:*

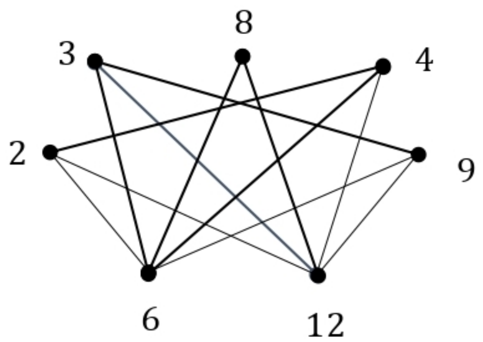

Figure 1 The graph $\mathrm{MG}(\mathbb{Z}_{24})$

The following main result determines whenever

Theorem 2.19. *Let* $\min(\mathrm{S}) \neq \emptyset$. *If* $\mathrm{V}(\mathrm{MG}(\mathrm{S})) = \min(\mathrm{S}) \cup \max(\mathrm{S})$, *then:*

1. *The graph* $\mathrm{MG}(\mathrm{S})$ *is split.*
2. *The graph* $\mathrm{MG}(\mathrm{S})$ *is perfect.*
3. *The clique number of* $\mathrm{MG}(\mathrm{S})$ *is* $\omega(\mathrm{MG}(\mathrm{S})) = \max\{|\min(\mathrm{S})|, |\min(\mathrm{S})| + 1\}$.
4. *If* $J(\mathrm{S}) \neq \{0\}$ *then* $\max(\mathrm{S})$ *contains a cut vertex of* $\mathrm{MG}(\mathrm{S})$.

Proof .

1. *Let A be the induced subgraph of* $\mathrm{MG}(\mathrm{S})$ *by* $\min(\mathrm{S})$. *Let* $\mathrm{N}, \mathrm{T} \in \min(\mathrm{S})$ *with* $\mathrm{N} \neq \mathrm{T}$. *Evidently,* $\mathrm{N} + \mathrm{T} \neq \mathrm{S}$. *If we assume that* $\mathrm{N} + \mathrm{T} \in \min(\mathrm{S})$, *then* $\mathrm{N} = \mathrm{N} + \mathrm{T} = \mathrm{T}$, *which is a contradiction. Hence* $\mathrm{N} + \mathrm{T} \in \mathrm{MG}(\mathrm{S})$. *Thus* A *is a complete graph. From Remark 2.6 , the vertices in* $\max(\mathrm{S})$ *are independent. Thus,* $\mathrm{MG}(\mathrm{S})$ *is a split graph.*
2. *Let C :* $\mathrm{I}_1 - \mathrm{I}_2 - \cdots - \mathrm{I}_{2\mathrm{n}+1} - \mathrm{I}_1$ *be an induced cycle in* $\mathrm{MG}(\mathrm{S})$ *with* $\mathrm{n} \geq 2$. *If C does not contain any maximal ideal vertex, then by (1),* $\{\mathrm{I}_1, \mathrm{I}_3\} \in \mathrm{E}(\mathrm{MG}(\mathrm{S}))$, *which is a contradiction. Let $I_1 \in$* $\max(\mathrm{S})$. *Obviously,* $\mathrm{I}_{2\mathrm{n}+1}, \mathrm{I}_2 \notin \max(\mathrm{S})$. *Then they are adjacent in* $\mathrm{MG}(\mathrm{S})$, *which is a contradiction. Now, suppose that C' is an induced odd cycle in* $\overline{\mathrm{MG}(\mathrm{S})}$ *of length* $\mathrm{n} \geq 5$. *Then C' contains at least* $\mathrm{P}, \mathrm{Q} \in \max(\mathrm{S})$ *with* $\mathrm{P} \neq \mathrm{Q}$ *such that they are not adjacent in C'. From Lemma* 2.3, *P and Q are adjacent in $\overline{MG(S)}$. This contradicts Lemma2.3. Hence, by the strong perfect graph theorem in* [6], $\mathrm{MG}(\mathrm{S})$ *is a perfect graph.*
3. *The proof follows from the first part of Theorem 2.12 in* [7].
4. *Using the same argument as in Theorem 2.19(1) , one can show that* $\mathrm{J}(\mathrm{S}) + \mathrm{I} = \mathrm{M}$ *for some $I \in \min(S)$ and $M \in \max(S)$. It is easy to see that I is not adjacent to any vertex belong to the graph induced by* $\mathrm{MG}(\mathrm{S}) - \mathrm{M}$. *This ends the proof.*

□

Example 2.20. *Consider $S = \mathbb{Z}_{18}$ as the semiring of integers modulo* 18. *The following graph shows that* $\mathrm{MG}(\mathbb{Z}_{18})$ *is a split and perfect graph. Also* $\omega(\mathrm{MG}(\mathbb{Z}_{18})) = |\min(\mathbb{Z}_{18})| + 1 = 3$.

