Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 1115-1121
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23061.2471

On subgroups of the unitary group especially of degree 2

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The point of the current investigation is to research one of the extremely significant groups exceedingly associated with the classical group which is called the special unitary groups $SU_2(K)$ particularly of degree 2. Let K be a field of characteristic, not equal 2, our principal objective that to depicting subgroups of $SU_2(K)$ over a field K contains all elementary unitary transvections.

Keywords: Unitary group, special unitary group, unitary transvection
2010 MSC: Primary 14Lxx; Secondary 20G40.

1. Introdaction

Bashkirov in [3] described subgroups of the special linear group SL_2 for arbitrary (infinite) fields of degree 2, and afterward by his articles in ([6], [7], [8]), portrayed subgroups of GL the general linear group of degree 4, degree 7, and degree 2 respectively over various fields. Sabbar in [18] extra some of the consequences of Bashkirov's outcomes when characterized subgroups of $PSL_2(K)$ over a field K of degree 2, under his supervision. In the current investigation, the essential request wich dependent on past investigations and identified with portray subgroups of $SU_2(K)$ particularly of degree 2.

L. E. Dickson's book [11] deliberated the generations of $SL_2(p^r)$ over the field of p^r of order p and obtained strong classical results. Dickson's theorem has been utilized to demonstrate numerous significant and fascinating consequences of a finite group theory. For instance, [24], utilized the previous theorem to distinguish the irreducible subgroups of the linear groups generated by transvections. In

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Received: February 2021 *Revised:* April 2021

[4] characterized the irreducible subgroups of linear groups generated by transvections containing a root k-subgroup, where K is algebraic over k and k is a subfield of K.

The generating set is formative by matrices for $SL_2(K)$, and $SU_2(K)$ which are recognized as transvections or most properly elementary transvections and elementary unitary transvections respectively. In [16] depicted subgroups of the SL_n containing the SU_n, so in [**?**]pecial orthogonal group. In [22] demonstrated overgroups of $SU_n(K)$ in $GL_n(K)$, so in [23] confirmed analogous result for the unitary group in $GL_2(K)$. In [5] described the subgroups of $GL_n(K)$ containing the SU_n over the skew field of quaternions. There are loads of studies that give us expanding conception about unitary transvections see, for example ([14], [1], [2]).

Definition 1.1. *H is a normal subgroup in the group G if $aH = Ha$ for all $a \in G$. On the other hand,*

$$aHa^{-1} \subseteq H.$$

Definition 1.2. *Let S_1 and S_2 be subgroups of the group F. Then S_1 is said to be a Conjugate of S_2 if there exists an $a \in F$ such that $S_1 = aS_2a^{-1}$.*

Lemma 1.3. *([3]) If α is an algebraic element over an infinite field $k \neq GF(3)$ then the group are generated by all matrices*

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \alpha r & 1 \end{pmatrix}$$

synchronizes with the group $SL_2(k(\alpha r))$.

Note these matrices are called elementary transvections. Now the next lemma extremely significant of the present paper, where the author has been achieved some results concerning the linear group.

Lemma 1.4. *([19]) Let K be a field has characteristic not equalize 2. If H is a normal subgroup of $SL_2(K)$ contains an elementary transvection $B_{12}(\lambda)$ or $B_{21}(\lambda)$, then $H = SL_2(K)$.*

The current investigation has been utilized the past lemma to portray subgroups of $SU(2, K)$ that contain an elementary unitary transvection. When K be a finite field of complex numbers such that $|K|$ great than 9, and K_0 is a finite field of real numbers such that $|K_0| \geq 4$. The fundamental outcome we endeavor to investigate is as per the following

Theorem 1.5. *Let V is a hyperbolic plan with Witt index $\upsilon \geq 1$, K be a finite field of characteristic $\neq 2$, and let M be a normal subgroup of $SU(2, K)$. If M contains all elementary unitary transvections then $M = SU(2, K)$.*

Definition 1.6. *A complex nonsingular square matrix A is said to be unitary by*

$$\overline{A}^T A = A^{-1}A = AA^{-1} = E \text{ (identity matrix)}$$

the subsequent equivalences hold

$$A \text{ is unitary} \Leftrightarrow A^{-1} = \overline{A}^T \Leftrightarrow \overline{A}^T A = I$$

Let be a matrix X is associated with a nondegenerate Hermitian form$B$. Then $X = \overline{X}^T$, and the isometry group of B ($U(n, B)$ comprising of all invertible matrices P which fulfills $\overline{P}^t AP = A$.)

