Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 1181-1194
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2019.15346.1807

The extended tanh method for solving conformable space-time fractional KdV equations

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this study, we obtain exact traveling wave solutions of the conformable space-time fractional Sawada-Kotera-Ito, Lax and Kaup-Kupershmidt equations by using the extended tanh method. The obtained traveling wave solutions are expressed by the hyperbolic, trigonometric, exponential and rational functions. Simulation of the obtained solutions are given at the end of the paper.

Keywords: conformable space-time fractional Sawada-Kotera-Ito equation, conformable space-time fractional Lax equation, conformable space-time fractional Kaup- Kupershmidt equation, extended tanh method, traveling wave solutions.
2010 MSC: Primary 35C07; Secondary 35G20.

1. Introduction

Nonlinear fractional partial differential equations have extensive application in many areas of science and engineering such as physics, chemistry, mechanics, electrical networks, astronomy, diffusion, viscoelastic fluid, entropy and biological sciences [1, 2, 3]. As a result, researchers have begun to search for methods that give exact and approximate solutions of nonlinear fractional partial differential equations (see, for example, [4, 5, 6, 7, 8, 9, 10, 11]).

The Korteweg-de Vries (KdV)-type equations are one of the important mathematical models of nonlinear partial differential equations. They are used to describe long wave motion in shallow water, one-dimensional nonlinear lattice, hydrodynamics, quantum mechanics, plasma physics, and optics [12, 13, 14, 15]. Due to the higher-order dispersion or other properties, the higher-order integrable members in the KdV hierarchy have also been studied, which can model some physical phenomena

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Received: July 2018 *Revised:* October 2019

such as the surface and internal waves, gravity-capillary waves, interaction between a water wave and a floating ice cover, etc. [16, 17].

The conformable space-time fractional Korteweg-de Vries (KdV) equation is given as follows:

$$\begin{aligned} & T_t^{\alpha}u + au^3T_x^{\beta}u + b(T_x^{\beta}u)^3 + cu.T_x^{\beta}u.T_x^{\beta}T_x^{\beta}u + du^2.T_x^{\beta}T_x^{\beta}T_x^{\beta}u + eT_x^{\beta}T_x^{\beta}u \\ . & \ T_x^{\beta}T_x^{\beta}T_x^{\beta}u + fT_x^{\beta}u.T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}u + gu.T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}u \\ + & \ T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}u = 0, \end{aligned} \tag{1.1}$$

where a, b, c, d, e, f and g are non-zero constants. α $(0 < \alpha \leq 1)$ and β $(0 < \beta \leq 1)$ are parameters describing the order of the conformable time fractional and the conformable space fractional, respectively. When $\alpha = 1$, $\beta = 1$, Eq.(1.1) corresponds to the classical seventh-order KdV equation. In fact, the seventh-order KdV was introduced by Pomeau et al.[18] and its structural stability was discussed under a singular perturbation. For $a = 252, b = 63, c = 378, d = 126, e = 63, f = 42$ and $g = 21$, Eq.(1.1) is called the conformable space-time fractional Sawada-Kotera-Ito equation [19, 20, 21]. For $a = 140, b = 70, c = 280, d = 70, e = 70, f = 42$ and $g = 14$, Eq.(1.1) is called the conformable space-time fractional Lax equation equation [19, 20, 21]. For $a = 2016, b = 630, c = 2268, d = 504, e = 252, f = 147$ and $g = 42$, Eq.(1.1) is called the conformable space-time fractional Kaup-Kupershmidt equation [19, 20, 21].

Seventh-order KdV equations have been widely studied in the literature. Exp-function method and modified Kudryashov method have been used to solve seventh-order Sawada-Kotera-Ito, Lax and Kaup-Kupershmidt equations in [21, 20], respectively. Seventh-order Sawada-Kotera-Ito equation has been solved by using G'/G-expansion method, Bell-polynomial approach, tanh-coth method in [22, 23, 24], respectively. Seventh-order Sawada-Kotera-Ito and Lax equations have been solved by using homotopy perturbation method, the Cole-Hopf transform and reconstruction of variational iteration method in [25, 26, 27], respectively. G'/G-expansion method to construct the closed form solutions of the seventh order conformable time fractional Sawada-Kotera-Ito equation has been presented in [28]. Lie symmetry analysis of the seventh-order time fractional Sawada-Kotera-Ito equation with Riemann-Liouville derivative has been performed and obtained exact traveling wave solutions by using the sub-equation method in [29].