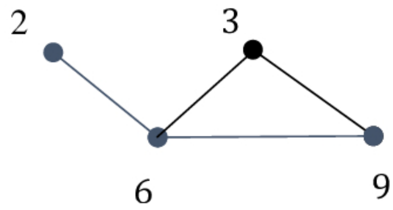

Figure 2 The graph $MG(\mathbb{Z}_{18})$

In the next result, we find the girth of MG(S).

Theorem 2.21. *Let $J(S) \neq (0)$. Then gr(MG(S)) $\in \{3, 4, \infty\}$.*

Proof . If MG(S) contains an edge $\{R, T\}$ with R and T $\notin$ max(S), then $R, T \neq R + T \in$ max(S) Thus $R + T$ is adjacent to both S and T. This means that $C : R - T - \{R + T\} - R$ is a cycle in MG(S) In this case, gr(MG(S)) $= 3$. Suppose that for every $\{I, J\} \in E(MG(S))$, either $I \in$ max(S) or $J \in$ max(S). If MG(S) does not possess any cycle, then gr(MG(S)) $= \infty$. Now, suppose that $C_n : I_1 - I_2 - \cdots - I_n - I_1$ is a cycle in MG(S) of length n. Since the maximal ideals are not adjacent in MG(S), the vertices of C are alternatively maximal and non-maximal ideals. Consequently, $J(S) \notin$ max(S). Let $I_1 \in$ max(S). From Lemma 2.3, J(S) is adjacent to each of I_1, I_3 and I_5. If $I_2 = J(S)$, then $C' : I_2 - I_3 - I_4 - I_5 - I_2$ is a cycle in MG(S). If $J(S) \neq I_2$, then $C'' : J(S) - I_1 - I_2 - I_3 - J(S)$ is a cycle in MG(S). From both cases, we have proved that gr(MG(S)) is either 3 or 4. □

The next result shows the upper bound of clique number of MG(S).

Proposition 2.22. *The clique of* MG(S) *contains in an its induced subgraph by* $\{I \in V(MG(S)) : I \subseteq M\}$, *for precisely one* $M \in$ max(S).

Proof . Let H be the clique of MG(S). Since any two maximal ideals are not adjacent in MG(S), then H has only one maximal ideal. The adjacency of every two vertices of H and Proposition 2.10 explains that there is precisely one $M \in$ max(S) such that H is a subgraph of the graph induced by $\{I \in V(MG(S)) : I \subseteq M\}$. □

Example 2.23. *Consider the following four ideals of the semiring of nonnegative integers* $\mathbb{N}$:

1. $I = 2\mathbb{N}$,
2. $J = 3\mathbb{N}$,
3. $K = \mathbb{N}\backslash\{1\}$,
4. $L = \mathbb{N}\backslash\{1, 2\}$.

Now, let $\{H_i \mid i \in \Omega\}$ be a set of all ideals of a semiring $\mathbb{N}$ *which are not identical to any one of above four ideals. In the following figure we explain that the girth* gr(MG($\mathbb{N}$)) *and the clique of* MG($\mathbb{N}$)

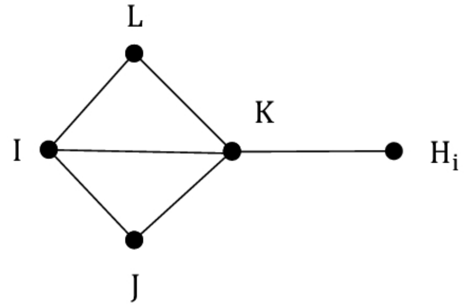

Figure 3 The graph MG($\mathbb{N}$)

Note that $\mathrm{gr}(\mathrm{MG}(\mathbb{N})) = 3$*. Also,* $\{I, J, K\}$ *and* $\{I, L, K\}$ *are the cliques with three elements in* $\mathrm{MG}(\mathbb{N})$*. The clique number of* $\mathrm{MG}(\mathbb{N})$ *is* $\omega(\mathrm{MG}(\mathbb{N})) = 3$*. It is easy to see that* $\min(\mathbb{N}) = \emptyset$ *and so this example shows that the condition "* $\min(\mathrm{S}) \neq \emptyset$*" in Theorem 2.19 is not superfluous.*

The following remark is clear.