The set of all unitary group is defined

$$U(n,K)=\{A\in GL(n,K):\overline{A}^{T}A=A\overline{A}^{T}=I_n\}$$

The $SU(n,K)$ is the subgroup of $U(n,K)$ consisting of all elements of the unitary group which has determinant 1.

$$SU(V,h)=U(V,h)\cap SL(V)$$

$$SU(n,K)=\{A\in SL(n,K):\overline{A}^{T}A=A\overline{A}^{T}=I_n,\ \det A=1\}$$

In general, the complex matrices of the general unitary group has 2-dimension $GU_2(\mathbb{C})$ over a field $\mathbb{C}$ (complex number) has the form

$$GU_2(\mathbb{C})=\left\{U=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL_2(\mathbb{C}):a,b,c,d\in\mathbb{C},\ \overline{U}^{T}U=I_2\right\}$$

so the real matrix of the special unitary group has 2-dimension $SL_2(\mathbb{C})$ over a field $\mathbb{C}$ has the form

$$SU_2(\mathbb{C})=\left\{U=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL_2(\mathbb{C}):a,b,c,d\in\mathbb{C},\ \overline{U}^{T}U=I_2,\ \det U=1\right\}$$

For example, some subgroups belong to the general unitary group $GU_2(\mathbb{C})$ within the same time these subgroups belong to $SU_2(\mathbb{C})$ over a field $\mathbb{C}$, for instance.

$$U_1=\left\{\begin{pmatrix} \cos\theta & -i\sin\theta\\ -i\sin\theta & \cos\theta\end{pmatrix},\theta\in\mathbb{C}\right\},\quad U_2=\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}}\\ i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\right\}$$

Let A is a matrix of the $SU(2,K)$ by form,

$$A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$$

Where $a\overline{a}+b\overline{b}=1$ (norm of the first row) and $a\overline{c}+b\overline{d}=0$ orthogonality condition (for the two- row vectors) implies for some scalar λ, we have $\overline{c}=-\lambda b,\ \overline{d}=\lambda a$. Therefore the determinant condition gives

$$detA=ad-bc=\lambda(a\overline{a}+b\overline{b})=1,\ \text{where } \lambda=1$$

The formula of matrices for $SU(2,K)$ as subsequent

$$A=\begin{bmatrix} a & b\\ -\overline{b} & \overline{a}\end{bmatrix}\ \text{with } a\overline{a}+b\overline{b}=1$$

The scalar transformation aI is in $U(n,K)$ if and only if $a\overline{a}=1$ Thus the group $PU(n,K)$ of collimations of $P(V)$ induced by $U(n,K)$ is isomorphic to

$$U(n,K)/\{a1|\ a\overline{a}=1\}$$

The group $PSU(n,K)$ of collimations of $P(V)$ induced by $SU(n,K)$ is isomorphic to

$$SU(n,K)/\{a1|\ a\overline{a}=1\ \text{and } a^n=1\}.$$

2. Preliminary Resuls

The formulation of a linear transvection in $SL(n,K)$ is a map

$$: v \mapsto v + \theta(v).u,$$

When u is a non-zero vector in V and θ is a linear form on V with $\theta(u) = 0$. The commutative subgroup of $SL(n,K)$ is generated by all transvections for any pair dimension 1 and $n-1$. A linear transvection given above it is lie in $SU(n,K)$ if and only if u is isotropic and $\theta(v) = \lambda(u,v)$ for some $\lambda \in K^*$ such that $\lambda = -\overline{\lambda}$. Unitary transvection exists if Witt index ν great than zero or $\nu \geq 1$ and then are of the form.

$$: v \mapsto v + a\beta(v,u)u,$$

Where $a \in K$ is an arbitrary symmetric element that satisfies $a + \overline{a} = 0$ and u is an arbitrary isotropic vector. Conversely, every transvection of this form is in the unitary group. In [15], proved the following.