The objective of this paper is to employ the extended tanh method (see, for example, [30, 31]) to obtain the exact traveling wave solutions of the conformable space-time fractional Sawada-Kotera-Ito, conformable space-time fractional Lax and conformable space-time fractional Kaup- Kupershmidt equations. The paper is organized as follows. In Section 2, definition and properties of the conformable fractional derivative are presented. In Section 3, 4 and 5, exact traveling wave solutions of the conformable space-time fractional Sawada-Kotera-Ito, conformable space-time fractional Lax and conformable space-time fractional Kaup- Kupershmidt equations via the extended tanh method are obtained, respectively. Finally, conclusion is given in Section 6.

2. Description of conformable fractional derivative and its properties

For a function $f : (0, \infty) \to R$, the conformable fractional derivative of f of order $0 < \alpha < 1$ is defined as (see, for example, [32])

$$T_t^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}. \tag{2.1}$$

Some important properties of the the conformable fractional derivative are as follows:

$$T_t^{\alpha}(af+bg)(t) = aT_t^{\alpha}f(t)+bT_t^{\alpha}g(t), \ \forall a,b\in R, \tag{2.2}$$

$$T_t^{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha}, \tag{2.3}$$

$$T_t^{\alpha}(f(g(t)) = t^{1-\alpha}g^{'}(t)f^{'}(g(t)). \tag{2.4}$$

3. Analytic solutions to the conformable space-time fractional Sawada-Kotera-Ito equation

Firstly, we consider the conformable space-time fractional Sawada-Kotera-Ito equation as follows [19, 20, 21]

$$\begin{aligned} & T_t^{\alpha}u+252u^3T_x^{\beta}u+63(T_x^{\beta}u)^3+378uT_x^{\beta}uT_x^{\beta}T_x^{\beta}u+126u^2T_x^{\beta}T_x^{\beta}T_x^{\beta}u+63 \\ . \ & T_x^{\beta}T_x^{\beta}uT_x^{\beta}T_x^{\beta}T_x^{\beta}u+42T_x^{\beta}uT_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}u+21uT_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}u \\ + \ & T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}T_x^{\beta}u=0, \ \ 0<\alpha\leq 1, \ \ 0<\beta\leq 1. \end{aligned} \tag{3.1}$$

Using the following transformation

$$u(x,t)=U(\xi), \ \ \xi=k\frac{t^{\alpha}}{\alpha}+m\frac{x^{\beta}}{\beta}, \tag{3.2}$$

where k, m, are constants, Eq.(3.1) can be written as the following differential equations

$$\begin{aligned} & kU'+252mU^3U'+63m^3(U')^3+378m^3UU'U''+126m^3U'''U^2+63m^5 \\ . \ & U''U'''+42m^5U'U^{(4)}+21m^5UU^{(5)}+m^7U^{(7)}=0. \end{aligned} \tag{3.3}$$

Let us suppose that the solution of Eq.(3.3) can be expressed in the following form:

$$U(\xi)=\sum_{i=0}^{N}a_i\phi(\xi)^i, \tag{3.4}$$

where $\phi(\xi)$ satisfies the linear ordinary differential equation in the form

$$\phi'=b+\phi^2, \tag{3.5}$$

where b is nonzero constant, a_i are arbitrary constants to be determined. Eq.(3.5) has different solutions as follows (see, for example, [30, 31]):
When $b>0$

$$\phi=\sqrt{b}\tan(\sqrt{b}\xi) \ \ or \ \ \phi=-\sqrt{b}\cot(\sqrt{b}\xi). \tag{3.6}$$

When $b<0$

$$\phi=-\sqrt{-b}\tanh(\sqrt{-b}\xi) \ \ or \ \ \phi=-\sqrt{-b}\coth(\sqrt{-b}\xi). \tag{3.7}$$

When $b=0$

$$\phi=-\frac{1}{(\xi+D)}. \tag{3.8}$$

Here D is nonzero constant. Substituting Eq.(3.4) into Eq.(3.3) and then by balancing the highest order derivative term and nonlinear term in result equation, the value of N can be determined as 2. Therefore, Eq.(3.4) reduces to

$$U(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2. \tag{3.9}$$

Substituting (3.9) into (3.3), collecting all the terms with the same power of ϕ, we can obtain a set of algebraic equations for the unknowns a_0, a_1, a_2, b, k, m:

$$504a_2^4m + 8064a_2^3m^3 + 34272a_2^2m^5 + 40320a_2m^7 = 0,$$

$$41764a_1a_2^3m + 15876a_1a_2^2m^3 + 29988a_1a_2m^5 + 5040a_1m^7 = 0,$$

$$\begin{aligned}
& 378a_2(2a_1^2m^3 + 28ba_2^2m^3) + 4284a_1^2m^5 + 4536a_0a_2^2m^3 + 756a_1^2a_2^2m + 5670 \\
. \quad & a_1^2a_2m^3 + 6552a_2^3bm^3 + 92736a_2^2bm^5 + 504a_2m(a_2(a_1^2 + 2a_0a_2) + a_0a_2^2 \\
+ \quad & 2a_1^2a_2) + 3024a_2m^3(a_1^2 + 2a_0a_2) + 15120a_0a_2m^5 + 504a_2^4bm + 120960a_2bm^7 \\
= \quad & 0,
\end{aligned}$$

