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Asymptotic behavior of generalized quadratic mappings

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Abstract

We show in this paper that a mapping f satisfies the following functional equation

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) = 2^d \sum_{i=1}^{d+1} f(x_i),$$

if and only if it is quadratic. In addition, we investigate generalized Hyers-Ulam stability problem for the equation, and thus obtain an asymptotic property of quadratic mappings as applications.

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1. Introduction

The following stability problem had been formulated by Ulam [22] concerning the stability of group homomorphisms. Thus one can ask stability of general functional equations as follows: if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality must be close to the solutions of the given equation? If the answer is affirmative, we would say that a given functional equation is stable [15]. Gruber [7] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types.

We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [23] used a stability property of the functional equation $f(x-$

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$y)+f(x+y)=2f(x)$ to prove a conjecture of Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials.

Now, a square norm on an inner product space satisfies the important parallelogram equality $\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2)$ for all vectors x,y. If $\triangle ABC$ is a triangle in a finite dimensional Euclidean space and I is the center of the side $\overline{BC}$, then the following identity $\|\overrightarrow{AB}\|^2+\|\overrightarrow{AC}\|^2=2(\|\overrightarrow{AI}\|^2+\|\overrightarrow{CI}\|^2)$ holds for all vectors A,B and C. The following functional equation which was motivated by these equalities

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y) \tag{1.1}$$

is called a quadratic functional equation, and every solution of the equation (1.1) is said to be a quadratic mapping. The quadratic functional equation and several other functional equations are useful to characterize inner product spaces [1, 2, 16, 20].

Skof [21] was the first author to solve the Ulam problem for additive mappings on a restricted domain. Jung [9] and Rassias [14] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. The stability problems of several functional equations have been extensively investigated by a number of authors [4, 5, 8, 11, 17] and there are many interesting results concerning this problem [18, 19].

On the other hand, Jung [10] and Bae et.al. [3] have generalized the equation (1.1) to the equation of 3-variables

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)=4f(x)+4f(y)+4f(z), \tag{1.2}$$

which was motivated by a parallelepiped equality

$$\|x+y+z\|^2+\|x-y+z\|^2+\|x+y-z\|^2+\|x-y-z\|^2=4\|x\|^2+4\|y\|^2+4\|z\|^2,$$

for all vectors x,y and z in an inner product space. And then they have investigated the general solution and stability problem for the functional equation. Now we are going to extend the equations (1.1) and (1.2) to a more generalized equation with $(d+1)$-variables. For this purpose, we employ the operator $\biguplus_{x_2} f(x_1)$, which is defined in [13] as follows

$$\biguplus_{x_2} f(x_1)=f(x_1+x_2)+f(x_1-x_2)$$

for a given mapping $f:E_1\to E_2$ between vector spaces. Similarly, we define

$$\biguplus_{x_2,x_3}^{2} f(x_1)=\biguplus_{x_3}\Big(\biguplus_{x_2} f(x_1)\Big)$$

and inductively

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1)=\biguplus_{x_{d+1}}\Big(\biguplus_{x_2,\cdots,x_d}^{d-1} f(x_1)\Big)$$

for all natural number d. Then it follows from definition that

$$\biguplus_{x_2,x_3}^{2} f(x_1)=\biguplus_{x_3,x_2}^{2} f(x_1),\qquad \biguplus_{x_2,\cdots,x_{k+1},\underbrace{0,\cdots,0}_{d-k}}^{d} f(x_1)=2^{d-k}\biguplus_{x_2,\cdots,x_{k+1}}^{k} f(x_1),\quad\text{and} \tag{1.3}$$

$$\biguplus_{x_2,\cdots,x_d}^{d-1} f(x_1+x_{d+1})+\biguplus_{x_2,\cdots,x_d}^{d-1} f(x_1-x_{d+1})=\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1).$$

In [12], the general solution of the following functional equation

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) + 2^d(d-1)\sum_{i=1}^{d+1} f(x_i) = 2^{d-1}\sum_{1\le i<j\le d+1}\Big(\biguplus_{x_j} f(x_i)\Big),$$

has been determined and then the generalized Hyers-Ulam stability problem for the equation has been investigated. Now, we consider the following functional equation,

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) = 2^d\sum_{i=1}^{d+1} f(x_i), \tag{1.4}$$

for all $(d+1)$-variables $x_1,\cdots,x_{d+1}$, where $d\ge 1$ is a natural number, which is motivated by a $(d+1)$-dimensional parallel polyhedron equality

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} \|x_1\|^2 = 2^d\sum_{i=1}^{d+1} \|x_i\|^2$$

generated by $x_1,\cdots,x_{d+1}$ in an inner product space. As a special case, we note that the equation (1.4) reduces to the equation (1.1) in case $d=1$ and (1.2) in case $d=2$.