Remark 2.24. *Let* S *be a semiring and* I, J *be two ideals of* S*. If* M *is a maximal ideal of* S*, then* $\mathrm{I} \cap \mathrm{J} \subseteq \mathrm{M}$ *implies* $\mathrm{I} \subseteq \mathrm{M}$ *or* $\mathrm{J} \subseteq \mathrm{M}$.

Lemma 2.25. *Suppose* I_1 *and* I_2 *are two ideals of a semiring* S *such that* $\mathrm{I}_1 \cap \mathrm{I}_2 \neq (0)$*, then at least one of them is non-isolated vertex in* MG(S).

Proof . Suppose I_1 and I_2 are two ideals of a semiring S such that $\mathrm{I}_1 \cap \mathrm{I}_2 \neq (0)$. Then $\mathrm{I}_1 \cap \mathrm{I}_2 \subseteq \mathrm{M}$ for some $\mathrm{M} \in \max(\mathrm{S})$. By Remark 2.24 , either $\mathrm{I}_1 \subseteq \mathrm{M}$ or $\mathrm{I}_2 \subseteq \mathrm{M}$. Without loss of generality we may take $\mathrm{I}_1 \subseteq \mathrm{M}$. Now, we have two cases either $\mathrm{I}_1 \subset \mathrm{M}$ and so I_1 is adjacent to M. Or $\mathrm{I}_1 = \mathrm{M}$ and thus I_1 is adjacent to $\mathrm{I}_1 \cap \mathrm{I}_2$. This ends the proof. □

Theorem 2.26. *If* $S = S_1 \times S_2 \times \cdots \times S_n$ $(n \in \mathbb{N})$*, where* (S_i, M_i) *is a local semiring for* $1 \leq i \leq n$*. Then the following statements are equivalent:*

(1) $\mathrm{MG}(S)$ *is complete;*
(2) $\mathrm{MG}(S_i)$ *is complete for all* $1 \leq i \leq n$.

Proof . (1) $\Rightarrow$ (2) Assume that S is a product of local Artinian semirings S_i with maximal ideals M_i. We show that $\mathrm{MG}(S_i)$ is complete. Let I, J be two non-trivial ideals of S_i, then $S_1 \times \cdots \times S_{i-1} \times I \times S_{i+1} \times \cdots \times S_n$ and $S_1 \times \cdots \times S_{i-1} \times J \times S_{i+1} \times \cdots \times S_n$ are non-trivial ideals of S. As $\mathrm{MG}(S)$ is complete, I and J are adjacent in S_i. Therefore $\mathrm{MG}(S_i)$ is complete.

(2) $\Rightarrow$ (1) Let $I = I_1 \times \cdots \times I_n, J = J_1 \times \cdots \times J_n$ be two non-trivial ideals of $S_1 \times \cdots \times S_n$. Set

$$S_I = \{i : I_i \text{ is non-trivial }\} \text{ and } S_J = \{i : J_i \text{ is non-trivial }\}.$$

If $S_I \cap S_J = \emptyset$, then I and J are adjacent. If $S_I \cap S_J \neq \emptyset$, then by assumption, for each $i \in S_I \cap S_J, I_i$ and J_i are adjacent in $\mathrm{MG}(S_i)$. Thus I and J are adjacent. So $\mathrm{MG}(S)$ is complete. □

3. Planar Property

In this section, we will investigate planar property of the maximal ideal graph. In the beginning, we find the clique number of $\mathrm{MG}(S)$.

Proposition 3.1. *Let S be a semiring. If the subgraph induced by $\{I \in V(\mathrm{MG}(S)) : I \subseteq M\}$ is planar, for every $M \in \max(S)$, then $\omega(\mathrm{MG}(S)) \in \{2, 3, 4\}$.*

Proof . The proof follows from Proposition2.22 and Koratowski's theorem [12]. □

In the next theorem, we show that $\mathrm{MG}(S)$ is a planar graph under some conditions on vertex set of $\mathrm{MG}(S)$.