$$\begin{aligned}
& 378a_1(2a_1^2m^3 + 28ba_2^2m^3) + 63a_1^3m^3 + 504a_2m(a_1(a_1^2 + 2a_0a_2) + 4a_0a_1a_2) \\
+ \quad & 252a_1m(a_2(a_1^2 + 2a_0a_2) + a_0a_2^2 + 2a_1^2a_2) + 756a_1m^3(a_1^2 + 2a_0a_2) + 2520a_0 \\
. \quad & a_1m^5 + 13440a_1bm^7 + 9828a_0a_1a_2m^3 + 1764a_1a_2^3bm + 74928a_1a_2bm^5 \\
+ \quad & 21672a_1a_2^2bm^3 = 0,
\end{aligned}$$

$$\begin{aligned}
& 378a_2(4a_1^2bm^3 + 20a_2^2b^2m^3) + 378a_0(2a_1^2m^3 + 28ba_2^2m^3) + 126b(12a_1^2m^5 \\
+ \quad & 144ba_2^2m^5) + 252a_1m(a_1(a_1^2 + 2a_0a_2) + 4a_0a_1a_2) + 1512a_0a_1^2m^3 + 3024a_0^2 \\
. \quad & a_2m^3 + 7980a_1^2bm^5 + 129024a_2b^2m^7 + 504a_2m(a_0(a_1^2 + 2a_0a_2) + 2a_0a_1^2 \\
+ \quad & a_0^2a_2) + 3528a_2^3b^2m^3 + 68544a_2^2b^2m^5 + 5040a_2bm^3(a_1^2 + 2a_0a_2) + 35280a_0 \\
. \quad & a_2bm^5 + 756a_1^2a_2^2bm + 11466a_1^2a_2bm^3 + 504a_2bm(a_2(a_1^2 + 2a_0a_2) + a_0a_2^2 \\
+ \quad & 2a_1^2a_2) = 0,
\end{aligned}$$

$$\begin{aligned}
& 378a_1(4a_1^2bm^3 + 20a_2^2b^2m^3) + 756a_0^2a_1m^3 + 189a_1^3bm^3 + 12096a_1b^2m^7 \\
+ \quad & 252a_1m(a_0(a_1^2 + 2a_0a_2) + 2a_0a_1^2 + a_0^2a_2) + 11844a_1a_2^2b^2m^3 + 1008a_1bm^3 \\
. \quad & (a_1^2 + 2a_0a_2) + 1512a_0^2a_1a_2m + 5040a_0a_1bm^5 + 504a_2bm(a_1(a_1^2 + 2a_0a_2) \\
+ \quad & 4a_0a_1a_2) + 61824a_1a_2b^2m^5 + 252a_1bm(a_2(a_1^2 + 2a_0a_2) + a_0a_2^2 + 2a_1^2a_2) \\
+ \quad & 18396a_0a_1a_2bm^3 = 0,
\end{aligned}$$

$$\begin{aligned}
& 378a_0(4a_1^2bm^3 + 20a_2^2b^2m^3) + 126b(4a_1^2bm^5 + 32a_2^2b^2m^5) + 2a_2k + 378a_2 \\
. \quad & (2a_1^2b^2m^3 + 4a_2^2b^3m^3) + 63b^2(12a_1^2m^5 + 144ba_2^2m^5) + 756a_0^2a_1^2m + 56320 \\
. \quad & a_2b^3m^7 + 5208a_1^2b^2m^5 + 504a_2^3b^3m^3 + 18480a_2^2b^3m^5 + 504a_0^3a_2m + 6930 \\
. \quad & a_1^2a_2b^2m^3 + 252a_1bm(a_1(a_1^2 + 2a_0a_2) + 4a_0a_1a_2) + 2016a_2b^2m^3(a_1^2 + 2a_0 \\
. \quad & a_2) + 2016a_0a_1^2bm^3 + 5040a_0^2a_2bm^3 + 25872a_0a_2b^2m^5 + 504a_2bm(a_0(a_1^2 \\
+ \quad & 2a_0a_2) + 2a_0a_1^2 + a_0^2a_2) = 0,
\end{aligned}$$