In this paper, we establish new theorems about the Ulam stability of Eq. (1.4) and apply our results to the asymptotic behavior of functional equations on restricted domains.

2. Generalized stability of Eq. (1.4)

First of all, we present the general solution of Eq. (1.4) as follows.

Lemma 2.1. *Let E_1 and E_2 be vector spaces. A mapping $f: E_1\to E_2$ satisfies the functional equation* (1.4) *if and only if the mapping f satisfies the functional equation* (1.1).

Proof . We first assume that f is a solution of the functional equation (1.4). Set $x_i := 0$ in (1.4) for all $i=1,\cdots,d+1$ to get $f(0)=0$. Putting $x_i := 0$ in (1.4) for all $i=3,\cdots,d+1$, we get $f(x_1+x_2)+f(x_1-x_2)=2[f(x_1)+f(x_2)]$ for all $x_1,x_2\in E_1$ and so the mapping is quadratic.

Conversely, suppose the mapping f satisfies the functional equation (1.1). Then we first assume by induction that f satisfies the equation

$$\biguplus_{x_2,\cdots,x_d}^{d-1} f(x_1) = 2^{d-1}\sum_{i=1}^{d} f(x_i) \tag{2.1}$$

for all d-variables $x_1,\cdots,x_d\in E_1$. Putting $x_1 := x_1+x_{d+1}$ in (2.1), we get

$$\biguplus_{x_2,\cdots,x_d}^{d-1} f(x_1+x_{d+1}) = 2^{d-1}\Big[f(x_1+x_{d+1})+\sum_{i=2}^{d} f(x_i)\Big] \tag{2.2}$$

for all $(d+1)$-variables $x_1,\cdots,x_{d+1}\in E_1$. Replacing x_{d+1} by $-x_{d+1}$ in (2.2), we get

$$\biguplus_{x_2,\cdots,x_d}^{d-1} f(x_1-x_{d+1}) = 2^{d-1}\Big[f(x_1-x_{d+1})+\sum_{i=2}^{d} f(x_i)\Big] \tag{2.3}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in E_1$. Adding the equation (2.2) to (2.3) and then utilizing (1.3), we lead to the relation

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) = 2^{d-1}\Big[f(x_1+x_{d+1}) + f(x_1-x_{d+1}) + 2\sum_{i=2}^{d} f(x_i)\Big] = 2^d \sum_{i=1}^{d+1} f(x_i)$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in E_1$. This completes the proof. □

From now on, we investigate the generalized Hyers-Ulam stability problem for the equation (1.4). Thus we give conditions in order for a true mapping near an approximate mapping of the equation (1.4) to exist. From now on, let X be a normed space and Y a Banach space unless we give any specific reference. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and d a positive integer with $d \geq 1$. Now before taking up the main subject, given a mapping $f : X \to Y$, we define the difference operator $Df : X^{d+1} \to Y$ of the equation (1.4) by

$$Df(x_1, x_2, \cdots, x_{d+1}) := \biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) - 2^d \sum_{i=1}^{d+1} f(x_i)$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, which acts as a perturbation of the equation (1.4).