Theorem 3.2. *If $V(\mathrm{MG}(S)) = \min(S) \cup \max(S)$ is finite and $|\max(S)| \leq 3$, then the graph $\mathrm{MG}(S)$ is planar.*

Proof . To show that $\mathrm{MG}(S)$ is planar, we refer to Koratowski's theorem. Since $|\max(S)| \leq 3$, then any subgraph of $\mathrm{MG}(S)$ induced by five vertices is not complete. This means that $\mathrm{MG}(S)$ does not contain any complete subgraph K_5. If we assume that $\mathrm{MG}(S)$ contains a $K_{3.3}$ with partite sets $V_1 = \{I_1, I_2, I_3\}$ and $V_2 = \{J_1, J_2, J_3\}$, then by Lemma 2.3 either $V_1 \subseteq \max(S)$ or $V_2 \subseteq \max(S)$. Assume that $V_1 \subseteq \max(S)$. Then $V_2 \subseteq \min(S)$. From Proposition 2.10, any two of J_1, J_2 and J_3 are independent. This contradicts that each minimal ideal is adjacent in $\mathrm{MG}(S)$. Thus, $\mathrm{MG}(S)$ is a planar graph. □

The next result exhibits that the planarity of $\mathrm{MG}(S)$ limits the order of $\max(S)$.

Proposition 3.3. *Let $\mathrm{J}(S) \neq (0)$. If $\mathrm{MG}(S)$ is planar graph, then $|\max(S)| \leq 4$.*

Proof . Let $\mathrm{MG}(S)$ be a planar graph. Assume by contrary that $\mathrm{MG}(S)$ has at least five distinct maximal ideals, say M, N, P, Q and T. Obviously, any one of the vertices MNP, $MNPQ$ and $MNPQT$ are non-zero ideals and adjacent to each of ideals M, N and P in $\mathrm{MG}(S)$. Therefore, $\mathrm{MG}(S)$ contains a complete bipartite graph $K_{3,3}$. This contradicts the Koratowski's theorem. Thus, $|\max(S)| \leq 4$. □

Theorem 3.4. *Let $S \cong S_1 \times S_2 \times \cdots \times S_n$, with $S_1, S_2, \ldots, S_n$ are distinct semifields. Then $MG(S)$ is planar graph if and only if $n \leq 4$.*

Proof . Let $\mathrm{MG}(S)$ be a planar graph. Suppose that $n > 4$. Obviously, $(0) \times S_2 \times \cdots \times S_n \in \max(S)$ and the sum of every two of ideals $(0) \times S_2 \times \cdots \times S_n, (0) \times (0) \times S_3 \times \cdots \times S_n, (0) \times S_2 \times (0) \times S_4 \times \cdots \times S_n, (0) \times S_2 \times S_3 \times (0) \times S_5 \times \cdots \times S_n, (0) \times S_2 \times S_3 \times S_4 \times (0) \times \cdots \times S_n$ is equal to $(0) \times S_2 \times \cdots \times S_n$. Then $\mathrm{MG}(S)$ contains a complete subgraph of order 5. This contradicts the planarity of $\mathrm{MG}(S)$. Thus, $n \leq 4$.

Conversely, let $n \leq 4$. Evidently, $E(\mathrm{MG}(S)) = \emptyset$, when $n \in \{1, 2\}$. Now, suppose that $n = 3$. Then $V(\mathrm{MG}(S))$ consists of $I_1 = S_1 \times (0) \times (0), I_2 = (0) \times S_2 \times (0), I_3 = (0) \times (0) \times S_3$, $I_4 = S_1 \times S_2 \times (0), I_5 = S_1 \times (0) \times S_3$ and $I_6 = (0) \times S_2 \times S_3$. Clearly, $\mathrm{MG}(S)$ is planar graph, when $n = 3$.

Suppose that $n = 4$. The maximal ideal vertices of $\mathrm{MG}(S)$ are $(0) \times S_2 \times S_3 \times S_4, S_1 \times (0) \times S_3 \times S_4, S_1 \times S_2 \times (0) \times S_4$ and $S_1 \times S_2 \times S_3 \times (0)$, and the other vertices are $(0) \times (0) \times S_3 \times S_4, (0) \times S_2 \times (0) \times S_4, (0) \times S_2 \times S_3 \times (0), S_1 \times (0) \times (0) \times S_4, S_1 \times S_2 \times (0) \times (0), S_1 \times (0) \times S_3 \times (0)$ $S_1 \times (0) \times (0) \times (0), (0) \times S_2 \times (0) \times (0), (0) \times (0) \times S_3 \times (0), (0) \times (0) \times (0) \times S_4$. This graph does not contain K_5. Also, for every three distinct vertices I, J and K of $\mathrm{MG}(S)$, there exists at most two vertices adjacent to each of I, J and K. Thus $\mathrm{MG}(S)$ does not contain $K_{3,3}$. In this case, $\mathrm{MG}(S)$ is a planar graph. □

For the important classes of additively regular (additively idempotent) semirings see [13], as a special case of Theorem 3.4 we obtain the following result.