$$\begin{aligned}
& a_1k + 378a_1(2a_1^2b^2m^3 + 4a_2^2b^3m^3) + 3968a_1b^3m^7 + 189a_1^3b^2m^3 + 252a_0^3a_1 \\
. \quad & m + 1512a_1a_2^2b^3m^3 + 252a_1b^2m^3(a_1^2 + 2a_0a_2) + 1008a_0^2a_1bm^3 + 2856a_0a_1 \\
. \quad & b^2m^5 + 17808a_1a_2b^3m^5 + 252a_1bm(a_0(a_1^2 + 2a_0a_2) + 2a_0a_1^2 + a_0^2a_2) \\
+ \quad & 9324a_0a_1a_2b^2m^3 + 1512a_0^2a_1a_2bm = 0,
\end{aligned}$$

$$\begin{aligned}
& 378a_0(2a_1^2b^2m^3 + 4a_2^2b^3m^3) + 63b^2(4a_1^2bm^5 + 32a_2^2b^2m^5) + 7936a_2b^4m^7 \\
+ \quad & 2a_2bk + 1008a_1^2b^3m^5 + 1344a_2^2b^4m^5 + 504a_0a_1^2b^2m^3 + 2016a_0^2a_2b^2m^3 \\
+ \quad & 1134a_1^2a_2b^3m^3 + 504a_0^3a_2bm + 756a_0^2a_1^2bm + 5712a_0a_2b^3m^5 = 0,
\end{aligned}$$

$$\begin{aligned}
& 252a_0^3a_1bm + 252a_0^2a_1b^2m^3 + 336a_0a_1b^3m^5 + 756a_2a_0a_1b^3m^3 + 63a_1^3b^3m^3 \\
+ \quad & 272a_1b^4m^7 + 924a_2a_1b^4m^5 + ka_1b = 0.
\end{aligned}$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solutions: $a_0 = \frac{-8bm^2}{3}, a_1 = 0, a_2 = -4m^2, k = \frac{-256}{3}b^3m^7$:

When $b > 0$,

$$u_1(x,t) = \frac{-8bm^2}{3} - 4m^2\Big(\sqrt{b}\tan(\sqrt{b}(\frac{-256}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \tag{3.10}$$

$$u_2(x,t) = \frac{-8bm^2}{3} - 4m^2\Big(\sqrt{b}\cot(\sqrt{b}(\frac{-256}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \tag{3.11}$$

When $b < 0$,

$$u_3(x,t) = \frac{-8bm^2}{3} - 4m^2\Big(\sqrt{-b}\tanh(\sqrt{-b}(\frac{-256}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \tag{3.12}$$

$$u_4(x,t) = \frac{-8bm^2}{3} - 4m^2\Big(\sqrt{-b}\coth(\sqrt{-b}(\frac{-256}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \tag{3.13}$$

When $b = 0$,

$$u_5(x,t) = -\frac{1}{\frac{-256}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + D}. \tag{3.14}$$

Fig.1 shows 3D plot of the traveling wave solution $u_3(x,t)$ in Eq.(3.1) for $\alpha = 0.5$, $\beta = 1$, $m = 1$, $b = -1/4$. Note that all of the solutions which are given by Eqs.(3.10)-(3.14) satisfy the conformable space-time fractional Sawada-Kotera-Ito equation Eq.(3.1). This has been seen by substituting the obtained solutions into the Eq.(3.1) and using the symbolic toolbox of MATLAB.

4. Analytic solutions to the conformable space-time fractional Lax equation

Conformable space-time fractional Lax equation is given in the following form [19, 20, 21]

$$\begin{aligned} & T_t^\alpha u + 140u^3T_x^\beta u + 70(T_x^\beta u)^3 + 280uT_x^\beta uT_x^\beta T_x^\beta u + 70u^2T_x^\beta T_x^\beta T_x^\beta u \\ + \ & 70T_x^\beta T_x^\beta uT_x^\beta T_x^\beta T_x^\beta u + 42T_x^\beta uT_x^\beta T_x^\beta T_x^\beta T_x^\beta u + 14uT_x^\beta T_x^\beta T_x^\beta T_x^\beta T_x^\beta u \\ + \ & T_x^\beta T_x^\beta T_x^\beta T_x^\beta T_x^\beta T_x^\beta T_x^\beta u = 0, \ \ 0 < \alpha \le 1, \ \ 0 < \beta \le 1. \end{aligned} \tag{4.1}$$

Substituting Eq.(3.2) into Eq.(4.1), Eq.(4.1) can be transformed to the following differential equation

$$\begin{aligned} & kU' + 140mU^3U' + 70m^3(U')^3 + 280m^3UU'U'' + 70m^3U'''U^2 + 70m^5U''U''' \\ + \ & 42m^5U'U^{(4)} + 14m^5UU^{(5)} + m^7U^{(7)} = 0. \end{aligned} \tag{4.2}$$