Lemma 2.2. *Suppose that there exists a mapping $\phi : X^3 \to \mathbb{R}^+$ for which $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, x_3)\| \leq \phi(x_1, x_2, x_3) \tag{2.4}$$

for all $x_1, x_2, x_3 \in X$, and the series

$$\sum_{i=0}^{\infty} \frac{\phi(3^i x_1, 3^i x_2, 3^i x_3)}{9^i}$$

converges for all $x_1, x_2, x_3 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ defined by $Q(x) := \lim_{n\to\infty} \frac{f(3^n x)}{9^n}$ for all $x \in X$, which satisfies the equation (1.2) and the inequality

$$\left\| f(x) - \frac{f(0)}{2} - Q(x) \right\| \leq \frac{1}{9} \sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i} + \frac{1}{18} \sum_{i=0}^{\infty} \frac{\hat{\phi}(3^i x)}{9^i} \tag{2.5}$$

for all $x \in X$, where the mapping $\hat{\phi} : X \to Y$ is given by

$$\hat{\phi}(x) := \min\{\phi(0, x, 0), \phi(0, 0, x)\} \quad \text{and} \quad 8\|f(0)\| \leq \phi(0, 0, 0)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$, that is, $Q(tx) = t^2 Q(x)$ for all $x \in X$ and all $t \in \mathbb{R}$.

Proof . If we replace (x_1, x_2, x_3) by $(0, x, 0)$ or $(0, 0, x)$ in relation (2.4), one has the approximate even condition of f as follows

$$2\|f(-x) - f(x) - 4f(0)\| \leq \hat{\phi}(x) := \min\{\phi(0, x, 0), \phi(0, 0, x)\} \tag{2.6}$$

for all $x \in X$. If we put (x, x, x) into (x_1, x_2, x_3) in (2.4), we have

$$\|f(3x) + 2f(x) + f(-x) - 12f(x)\| \leq \phi(x, x, x) \tag{2.7}$$

for all $x \in X$. Associating (2.6) with (2.7), we obtain

$$\|f(3x) - 9f(x) + 4f(0)\| \leq \phi(x,x,x) + \frac{1}{2}\hat{\phi}(x), \text{ or } \|q(3x) - 9q(x)\| \leq \phi(x,x,x) + \frac{1}{2}\hat{\phi}(x) \quad (2.8)$$

for all $x \in X$, where $q(x) := f(x) - \frac{f(0)}{2}$.

We define a sequence $\{Q_n(x)\}$ by

$$Q_n(x) := \frac{q(3^n x)}{9^n}, \quad x \in X,$$

and claim that it is a convergent sequence. Now we figure out by (2.8)

$$\|Q_{i+1}(x) - Q_i(x)\| = \frac{1}{9^i}\left\|\frac{q(3^{i+1}x)}{9} - q(3^i x)\right\| \leq \frac{1}{9^{i+1}}\left[\phi(3^i x, 3^i x, 3^i x) + \frac{1}{2}\hat{\phi}(3^i x)\right]$$

and so we see that for any integers m, n with $n > m \geq 0$,

$$\|Q_m(x) - Q_n(x)\| \leq \sum_{i=m}^{n-1} \|Q_{i+1}(x) - Q_i(x)\| \leq \frac{1}{9}\sum_{i=m}^{n-1} \frac{1}{9^i}\left[\phi(3^i x, 3^i x, 3^i x) + \frac{1}{2}\hat{\phi}(3^i x)\right] \quad (2.9)$$

for all $x \in X$. The right hand side of the above inequality tends to 0 as $m \to \infty$ and thus the sequence $\{Q_n(x)\}$ is Cauchy in Y, as desired. Therefore, we may define a mapping $Q : X \to Y$ as

$$Q(x) = \lim_{n\to\infty} \frac{q(3^n x)}{9^n} = \lim_{n\to\infty} \frac{f(3^n x)}{9^n}$$

for all $x \in X$, and then by letting $n \to \infty$ in (2.9) with $m = 0$ we arrive at the formula (2.5).

We claim that Q satisfies the equation (1.2). For this purpose, we calculate the following inequality from (2.4)

$$\|DQ_n(x_1, x_2, x_3)\| = \frac{1}{9^n}\|Df(3^n x_1, 3^n x_2, 3^n x_3)\| \leq \frac{1}{9^n}\phi(3^n x_1, 3^n x_2, 3^n x_3),$$

which yields by letting $n \to \infty$ that $DQ(x_1, x_2, x_3) = 0$ for all $x_1, x_2, x_3 \in X$. Hence the mapping Q is quadratic by Lemma 2.1.