Corollary 3.5. *Let S be an additively regular (or additively idempotent) subtractive semiring if $\mathrm{MG}(S)$ is planar graph then S is semisimple if and only if $S \cong D_1 \times \cdots \times D_n$ with the semifields $D_1, \ldots, D_n$ and $n \leq 4$.*

Proof . By Theorem 3.4 and [16, Theorem 4.14]. □

If I is an ideal of a semiring S, then an idempotent $g + I \in S/I$ can be lifted mod I if there is an idempotent $e \in S$ with $e + I = g + I$.

We now give the following definition similar to [17, P. 356].

Definition 3.6. *A semiring S is called semiperfect in case $S/J(S)$ is semisimple and every idempotent of $S/J(S)$ can be lifted mod $J(S)$. Clearly each local semiring is semiperfect.*

Theorem 3.7. *Let S be a semiring such that $|\max(S)| < \infty$ and $\omega(\mathrm{MG}(S)) < \infty$. Then the following holds.*

1) *S is semiperfect.*
2) *If S is a ring then $S = S_1 \times S_2 \times \cdots \times S_r$ where $r \geq 2, (S_i, M_i)$ is a local ring and $\mathrm{MG}(S)$ is finite.*
3) *If $V(\mathrm{MG}(S)) = \min(S) \cup \max(S)$, then S is Artinian.*
4) *$\omega(\mathrm{MG}(S)) \leq \max\{(\prod_{j=i,j\neq i}^{r} |\mathrm{Id}(S_\mathrm{i})|) - 1 : 1 \leq \mathrm{i} \leq \mathrm{r}\}$.*

Proof . (1) Since $\max(S)$ is finite. Therefore, $S/J(S)$ is semisimple. Now, we show that idempotent of $S/J(S)$ can be lifted. Let $g + J(S)$ be a nonzero idempotent of $S/J(S)$. Clearly $g \notin J(S)$, so $g^n \notin J(S)$ for each $n \in \mathbb{N}$. Hence $Sg \supseteq Sg^2 \supseteq Sg^3 \supseteq \cdots$ is a descending chain of proper ideals of S (if $Sg^n = S$, then $g + J(S) = 1 + J(S)$). Since $\omega(\mathrm{MG}(S)) \leq 4$ by Proposition3.1, so there exists $n \in \mathbb{N}$ such that $Sg^n = Sg^{n+1}$. Thus $g^n = g^{n+1}s$ for some $s \in \mathrm{S}$. Let $z = g^n s^n$. Then $z = (g^{n+1}s)\, s^n = g^{n+1}s^{n+1}$. This implies that $z = z^2$ and $g + J(S) = g^n + J(S) = g^{n+1}s + J(S) = (g^{n+1} + J(S))\,(s + J(S)) = (g + J(S))(s + J(S)) = gs + J(S)$. Thus, $g + J(S) = (g + J(S))^2 = (g + J(S))^n = (gs + J(S))^n = z + \mathrm{J}(S)$. Hence S is semiperfect.

(2) Suppose that S is a ring. By [17, Theorem 23.11], $S = S_1 \times S_2 \times \cdots \times S_r$, where (S_i, M_i) is a local ring for $1 \leq i \leq r$. Now, we will show that $\mathrm{MG}(S)$ is finite. It suffices to show that $\mathrm{Id}\,(S_i)$ is finite for all $1 \leq i \leq r$. Suppose, on the contrary, $\mathrm{Id}\,(S_i)$ is infinite for some $1 \leq i \leq r$. Put

$$\mathbb{E} = \{S_1 \times S_2 \times \cdots \times S_{i-1} \times F \times S_{i+1} \times \cdots \times S_r \mid F \in \mathrm{Id}\,(S_i)\}\,.$$