Let us suppose that the solution of Eq.(4.2) can be written in the form of Eq.(3.4). Substituting Eq.(3.4) into Eq.(4.2) and then by balancing the highest order derivative term and nonlinear term in result equation, the value of N can be determined as 2. Therefore, Eq.(3.4) reduces to

$$U(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2. \tag{4.3}$$

Substituting (4.3) into (4.2), collecting all the terms with the same power of ϕ, we can obtain a set of algebraic equations for the unknowns a_0, a_1, a_2, b, k, m:

$$280a_2^4m + 5600a_2^3m^3 + 30240a_2^2m^5 + 40320a_2m^7 = 0,$$

$$980a_1a_2^3m + 10780a_1a_2^2m^3 + 24696a_1a_2m^5 + 5040a_1m^7 = 0,$$

$$\begin{aligned}
&280a_2(2a_1^2m^3 + 28ba_2^2m^3) + 3528a_1^2m^5 + 3360a_0a_2^2m^3 + 420a_1^2a_2^2m + 4060\\
.\ &a_1^2a_2m^3 + 4480a_2^3bm^3 + 84000a_2^2bm^5 + 280a_2m(a_2(a_1^2 + 2a_0a_2) + a_0a_2^2\\
+\ &2a_1^2a_2) + 1680a_2m^3(a_1^2 + 2a_0a_2) + 10080a_0a_2m^5 + 280a_2^4bm + 120960a_2b\\
.\ &m^7 = 0,
\end{aligned}$$

$$\begin{aligned}
&280a_1(2a_1^2m^3 + 28ba_2^2m^3) + 70a_1^3m^3 + 280a_2m(a_1(a_1^2 + 2a_0a_2) + 4a_0a_1a_2)\\
+\ &140a_1m(a_2(a_1^2 + 2a_0a_2) + a_0a_2^2 + 2a_1^2a_2) + 420a_1m^3(a_1^2 + 2a_0a_2) + 1680a_0\\
.\ &a_1m^5 + 13440a_1bm^7 + 6160a_0a_1a_2m^3 + 980a_1a_2^3bm + 63056a_1a_2bm^5\\
+\ &14840a_1a_2^2bm^3 = 0,
\end{aligned}$$

$$\begin{aligned}
&280a_2(4a_1^2bm^3 + 20a_2^2b^2m^3) + 280a_0(2a_1^2m^3 + 28ba_2^2m^3) + 140b(12a_1^2m^5\\
+\ &144ba_2^2m^5) + 140a_1m(a_1(a_1^2 + 2a_0a_2) + 4a_0a_1a_2) + 840a_0a_1^2m^3 + 1680a_0^2\\
.\ &a_2m^3 + 6328a_1^2bm^5 + 129024a_2b^2m^7 + 280a_2m(a_0(a_1^2 + 2a_0a_2) + 2a_0a_1^2\\
+\ &a_0^2a_2) + 2800a_2^3b^2m^3 + 61152a_2^2b^2m^5 + 2800a_2bm^3(a_1^2 + 2a_0a_2) + 23520a_0\\
.\ &a_2bm^5 + 420a_1^2a_2^2bm + 8540a_1^2a_2bm^3 + 280a_2bm(a_2(a_1^2 + 2a_0a_2) + a_0a_2^2\\
+\ &2a_1^2a_2) = 0,
\end{aligned}$$

$$\begin{aligned}
&280a_1(4a_1^2bm^3 + 20a_2^2b^2m^3) + 420a_0^2a_1m^3 + 210a_1^3bm^3 + 12096a_1b^2m^7\\
+\ &140a_1m(a_0(a_1^2 + 2a_0a_2) + 2a_0a_1^2 + a_0^2a_2) + 8820a_1a_2^2b^2m^3 + 560a_1bm^3(a_1^2\\
+\ &2a_0a_2) + 840a_0^2a_1a_2m + 3360a_0a_1bm^5 + 280a_2bm(a_1(a_1^2 + 2a_0a_2) + 4a_0a_1\\
.\ &a_2) + 53648a_1a_2b^2m^5 + 140a_1bm(a_2(a_1^2 + 2a_0a_2) + a_0a_2^2 + 2a_1^2a_2)\\
+\ &11760a_0a_1a_2bm^3 = 0,
\end{aligned}$$

$$\begin{aligned}
&280a_0(4a_1^2bm^3 + 20a_2^2b^2m^3) + 140b(4a_1^2bm^5 + 32a_2^2b^2m^5) + 2a_2k + 280a_2\\
.\ &(2a_1^2b^2m^3 + 4a_2^2b^3m^3) + 70b^2(12a_1^2m^5 + 144ba_2^2m^5) + 420a_0^2a_1^2m + 56320\\
.\ &a_2b^3m^7 + 4256a_1^2b^2m^5 + 560a_2^3b^3m^3 + 16576a_2^2b^3m^5 + 280a_0^3a_2m + 5460\\
.\ &a_1^2a_2b^2m^3 + 140a_1bm(a_1(a_1^2 + 2a_0a_2) + 4a_0a_1a_2) + 1120a_2b^2m^3(a_1^2 + 2a_0\\
.\ &a_2) + 1120a_0a_1^2bm^3 + 2800a_0^2a_2bm^3 + 17248a_0a_2b^2m^5 + 280a_2bm(a_0(a_1^2\\
+\ &2a_0a_2) + 2a_0a_1^2 + a_0^2a_2) = 0,
\end{aligned}$$