Proposition 2.1. *If $n \geq 2$ then, except $n = 3$ and $|K| = 4$, the special unitary group $SU(n,K)$ is generated by hyperbolic rotation, i.e, $R = SU(n,K)$.*

The following lemma has vital on the construction of subgroups of unitary groups in [12], supposes that $n = 2$, $\upsilon \geq 1$, and S the set of the symmetric elements. Let A be the subgroup of $U(2,K)$ generated by unitary transvection as a transform vuv^{-1}, it is clear that A is a normal subgroup of the unitary group $U(2,K)$. There is a basis of vector space V consisting of 2 isotropic vector e_1, e_2 such that $B(e_1, e_2) = 1$, the elementary unitary transvection of vector e_2 have matrices of the type

$$\beta(\gamma) = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

the elementary unitary transvection of vector e_1 have matrices of the type

$$C(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

where $\gamma,\ \lambda \in S$
By above information Dieudonne in [12], proved the following lemma.

Lemma 2.2. *Let $n = 2$ and witt index $\upsilon \geq 1$. Than the subgroup of the unitary group $U(2,K)$ is generated by the transvection $\beta(\gamma),\ C(\lambda)$.*

In [2] introduced a definition of elementary unitary transvections for n is an event such that $n \geq 2$. Therefore, if $n = 2$, then $SU(2,K)$ is generated by two elementary unitary transvections. By Lemma 1.3, and Lemma 2.2, conclude the following lemma.

Lemma 2.3. *Let $n = 2$ with Witt index $\upsilon \geq 1$. If $t_{12}(\alpha)$, and $t_{21}(\eta)$, $(\alpha, \eta \in K)$ two elementary unitary transvections, then the subgroups of $SU(2,K)$ is generated by these transvections i.e.*

$$SU(2,K) = \langle t_{12}(\alpha), t_{21}(\eta) \rangle$$

3. Proof the main result and discussion

Proof. If M contains an elementary unitary transvection $E_{21}(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$, then M contains the inverse of elementary unitary transvection.

$$E_{12}(\lambda)^{-1} = E_{12}(-\lambda) = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}$$

Let S be an element of $SU(2,K)$, when

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Thus, the product of the conjugate is

$$\begin{aligned} SE_{21}(\lambda)S^{-1} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} \\ &= E_{12}(\lambda)^{-1} = E_{12}(-\lambda) \end{aligned}$$

Now, we want to show that M contains every elementary unitary transvection. Assume that $E_{12}(\lambda) \in M$ for some $\lambda \in K^*$, and also that if $r \in K_0^*$ with $r\overline{r} = 1$, such that

$$A = \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix} \in SU(2,K), \text{that implies } \overline{A}^T = \begin{bmatrix} \overline{r} & 0 \\ 0 & \overline{r}^{-1} \end{bmatrix} = A^{-1}$$

by definition of the unitary group.Thus, the product of the conjugate is

$$\begin{aligned} AE_{12}(\lambda)A^{-1} &= \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \overline{r} & 0 \\ 0 & \overline{r}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} r & r\lambda \\ 0 & r^{-1} \end{bmatrix}\begin{bmatrix} \overline{r} & 0 \\ 0 & \overline{r}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} r\overline{r} & \lambda r\overline{r}^{-1} \\ 0 & (r\overline{r})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \lambda r\overline{r}^{-1} \\ 0 & 1 \end{bmatrix} \\ &= E_{12}(\lambda r\overline{r}^{-1}) \end{aligned}$$

These conjugates are also in M. Since M is a normal subgroup of $SU(2,K)$, therefore $E_{12}(\lambda r\overline{r}^{-1}) \in M$.