Then $\mathbb{E}$ is an infinite clique in $\mathrm{MG}(S)$, which is a contradiction. Thus $\mathrm{Id}\,(S_i)$ is finite for all $1 \leq i \leq r$. Hence $\mathrm{Id}(S)$ is finite and so $\mathrm{MG}(S)$ is finite. (3) Since $\omega(\mathrm{MG}(S)) < \infty$, by Theorem 2.19(3), then $|\min(S)| < \infty$. So, we have $\mathrm{Id}(S)$ is finite. Therefore, S is artinian. (4) Put

$$C_j = \{I \leq S : I = I_1 \times I_2 \times \cdots \times I_{j-1} \times M_j \times I_{j+1} \times \cdots \times I_r, I_t \in \mathrm{Id}\,(S_t)\,,\ \text{for } 1 \leq t \neq j \leq r\},$$

for each $1 \leq j \leq r$. As $0 \times 0 \times \cdots \times M_j \times \cdots \times 0 \subseteq I$ for each I $\in C_j$. By Proposition 2.22, we have that the clique of MG(S) contains in an its induced subgraph by C_j. Since $|C_j| = \left(\prod_{i=1,j\neq i}^{r} |\mathrm{Id}\,(S_i)|\right) - 1$, therefore

$$\omega(MG(S)) \leq \max\left\{\left(\prod_{j=1,j\neq 1}^{r} |\mathrm{Id}\,(S_i)|\right) - 1\right\}.$$

□

Theorem 3.8. *Let S be a semiring such that $J(S) \neq (0)$. If $MG(S)$ is a planar graph, then the following holds.*

1) $\chi(\mathrm{MG}(S))$ *is finite;*

2) $\chi(\mathrm{MG}(S)) = \omega(\mathrm{MG}(S))$.

Proof . For(1) By Propositions 3.1 and 3.3, $\chi(\mathrm{MG}(S))$ is finite.

For(2) It is known that $\omega(\mathrm{MG}(S)) \leq \chi(\mathrm{MG}(S))$. Without loss of generality we may take that $\{I_1, I_2, I_3, I_4\}$ to be a maximal clique with four elements in MG(S), by Propositions 3.1 and 3.3. Where for all $1 \leq i \leq 4, I_i \subseteq M_j$ for only one vertex M_j belong to max(S) such that max(S) has at most four element which are $\{M_1, M_2, M_3, M_4\}$. Since any two vertices of the clique are not adjacent if we consider they subsets of M_k, for $k \neq j$ by Proposition 2.10. So for all $k \neq j$, then we can coloured by at most 4-colours the vertices are adjacent to M_k. Then we can 4-colour the vertices of MG(S). Thus, $\chi(\mathrm{MG}(S)) = \omega(\mathrm{MG}(S))$. □

4. The Connectivity of MG(S)

We begin this section with the next result.

Theorem 4.1. *Let $S \cong S_1 \times S_2 \times \cdots \times S_n$, where (S_i, P_i) is a local semiring with* MG(S) *is a non-empty graph. Then every two vertices are disconnected if and only if $S = S_1 \times S_2$, where S_1 and S_2 are semifields.*

Proof . If $S = S_1 \times S_2$ where S_1 and S_2 are semifields, then $V(\mathrm{MG}(S)) = \{(0) \times S_2, S_1 \times (0)\}$. Evidently, $(0) \times S_2$ and $S_1 \times (0)$ are not adjacent in MG(S).

Conversely, suppose that any two vertices are disconnected. Since S is a finite non-local semiring, then $S \cong S_1 \times S_2 \times \cdots \times S_n$, where (S_i, P_i) is a local semiring for every $i = 1, 2, \ldots, n$ and $n \geq 2$. If $P_1 \neq (0)$, then $(P_1 \times S_2 \times \cdots \times S_n) + (P_1 \times P_2 \times \cdots \times S_n) \in \max(S)$, which is a contradiction. Hence $P_1 = (0)$. Similarly, $P_2 = P_3 = \cdots = P_n = (0)$. Thus $S_1, S_2, \ldots, S_n$ are semifields. If $n \geq 3$, then $(0) \times S_2 \times \cdots \times S_n$ and $(0) \times (0) \times S_3 \times \cdots \times S_n$ are adjacent in MG(S), which is a contradiction. Hence, $n = 2$. □

In the next main result, we investigate the connectivity of MG(S).

Theorem 4.2. *If every two distinct maximal ideals of S have a non-zero intersection, then* MG(S) *is connected with* $\mathrm{diam}(\mathrm{MG}(S)) \leq 3$.