To prove the afore-mentioned uniqueness, let $\phi_1 : X^3 \to \mathbb{R}^+$ be a mapping such that the functional inequality

$$\|Df(x_1, x_2, x_3)\| \leq \phi_1(x_1, x_2, x_3)$$

holds for all $x_1, x_2, x_3 \in X$ and the series

$$\sum_{i=0}^{\infty} \frac{\phi_1(3^i x_1, 3^i x_2, 3^i x_3)}{9^i}$$

converges for all $x_1, x_2, x_3 \in X$, and assume that there exists a quadratic mapping $Q_1 : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\left\|f(x) - \frac{f(0)}{2} - Q_1(x)\right\| \leq \frac{1}{9}\sum_{i=0}^{\infty} \frac{\phi_1(3^i x, 3^i x, 3^i x)}{9^i} + \frac{1}{18}\sum_{i=0}^{\infty} \frac{\hat{\phi_1}(3^i x)}{9^i} \quad (2.10)$$

for all $x \in X$, where the mapping $\hat{\phi_1} : X \to Y$ is given by

$$\hat{\phi_1}(x) := \min\{\phi_1(0, x, 0), \phi_1(0, 0, x)\}.$$

Since Q and Q_1 are quadratic, we see that the equation $Q(x) = 3^{-2n}Q(3^n x)$, $Q_1(x) = 3^{-2n}Q_1(3^n x)$ hold for all $x \in X$ and all $n \in \mathbb{N}$. Thus it follows from inequalities (2.5) and (2.10) that

$$\begin{aligned}\left\|\frac{f(3^n x)}{3^{2n}} - Q_1(x)\right\| &= \frac{1}{3^{2n}}\|f(3^n x) - Q_1(3^n x)\| \\ &\leq \frac{1}{3^{2n}}\left(\left\|f(3^n x) - \frac{f(0)}{2} - Q_1(3^n x)\right\| + \frac{\|f(0)\|}{2}\right) \\ &\leq \frac{1}{9}\sum_{i=n}^{\infty}\frac{\phi_1(3^i x, 3^i x, 3^i x)}{9^i} + \frac{1}{18}\sum_{i=n}^{\infty}\frac{\hat{\phi}_1(3^i x)}{9^i} + \frac{\|f(0)\|}{9^n 2}\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore letting $n \to \infty$, one has $Q(x) - Q_1(x) = 0$ for all $x \in X$, completing the proof of uniqueness. The proof is complete. □

The following lemma, which is similarly proved by way of (2.8), is an alternative result of the previous lemma.

Lemma 2.3. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, x_3)\| \leq \phi(x_1, x_2, x_3)$$

for all $x_1, x_2, x_3 \in X$. If the upper bound $\phi : X^3 \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=1}^{\infty} 9^i \phi\left(\frac{x_1}{3^i}, \frac{x_2}{3^i}, \frac{x_3}{3^i}\right)$$

converges for all $x_1, x_2, x_3 \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ defined by $Q(x) := \lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right)$ for all $x \in X$, which satisfies the equation (1.2) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{1}{9}\sum_{i=1}^{\infty} 9^i \phi\left(\frac{x}{3^i}, \frac{x}{3^i}, \frac{x}{3^i}\right) + \frac{1}{18}\sum_{i=1}^{\infty} 9^i \hat{\phi}\left(\frac{x}{3^i}\right)$$

for all $x \in X$, where the mapping $\hat{\phi} : X \to Y$ is given by

$$\hat{\phi}(x) := \min\{\phi(0, x, 0), \phi(0, 0, x)\}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

We are going to investigate the generalized Hyers-Ulam stability problem for the functional equation (1.4) with $d \geq 2$. That is, the following theorem says that if f is an approximate solution of the equation (1.4) with its difference operator Df bounded by a convergent series, then we can find a solution Q of the equation near f.