$$\begin{aligned}
&a_1k + 280a_1(2a_1^2b^2m^3 + 4a_2^2b^3m^3) + 3968a_1b^3m^7 + 210a_1^3b^2m^3 + 140a_0^3\\
.\ &a_1m + 1400a_1a_2^2b^3m^3 + 140a_1b^2m^3(a_1^2 + 2a_0a_2) + 560a_0^2a_1bm^3 + 1904a_0\\
.\ &a_1b^2m^5 + 16240a_1a_2b^3m^5 + 140a_1bm(a_0(a_1^2 + 2a_0a_2) + 2a_0a_1^2 + a_0^2a_2)\\
+\ &6160a_0a_1a_2b^2m^3 + 840a_0^2a_1a_2bm = 0,
\end{aligned}$$

$$\begin{aligned}
&280a_0(2a_1^2b^2m^3 + 4a_2^2b^3m^3) + 70b^2(4a_1^2bm^5 + 32a_2^2b^2m^5) + 7936a_2b^4m^7\\
+\ &2a_2bk + 896a_1^2b^3m^5 + 1344a_2^2b^4m^5 + 280a_0a_1^2b^2m^3 + 1120a_0^2a_2b^2m^3\\
+\ &980a_1^2a_2b^3m^3 + 280a_0^3a_2bm + 420a_0^2a_1^2bm + 3808a_0a_2b^3m^5 = 0,
\end{aligned}$$

$$\begin{aligned}
&140a_0^3a_1bm + 140a_0^2a_1b^2m^3 + 224a_0a_1b^3m^5 + 560a_2a_0a_1b^3m^3 + 70a_1^3b^3m^3\\
+\ &272a_1b^4m^7 + 952a_2a_1b^4m^5 + ka_1b = 0.
\end{aligned}$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solutions:

$a_1 = 0, a_2 = -2m^2, k = -4(35a_0^3m + 140a_0^2bm^3 + 196a_0b^2m^5 + 96b^3m^7)$:
When $b > 0$,

$$\begin{aligned} u_1(x,t) &= a_0 - 2m^2\Big(\sqrt{b}\tan(\sqrt{b}(-4(35a_0^3m + 140a_0^2bm^3 + 196a_0b^2m^5 \\ &+ 96b^3m^7)\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \end{aligned} \tag{4.4}$$

$$\begin{aligned} u_2(x,t) &= a_0 - 2m^2\Big(\sqrt{b}\cot(\sqrt{b}(-4(35a_0^3m + 140a_0^2bm^3 + 196a_0b^2m^5 \\ &+ 96b^3m^7)\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \end{aligned} \tag{4.5}$$

When $b < 0$,

$$\begin{aligned} u_3(x,t) &= a_0 - 2m^2\Big(\sqrt{-b}\tanh(\sqrt{-b}(-4(35a_0^3m + 140a_0^2bm^3 + 196a_0b^2m^5 \\ &+ 96b^3m^7)\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \end{aligned} \tag{4.6}$$

$$\begin{aligned} u_4(x,t) &= a_0 - 2m^2\Big(\sqrt{-b}\coth(\sqrt{-b}(-4(35a_0^3m + 140a_0^2bm^3 + 196a_0b^2m^5 \\ &+ 96b^3m^7)\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \end{aligned} \tag{4.7}$$

When $b = 0$,

$$u_5(x,t) = -\frac{1}{-4(35a_0^3m + 140a_0^2bm^3 + 196a_0b^2m^5 + 96b^3m^7)\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + D}. \tag{4.8}$$

Fig.2 shows 3D plot of the traveling wave solution $u_3(x,t)$ in Eq.(4.1) for $\alpha = 0.25$, $\beta = 0.5$, $m = 0.07$, $a_0 = 0.25$, $b = -0.05$. Note that all of the solutions which are given by Eqs.(4.4)-(4.8) satisfy the conformable space-time fractional Lax equation Eq.(4.1). This has been seen by substituting the obtained solutions into the Eq.(4.1) and using the symbolic toolbox of MATLAB.