Proof . Let $K, L \in V(\mathrm{MG}(S))$ with $K \neq L$. If $\{K, L\} \in E(\mathrm{MG}(S))$, then they are connected. Suppose that $\{K, L\} \notin E(\mathrm{MG}(S))$. So, either $K + L = S$ or $K + L \subset P$, for some $P \in \max(S)$. If $K + L \subset P$, then by Lemma 2.3, $P_2 : -P - L$ is a path in MG(S). If $K + L = S$, then at least one of K and L is a maximal ideal and neither $K \subset L$ nor $L \subset K$. Assume that $K \in \max(S)$. If

$L \in \max(S)$, again by Lemma 2.3, $P'_2 : K - K \cap L - L$ is a path in $\mathrm{MG}(S)$. Let $L \notin \max(S)$. Then there exists $M \in \max(S)$ such that L is adjacent to M. If $M = K$, then K is adjacent to L. Let $M \neq K$. Then $P_3 : K - K \cap M - M - L$ is a path in $\mathrm{MG}(S)$. From each case, we have shown that K and L are connected and $d(K, L) \leq 3$. Thus $\mathrm{MG}(S)$ is connected with $\mathrm{diam}(\mathrm{MG}(S)) \leq 3$. □

The next result is clear.

Corollary 4.3. *Let S be a semiring such that $J(S) \neq (0)$. Then $\mathrm{MG}(S)$ is connected with $\mathrm{diam}(\mathrm{MG}(S)) \leq 3$.*

Remark 4.4. *For a graph G, it is well-known that if G contains a cycle, then $\mathrm{gr}(\mathrm{G}) \leq 2\,\mathrm{diam}(\mathrm{G}) + 1$. Thus, if S is any semiring with $\mathrm{J}(S) \neq (0)$ and $\mathrm{MG}(S)$ contains a cycle. Then by Corollary 4.3, $\mathrm{gr}(\mathrm{MG}(S)) \leq 7$.*

Note that the graph $\mathrm{MG}(S)$ may not be connected, whenever two distinct maximal ideals of S have a zero intersection. As in the following example.

Example 4.5. *(1) Consider $\mathbb{Z}_{15}$ as a semiring. Clearly, the graph $\mathrm{MG}(\mathbb{Z}_{15})$ is disconnected.*

(2) Consider $S = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a semiring. It is clear that $\max(S) = \{0 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus 0\}$, $\mathrm{J}(S) = 0$ and $\mathrm{MG}(S)$ is disconnected. See that $\mathrm{V}(\mathrm{MG}(S)) = \{0 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus 0\}$.

A semiring S is semidomain if $ab = ac$ implies $b = c$ for all $b, c \in S$ and all non-zero $a \in S$, or equivalent each non-zero principal ideal of S is invertible in S (see[4],[18]) . We say that a semidomain S is said to be a Dedekind semidomain if every non-zero ideal of S is invertible in S (see [3],[18]). Next, we turn to the following result.

Proposition 4.6. *If S is a Noetherian Dedekind semidomain in which every two distinct maximal ideals of S have a non-zero intersection, then $\mathrm{diam}(\mathrm{MG}(S)) \leq 2$.*

Proof . From Theorem 4.2, $d(P, Q) \leq 2$, for every $P, Q \in V(\mathrm{MG}(S))$ with $P \neq Q$, except for the possibility that $P + Q = S$ and $\{P, Q\} \nsubseteq \max(S)$. Now, suppose that $P + Q = S$ and $P \in \max(S)$ but $Q \notin \max(S)$. Then there exits $T \in \max(S)$ such that Q is adjacent to T. Since S is a Noetherian Dedekind semidomain, then $Q + (T \cap P) = (Q + T) \cap (Q + P) = T \cap S = T$ by [18] . Hence Q is adjacent to $T \cap P$. Since P is also adjacent to $T \cap P$, then $d(P, Q) \leq 2$. Thus, $\mathrm{diam}(\mathrm{MG}(S)) \leq 2$. □

Proposition 4.7. *Let S be a Dedekind semidomain. Then I and J are adjacent in $\mathrm{MG}(S)$ if and only if $I + J$ is a prime ideal in S.*

Proof . By Theorem 2.21 in [18] , each nonzero prime ideal of a Dedekind semidomain is maximal, this completes the proof. □

The following result is an immediate consequence of the proof of Theorem 4.2 .

Proposition 4.8. *If S is a semiring in which every two distinct ideals of S are non-comaximal ideals. Then $\mathrm{diam}(\mathrm{MG}(S)) \leq 2$.*

The next result discovers the characterizations of the cut-vertices of $\mathrm{MG}(S)$.