Theorem 2.4. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon(x_1, \cdots, x_{d+1}) \tag{2.11}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=0}^{\infty} \frac{\varepsilon(3^i x_1, \cdots, 3^i x_{d+1})}{3^{2i}}$$

converges for all $x_1, \cdots, x_{d+1} \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *defined by* $Q(x) = \lim_{n\to\infty} \frac{f(3^n x)}{9^n}$ *for all* $x \in X$, *which satisfies the equation* (1.4) *and the inequality*

$$\left\| f(x) + \frac{(d-4)f(0)}{4} - Q(x) \right\| \le \frac{1}{2^{d-2}9} \sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i} + \frac{1}{2^{d-2}18} \sum_{i=0}^{\infty} \frac{\hat{\phi}(3^i x)}{9^i} \tag{2.12}$$

for all $x \in X$, *where the mappings* $\phi : X^3 \to Y$ *and* $\hat{\phi} : X \to Y$ *are given by*

$$\phi(x, y, z) := \underbrace{\min}_{2\le i<j\le d+1} \Big\{ \varepsilon(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0, \overbrace{z}^{j}, 0, \cdots, 0) \Big\}, \tag{2.13}$$
$$\hat{\phi}(x) := \min\{\phi(0, x, 0), \phi(0, 0, x)\},$$

and $2^d d\|f(0)\| \le \varepsilon(0, \cdots, 0,)$. *Moreover, if* f *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in X$, *then the mapping* Q *is homogeneous of degree* 2 *over* $\mathbb{R}$.

Proof . Taking $(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0, \overbrace{z}^{j}, 0, \cdots, 0)$ instead of $(x_1, \cdots, x_{d+1})$ in (2.11), we obtain by virtue of (1.3)

$$2^{d-2}\|Df(x, y, z) - 4(d-2)f(0)\| \le \varepsilon(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0, \overbrace{z}^{j}, 0, \cdots, 0), \quad \text{or} \tag{2.14}$$
$$\|Dq(x, y, z)\| \le \frac{1}{2^{d-2}} \varepsilon(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0, \overbrace{z}^{j}, 0, \cdots, 0)$$

for all $x, y, z \in X$, and all i, j with $2 \le i < j \le d+1$, where $q(x) := f(x) + \frac{(d-2)f(0)}{2}$. Considering a mapping $\phi : X^3 \to Y$ defined by

$$\phi(x, y, z) := \min\Big\{ \varepsilon(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0, \overbrace{z}^{j}, 0, \cdots, 0) \mid 2 \le i < j \le d+1 \Big\},$$

we can rewrite the functional inequality (2.14) in the form

$$\|Dq(x, y, z)\| \le \frac{1}{2^{d-2}} \phi(x, y, z) \tag{2.15}$$

for all $x, y, z \in X$. Applying Lemma 2.2 to the inequality (2.15), we obtain the desired results. The proof is complete. □

Theorem 2.5. *Suppose that for a positive integer* $d \ge 2$, *a mapping* $f : X \to Y$ *satisfies*

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \le \varepsilon(x_1, \cdots, x_{d+1})$$

for all $(d+1)$*-variables* $x_1, \cdots, x_{d+1} \in X$, *and that* $\varepsilon : X^{d+1} \to \mathbb{R}^+$ *is a mapping such that the series*

$$\sum_{i=1}^{\infty} 9^i \varepsilon\left(\frac{x_1}{3^i}, \cdots, \frac{x_{d+1}}{3^i}\right)$$

converges for all $x_1, \cdots, x_{d+1} \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *which satisfies the equation* (1.4) *and the inequality*

$$\|f(x) - Q(x)\| \le \frac{1}{2^{d-2}9} \sum_{i=1}^{\infty} 9^i \phi\left(\frac{x}{3^i}, \frac{x}{3^i}, \frac{x}{3^i}\right) + \frac{1}{2^{d-2}18} \sum_{i=1}^{\infty} 9^i \hat{\phi}\left(\frac{x}{3^i}\right)$$

for all $x \in X$, where the mappings $\phi : X^3 \to Y$ and $\hat{\phi} : X \to Y$ are given by (2.13). *The mapping Q is defined by*

$$Q(x) = \lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Remark 2.6. *If $d = 2$ in Theorem* 2.4 *and Theorem* 2.5*, then the results exactly coincide with Lemma* 2.2 *and Lemma* 2.3*.*

The following two theorems are an answer to a generalized Hyers-Ulam stability problem of different types for the equation (1.4) with $d \geq 1$.