Now assume that $E_{12}(\lambda n\overline{n}^{-1}) \in M$ for some $n \in K_0^*$ with $n\overline{n} = 1$. Since M is a group, then every element of M has an inverse in M. The inverse of $E_{12}(\lambda n\overline{n}^{-1})$ is equal to $E_{12}(\lambda n\overline{n}^{-1})^{-1} = E_{12}(-\lambda n\overline{n}^{-1})$

$$\begin{aligned} E_{12}(\lambda n\overline{n}^{-1})E_{12}(-\lambda n\overline{n}^{-1}) &= \begin{bmatrix} 1 & \lambda n\overline{n}^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\lambda n\overline{n}^{-1} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\lambda n\overline{n}^{-1} + \lambda n\overline{n}^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \end{aligned}$$

Hence, the product

$$
\begin{aligned}
E_{12}(\lambda r\overline{r}^{-1})E_{12}(-\lambda n\overline{n^{-1}}) &= \begin{bmatrix} 1 & \lambda r\overline{r}^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\lambda n\overline{n}^{-1} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\lambda n\overline{n}^{-1}+\lambda r\overline{r}^{-1} \\ 0 & 1 \end{bmatrix} \\
&= E_{12}(-\lambda n\overline{n}^{-1}+\lambda r\overline{r}^{-1}) \\
&= E_{12}\lambda(-n\overline{n}^{-1}+r\overline{r}^{-1})
\end{aligned}
$$

is also in M. So, r and n can be chosen to be any elements in K_0^*, and we can show that all the elements in K can be represented as $\lambda(-n\overline{n}^{-1}+r\overline{r}^{-1})$. Since all the elements of K are of the form $-n\overline{n}^{-1}+r\overline{r}^{-1}$, they are also of the form $\lambda(-n\overline{n}^{-1}+r\overline{r}^{-1})$, and thus, M contains all the elementary unitary transvection $E_{12}(\omega)$. M also contains elementary unitary transvections $E_{21}(\omega)$ where $\omega \in K$. In this case, since M contains all elementary unitary transvections, than we obtain $M = SU(2, K)$.which finishes the confirmation of the theorem.

Through the previous consequence, has been accomplished the $SU_2(K)$ is generated by elementary unitary transvections. In [13] depicted the conjugacy classes of fixed point free elements in $GL_{2n}(K)$, $SL_{2n}(K)$, $PGL_{2n}(K)$, *and* $PSL_{2n}(K)$. In [20] we portrayed an essential component of the posterior investigation called a projective transvection, so in [21] has been described subgroups of the $PSL_2(K)$ that contains a projective root subgroup.

Now let Z be the center of $SU_2(k)$ the matrix g belongs to the Z as the form αI_n such that α is an element of k and $\alpha^n = 1$. On the other hand, the subgroup of all matrix αI_2 is the center of $SU_2(k)$ and $\alpha^2 = 1$, I_2 is an 2×2 identity matrix. Presume k has characteristic not equalize 2 the equation $\alpha^2 = 1$ has precisely two roots, ± 1, and subsequently the center of $SU_2(k)$ is the subgroup

$$Z = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \{\pm I_2\}$$

Let Z be the center of $SU_2(K)$. We knew $SU_2(K)/Z$ is the projective special unitary group and hZ the coset of $PSU_2(K)$, when $h \in SU_2(K)$, at that point, we can finish up as a prompt outcome of Theorem 1.5 by the accompanying outcomes.

Theorem 3.1. *Let V is a hyperbolic plan with Witt index $\upsilon \geq 1$, K be a finite field of characteristic $\neq 2$, and let W be a normal subgroup of $PSU(2, K)$. If W contains all projective unitary transvections, then $W = PSU(2, K)$.*

4. Conclusion

By existing investigation, are expanding our realization of transvection and unitary transvection. These parts urged us to portrayed subgroups that contains all elementary unitary transvections of $SU_2(K)$ over a field K.

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