Theorem 4.9. *Suppose that every two distinct maximal ideals of S have a non-zero intersection. If L is a cut-vertex of $\mathrm{MG}(S)$, then $L = P \cap Q$, for some $P, Q \in \max(S)$.*

Proof . If $L \in \max(S)$, then by putting $P = Q = L$, the proof will be completed. Now, suppose that $L \notin \max(S)$. Let J and K be two vertices in different components of $\mathrm{MG}(S)$ - L. We have three cases:
Case1: If $J, K \in \max(S)$, then $J \cap K \in N(J) \cap N(K)$. Since L is a cut-vertex of $\mathrm{MG}(S)$, then $L = J \cap K$.
Case 2: If $J \in \max(S)$ and $K \notin \max(S)$, then $K \in N(M)$, for some $M \in \max(S)$. Since $J \cap M$ is adjacent to J and M, then $L = J \cap M$.
Case3: If $J, K \notin \max(S)$, then $P \in N(J)$ and $Q \in N(K)$, for some $\mathrm{P}, \mathrm{Q} \in \max(S)$ such that P and Q are adjacent to J and K, respectively. Since L is a cut-vertex, then $P \neq Q$. By the same way of Case 2, we obtain that $L = P \cap Q$. □

Proposition 4.10. *Let S be a semiring with $\mathrm{MG}(S)$ connected and every two distinct maximal ideals of S have a non-zero intersection. Then each non-maximal ideal of S is not a cut vertex.*

Proof . Let I be a non-maximal ideal of S. Suppose, on the contrary, I is a cut vertex of $\mathrm{MG}(S)$, so $\mathrm{MG}(S)\backslash\{I\}$ is not connected. Thus there exist vertices J, K such that I lies on every path from K to J. We have three cases:
Case1: If $J, K \in \max(S)$. So $J \cap K \neq 0$. It is clear that $J - J \cap K - K$ is a path in $\mathrm{MG}(S)\backslash\{I\}$, a contradiction.
Case 2: If $J \in \max(S)$ and $K \notin \max(S)$, then $K \subset P$ for some $P \in \max(S)$. It is clear that $J - J \cap P - P - K$ is a path in $\mathrm{MG}(S)\backslash\{I\}$, which is a contradiction.
Case3: If $J, K \notin \max(S)$, then $J \subset Q$ and $K \subset M$ for some $Q, M \in \max(S)$. It is clear that $J - Q - Q \cap P - P - M$ is a path in $\mathrm{MG}(S)\backslash\{I\}$, a contradiction. So I is not a cut vertex of $\mathrm{MG}(S)$. □

In the next main result, we find the radius of $\mathrm{MG}(S)$.

Theorem 4.11. *Let $J(S) \neq (0)$. If $|\max(S)| \geq 2$, then $\mathrm{rad}(\mathrm{MG}(S)) = 2$.*

Proof . From Lemma2.3, $d(J(S), K) = 1$, for every $K \in \max(S)$. Since every vertex $I \notin \max(S)$ is adjacent to a vertex in $\max(S)$, then $d(J(S), I) \leq 2$. Assume that $P, Q \in \max(S)$ with $P \neq Q$. If PQ is adjacent to $J(S)$, then $J(S) + PQ = P$, for some $P \in \max(S)$. Since $J(S) + PQ \subseteq P, Q$, then $P = P = Q$. This contradicts that $P \neq Q$. Hence, PQ is not adjacent to $J(S)$. Thus the eccentricity of $J(S)$ is $e(J(S)) = 2$. If there exists $\mathrm{I} \in \mathrm{V}(\mathrm{MG}(S))$ with $e(I) = 1$, then I is adjacent to each vertex $J \in \max(S)$. Clearly, $I \notin \max(S)$. Since PQ is not adjacent to $J(S)$, for every $P, Q \in \max(S)$ with $P \neq Q$, then neither $I = J(S)$ nor $I = PQ$. Thus $I \nsubseteq J(S)$. Hence, $\mathrm{MG}(S)$ contains a $P \in \max(S)$ which is adjacent to I. This contradicts that $\mathrm{e(I)} = 1$. Therefore, $J(S)$ has the minimum eccentricity over all vertices of $\mathrm{MG}(S)$. Thus, $\mathrm{rad}(\mathrm{MG}(S)) = e(J(S)) = 2$. □

Proposition 4.12. *Let S be a semiring. Then $\mathrm{MG}(S)$ is a totally disconnected graph if and only if S has no proper non-maximal ideal.*

Proof . Suppose that S has no non-maximal proper ideal. Since any comaximal ideals of S are not adjacent in $\mathrm{MG}(S)$, so $\mathrm{MG}(S)$ is a totally disconnected graph. Conversely, suppose that $\mathrm{MG}(S)$ is a totally disconnected graph. Suppose I is a non-maximal proper ideal of S. So, I is a adjacent to maximal ideal of $\mathrm{MG}(S)$ by Lemma 2.3 (1), which is a contradiction, as needed. □

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