Theorem 2.7. *Suppose that for a positive integer $d \geq 1$, a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon(x_1, \cdots, x_{d+1})$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=0}^{\infty} \frac{\varepsilon(2^i x_1, \cdots, 2^i x_{d+1})}{2^{2i}}$$

converges for all $x_1, \cdots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ defined by $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ for all $x \in X$, which satisfies the equation (1.4) *and the inequality*

$$\left\| f(x) + \frac{(2d-3)f(0)}{3} - Q(x) \right\| \leq \frac{1}{2^{d+1}} \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i x)}{4^i}$$

for all $x \in X$, where the mapping $\phi : X^2 \to Y$ is given by

$$\phi(x, y) := \min\Big\{\varepsilon(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0) \mid 2 \leq i \leq d+1\Big\}. \tag{2.16}$$

Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Proof . If we take $(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0)$ instead of $(x_1, \cdots, x_{d+1})$ in (2.11), we obtain by virtue of (1.3)

$$2^{d-1}\|Df(x, y) - 2(d-1)f(0)\| \leq \varepsilon(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0), \quad \text{or} \tag{2.17}$$
$$\|Dq(x, y)\| \leq \frac{1}{2^{d-1}}\varepsilon(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0)$$

for all $x, y \in X$, and all i with $2 \leq i \leq d+1$, where $q(x) := f(x) + (d-1)f(0)$. Considering a mapping $\phi : X^2 \to Y$ defined by

$$\phi(x, y) := \min\Big\{\varepsilon(x, 0, \cdots, 0, \overbrace{y}^{i}, 0 \cdots, 0) \mid 2 \leq i \leq d+1\Big\},$$

we can rewrite the functional inequality (2.17) in the form

$$\|q(x+y)+q(x-y)-2q(x)-2q(y)\| \leq \frac{1}{2^{d-1}}\phi(x,y)$$

for all $x, y \in X$. Now applying the same procedure of direct method [6] to the last relation, we get the desired results. The proof is complete. □

Theorem 2.8. *Suppose that for a positive integer $d \geq 1$ a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon(x_1, \cdots, x_{d+1})$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=1}^{\infty} 4^i \varepsilon\left(\frac{x_1}{2^i}, \cdots, \frac{x_{d+1}}{2^i}\right)$$

converges for all $x_1, \cdots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ defined by $Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ for all $x \in X$, which satisfies the equation (1.4) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{1}{2^{d+1}} \sum_{i=1}^{\infty} 4^i \phi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)$$

for all $x \in X$, where the mapping $\phi : X^2 \to Y$ is given by (2.16). Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

3. Approximately quadratic mappings on restricted domains

In this section, we are planning to investigate the stability problem for the equation (1.4) on a restricted domain. As results, we have corollaries with regard to an asymptotic property of the equation (1.4).

Theorem 3.1. *Let d be a positive integer with $d \geq 2$. Suppose that there exist a nonnegative real number ε and a positive real r for which a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon \tag{3.1}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \geq r$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\left\|f(x) + \frac{(d-2)f(0)}{2} - Q(x)\right\| \leq \frac{15\varepsilon}{2^{d+1}} \tag{3.2}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Proof . If we take $(x_1, x_2, x_3, 0, \cdots, 0)$ instead of $(x_1, \cdots, x_{d+1})$ in (3.1) with $\sum_{i=1}^{3} \|x_i\| \geq r$, we obtain by virtue of (2.14)

$$\|Dq(x_1, x_2, x_3)\| \leq \frac{1}{2^{d-2}}\varepsilon \tag{3.3}$$

for all $x_1, x_2, x_3 \in X$ with $\sum_{i=1}^{3} \|x_i\| \geq r$. Specially, we have $8\|q(0)\| \leq \frac{\varepsilon}{2^{d-2}}$ by setting $x_2, x_3 := 0$ and $x_1 := t$ with $\|t\| \geq r$ in (3.3). Thus it follows from (3.3) that

$$\|Dq(x_1, x_2, x_3) - 8q(0)\| \leq \frac{2}{2^{d-2}}\varepsilon \leq \frac{10}{2^{d-2}}\varepsilon \tag{3.4}$$

for all $x_1, x_2, x_3 \in X$ with $\sum_{i=1}^{3} \|x_i\| \geq r$.

Now assume $\sum_{i=1}^{3} \|x_i\| < r$. And choose a $t \in X$ with $\|t\| \geq 2r$. Then it holds clearly

$$\|x_1 \pm 2t\| \geq r, \ \|x_2 \pm t\| \geq r, \ \text{and } \|x_3 \pm t\| \geq r.$$

Therefore from (3.1), (3.3) and the following functional identity

$$\begin{aligned}
&2\big[q(x_1+x_2+x_3) + q(x_1-x_2+x_3) + q(x_1+x_2-x_3) + q(x_1-x_2-x_3) \\
&\qquad\qquad -4q(x_1) - 4q(x_2) - 4q(x_3) - 8q(0)\big] \\
&= \big[q(x_1+x_2+x_3) + q(x_1-x_2+x_3-2t) + q(x_1+x_2-x_3+2t) \\
&\qquad\qquad +q(x_1-x_2-x_3) - 4q(x_1) - 4q(x_2+t) - 4q(x_3-t)\big] \\
&\quad +\big[q(x_1+x_2+x_3) + q(x_1-x_2+x_3+2t) + q(x_1+x_2-x_3-2t) \\
&\qquad\qquad +q(x_1-x_2-x_3) - 4q(x_1) - 4q(x_2-t) - 4q(x_3+t)\big] \\
&\quad +\big[q(x_1+x_2+x_3+2t) + q(x_1-x_2+x_3) + q(x_1+x_2-x_3) \\
&\qquad\qquad +q(x_1-x_2-x_3-2t) - 4q(x_1) - 4q(x_2+t) - 4q(x_3+t)\big] \\
&\quad +\big[q(x_1+x_2+x_3-2t) + q(x_1-x_2+x_3) + q(x_1+x_2-x_3) \\
&\qquad\qquad +q(x_1-x_2-x_3+2t) - 4q(x_1) - 4q(x_2-t) - 4q(x_3-t)\big] \\
&\quad +\big[-q(x_1+x_2+x_3-2t) - q(x_1-x_2+x_3-2t) - q(x_1+x_2-x_3-2t) \\
&\qquad\qquad -q(x_1-x_2-x_3-2t) + 4q(x_1-2t) + 4q(x_2) + 4q(x_3)\big] \\
&\quad +\big[-q(x_1+x_2+x_3+2t) - q(x_1-x_2+x_3+2t) - q(x_1+x_2-x_3+2t) \\
&\qquad\qquad -q(x_1-x_2-x_3+2t) + 4q(x_1+2t) + 4q(x_2) + 4q(x_3)\big] \\
&\quad +4\big[q(x_2+t) + q(x_2-t) + q(x_2+t) + q(x_2-t) - 4q(x_2) - 4q(t) - 4q(0)\big] \\
&\quad +4\big[q(x_3+t) + q(x_3-t) + q(x_3+t) + q(x_3-t) - 4q(x_3) - 4q(t) - 4q(0)\big] \\
&\quad +2\big[-q(x_1+2t) - q(x_1-2t) - q(x_1+2t) - q(x_1-2t) + 4q(x_1) + 4q(2t) + 4q(0)\big] \\
&\quad +4\big[-q(2t) - q(0) - q(2t) - q(0) + 4q(t) + 4q(t) + 4q(0)\big],
\end{aligned}$$

we get

$$\|Dq(x_1, x_2, x_3) - 8q(0)\| \leq \frac{10}{2^{d-2}}\varepsilon \tag{3.5}$$

for all $x_1, x_2, x_3 \in X$ with $\sum_{i=1}^{3} \|x_i\| < r$. Hence the last functional inequality holds for all $x_1, x_2, x_3 \in X$ in view of (3.4). Applying the same manner as (2.6) together with (2.7) to the last inequality (3.5), we obtain

$$\|q(3x) - 9q(x)\| \leq \frac{15}{2^{d-2}}\varepsilon,$$

for all $x \in X$. By the same manner as Lemma 2.2, we obtain the desired results. The proof is complete. □

Theorem 3.2. *Suppose that there exist a nonnegative real number ε and a positive real r for which a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \le \varepsilon \tag{3.6}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \ge r$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) *and the inequality*

$$\|f(x) + (d-1)f(0) - Q(x)\| \le \frac{3\varepsilon}{2^d}$$

for all $x \in X$.

Proof . Replacing $(x_1, \cdots, x_{d+1})$ in (3.6) by $(x_1, x_2, 0, \cdots, 0)$ with $\sum_{i=1}^{2} \|x_i\| \ge r$, we obtain by virtue of (2.17)

$$\|q(x_1 + x_2) + q(x_1 - x_2) - 2q(x_1) - 2q(x_2)\| \le \frac{1}{2^{d-1}}\varepsilon \tag{3.7}$$

for all $x_1, x_2 \in X$ with $\sum_{i=1}^{2} \|x_i\| \ge r$, where $q(x) := f(x) + (d-1)f(0)$. Now using the same argument as that of (3.7), we get

$$\|q(x_1 + x_2) + q(x_1 - x_2) - 2q(x_1) - 2q(x_2) - q(0)\| \le \frac{9\varepsilon}{2^{d-1} \cdot 2} \tag{3.8}$$

for all $x_1, x_2 \in X$ with $\|x_1\| + \|x_2\| < r$. Consequently, the last functional inequality holds for all $x_1, x_2 \in X$ in view of (3.7). Now letting $(x_1, x_2) := (x, x)$ in (3.8), we obtain

$$\|q(2x) - 4q(x)\| \le \frac{9\varepsilon}{2^{d-1} \cdot 2}$$

for all $x \in X$. Now applying a standard procedure of direct method to the last inequality, we obtain the desired results. □

We note that if we define

$$S_{d+1} = \{(x_1, \cdots, x_{d+1}) \in X^{d+1} : \|x_i\| < r, \forall i = 1, \cdots, d+1\}$$

for some fixed $r > 0$, then we have

$$\Big\{(x_1, \cdots, x_{d+1}) \in X^{d+1} : \sum_{i=1}^{d+1} \|x_i\| \ge (d+1)r\Big\} \subset X^{d+1} \setminus S_{d+1}.$$

Thus the following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *If a mapping $f : X \to Y$ satisfies the inequality* (3.1) *for all $(x_1, \cdots, x_{d+1}) \in X^{d+1} \setminus S_{d+1}$, then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation* (1.4) *and the inequality* (3.2).

From Theorem 3.1 and Theorem 3.2, we get the following corollaries concerning an asymptotic property of quadratic mappings.

Corollary 3.4. *A mapping $f : X \to Y$ with $f(0) = 0$ is quadratic if and only if*

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \to 0.$$

Corollary 3.5. *A mapping $f : X \to Y$ with $f(0) = 0$ is quadratic if and only if there exists a positive real $r > 0$ such that*

$$\sup_{x_1,\cdots,x_{d+1}} \left\{\|Df(x_1, x_2, \cdots, x_{d+1})\| : \sum_{i=1}^{d+1} \|x_i\| \geq r\right\}$$

is bounded for all infinitely many $d \geq 1$.

We also have by Theorem 2.7 the following result for an asymptotic property of quadratic mappings.

Corollary 3.6. *A mapping $f : X \to Y$ with $f(0) = 0$ is quadratic if and only if*

$$\sup_{x_1,\cdots,x_{d+1}} \|Df(x_1, x_2, \cdots, x_{d+1})\|$$

is bounded for all infinitely many $d \geq 1$.

Proof . Let $\sup_{x_1,\cdots,x_{d+1}} \|Df(x_1, x_2, \cdots, x_{d+1})\| \leq M < \infty$ for all infinitely many $d \geq 1$. Then for each $d \geq 1$, there exists a unique quadratic mapping $Q_d : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - Q_d(x)\| \leq \frac{M}{2^{d-1}3}$$

for all $x \in X$ by Theorem 2.7. Let m be a positive integer with the stated property and $m > d$. Then, we obtain

$$\|f(x) - Q_m(x)\| \leq \frac{M}{2^{m-1}3} \leq \frac{M}{2^{d-1}3}$$

for all $x \in X$. The uniqueness of Q_d implies that $Q_m = Q_d$ for m with $m > d$, and so

$$\|f(x) - Q_d(x)t\| \leq \frac{M}{2^{m-1}3}$$

for all $x \in X$. By letting $m \to \infty$, we conclude that f is itself quadratic.

The reverse assertion is trivial. □

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