5. Analytic solutions to the conformable space-time fractional Kaup-Kupershmidt equation

Finally, we consider conformable space-time fractional Kaup-Kupershmidt equation as follows [19, 20, 21]

$$\begin{aligned} &T_t^\alpha u + 2016u^3T_x^\beta u + 630(T_x^\beta u)^3 + 2268uT_x^\beta uT_x^\beta T_x^\beta u + 504u^2T_x^\beta T_x^\beta T_x^\beta u \\ &+ 252T_x^\beta T_x^\beta uT_x^\beta T_x^\beta T_x^\beta u + 147T_x^\beta uT_x^\beta T_x^\beta T_x^\beta T_x^\beta u + 42uT_x^\beta T_x^\beta T_x^\beta T_x^\beta T_x^\beta u \\ &+ T_x^\beta T_x^\beta T_x^\beta T_x^\beta T_x^\beta T_x^\beta T_x^\beta u = 0, \ \ 0 < \alpha \le 1, \ \ 0 < \beta \le 1. \end{aligned} \tag{5.1}$$

Using the transformation (3.2), Eq.(5.1) can be written in following form

$$\begin{aligned} &kU' + 2016mU^3U' + 630m^3(U')^3 + 2268m^3UU'U'' + 504m^3U'''U^2 \\ &+ 252m^5U''U''' + 147m^5U'U^{(4)} + 42m^5UU^{(5)} + m^7U^{(7)} = 0. \end{aligned} \tag{5.2}$$

Let us suppose that the solution of Eq.(5.2) can be expressed in the form of Eq.(3.4) .Substituting Eq.(3.4) into Eq.(5.2) and then by balancing the highest order derivative term and nonlinear term in result equation, the value of N can be determined as 2. Therefore, Eq.(3.4) can be written as follows

$$U(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2. \tag{5.3}$$

Substituting (5.3) into (5.2), collecting all the terms with the same power of ϕ, we can obtain a set of algebraic equations for the unknowns a_0, a_1, a_2, b, k, m:

$$
\begin{aligned}
& 4032a_2^4 m + 44352a_2^3 m^3 + 101808a_2^2 m^5 + 40320a_2 m^7 = 0,\\
& 14112a_1 a_2^3 m + 84672a_1 a_2^2 m^3 + 81144a_1 a_2 m^5 + 5040a_1 m^7 = 0,\\
& 2268a_2(2a_1^2 m^3 + 28ba_2^2 m^3) + 11592a_1^2 m^5 + 27216a_0 a_2^2 m^3 + 6048a_1^2 a_2^2 m\\
+ & 32508a_1^2 a_2 m^3 + 35280a_2^3 bm^3 + 285264a_2^2 bm^5 + 4032a_2 m(a_2(a_1^2 + 2a_0 a_2)\\
+ & a_0 a_2^2 + 2a_1^2 a_2) + 12096a_2 m^3(a_1^2 + 2a_0 a_2) + 30240a_0 a_2 m^5 + 4032a_2^4 bm\\
+ & 120960a_2 bm^7 = 0,\\
& 2268a_1(2a_1^2 m^3 + 28ba_2^2 m^3) + 630a_1^3 m^3 + 4032a_2 m(a_1(a_1^2 + 2a_0 a_2)\\
+ & 4a_0 a_1 a_2) + 2016a_1 m(a_2(a_1^2 + 2a_0 a_2) + a_0 a_2^2 + 2a_1^2 a_2) + 3024a_1 m^3\\
. & (a_1^2 + 2a_0 a_2) + 5040a_0 a_1 m^5 + 13440a_1 bm^7 + 46872a_0 a_1 a_2 m^3 + 14112a_1\\
. & a_2^3 bm + 208824a_1 a_2 bm^5 + 116928a_1 a_2^2 bm^3 = 0,\\
& 2268a_2(4a_1^2 bm^3 + 20a_2^2 b^2 m^3) + 2268a_0(2a_1^2 m^3 + 28ba_2^2 m^3) + 504b(12a_1^2 m^5\\
+ & 144ba_2^2 m^5) + 2016a_1 m(a_1(a_1^2 + 2a_0 a_2) + 4a_0 a_1 a_2) + 6048a_0 a_1^2 m^3 + 12096\\
. & a_0^2 a_2 m^3 + 20496a_1^2 bm^5 + 129024a_2 b^2 m^7 + 4032a_2 m(a_0(a_1^2 + 2a_0 a_2) + 2a_0\\
. & a_1^2 + a_0^2 a_2) + 23184a_2^3 b^2 m^3 + 206640a_2^2 b^2 m^5 + 20160a_2 bm^3(a_1^2 + 2a_0 a_2)\\
+ & 70560a_0 a_2 bm^5 + 6048a_1^2 a_2^2 bm + 69300a_1^2 a_2 bm^3 + 4032a_2 bm(a_2(a_1^2\\
+ & 2a_0 a_2) + a_0 a_2^2 + 2a_1^2 a_2) = 0,\\
& 2268a_1(4a_1^2 bm^3 + 20a_2^2 b^2 m^3) + 3024a_0^2 a_1 m^3 + 1890a_1^3 bm^3 + 12096a_1 b^2 m^7\\
+ & 2016a_1 m(a_0(a_1^2 + 2a_0 a_2) + 2a_0 a_1^2 + a_0^2 a_2) + 71568a_1 a_2^2 b^2 m^3 + 4032a_1 bm^3\\
. & (a_1^2 + 2a_0 a_2) + 12096a_0^2 a_1 a_2 m + 10080a_0 a_1 bm^5 + 4032a_2 bm(a_1(a_1^2\\
+ & 2a_0 a_2) + 4a_0 a_1 a_2) + 179592a_1 a_2 b^2 m^5 + 2016a_1 bm(a_2(a_1^2 + 2a_0 a_2) + a_0 a_2^2\\
+ & 2a_1^2 a_2) + 90216a_0 a_1 a_2 bm^3 = 0,\\
& 2268a_0(4a_1^2 bm^3 + 20a_2^2 b^2 m^3) + 504b(4a_1^2 bm^5 + 32a_2^2 b^2 m^5) + 2a_2 k + 2268\\
. & a_2(2a_1^2 b^2 m^3 + 4a_2^2 b^3 m^3) + 252b^2(12a_1^2 m^5 + 144ba_2^2 m^5) + 6048a_0^2 a_1^2 m\\
+ & 56320a_2 b^3 m^7 + 13944a_1^2 b^2 m^5 + 5040a_2^3 b^3 m^3 + 56112a_2^2 b^3 m^5 + 4032a_0^3 a_2 m\\
+ & 45108a_1^2 a_2 b^2 m^3 + 2016a_1 bm(a_1(a_1^2 + 2a_0 a_2) + 4a_0 a_1 a_2) + 8064a_2 b^2 m^3(a_1^2\\
+ & 2a_0 a_2) + 8064a_0 a_1^2 bm^3 + 20160a_0^2 a_2 bm^3 + 51744a_0 a_2 b^2 m^5 + 4032a_2 bm(a_0\\
. & (a_1^2 + 2a_0 a_2) + 2a_0 a_1^2 + a_0^2 a_2) = 0,\\
& a_1 k + 2268a_1(2a_1^2 b^2 m^3 + 4a_2^2 b^3 m^3) + 3968a_1 b^3 m^7 + 1890a_1^3 b^2 m^3 + 2016\\
. & a_0^3 a_1 m + 12096a_1 a_2^2 b^3 m^3 + 1008a_1 b^2 m^3(a_1^2 + 2a_0 a_2) + 4032a_0^2 a_1 bm^3\\
+ & 5712a_0 a_1 b^2 m^5 + 55272a_1 a_2 b^3 m^5 + 2016a_1 bm(a_0(a_1^2 + 2a_0 a_2) + 2a_0 a_1^2\\
+ & a_0^2 a_2) + 47880a_0 a_1 a_2 b^2 m^3 + 12096a_0^2 a_1 a_2 bm = 0,\\
& 2268a_0(2a_1^2 b^2 m^3 + 4a_2^2 b^3 m^3) + 252b^2(4a_1^2 bm^5 + 32a_2^2 b^2 m^5) + 7936a_2 b^4 m^7\\
+ & 2a_2 bk + 3024a_1^2 b^3 m^5 + 4704a_2^2 b^4 m^5 + 2016a_0 a_1^2 b^2 m^3 + 8064a_0^2 a_2 b^2 m^3\\
+ & 8316a_1^2 a_2 b^3 m^3 + 4032a_0^3 a_2 bm + 6048a_0^2 a_1^2 bm + 11424a_0 a_2 b^3 m^5 = 0,\\
& 2016a_0^3 a_1 bm + 1008a_0^2 a_1 b^2 m^3 + 672a_0 a_1 b^3 m^5 + 4536a_2 a_0 a_1 b^3 m^3 + 630a_1^3\\
. & b^3 m^3 + 272a_1 b^4 m^7 + 3360a_2 a_1 b^4 m^5 + ka_1 b = 0.
\end{aligned}
$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solutions:

$a_0 = -\frac{bm^2}{3}, a_1 = 0, a_2 = -\frac{m^2}{2}, k = \frac{-4}{3}b^3m^7$:
When $b > 0$,

$$u_1(x,t) = -\frac{bm^2}{3} - \frac{m^2}{2}\Big(\sqrt{b}\tan(\sqrt{b}(\frac{-4}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \quad (5.4)$$

$$u_2(x,t) = -\frac{bm^2}{3} - \frac{m^2}{2}\Big(\sqrt{b}\cot(-\sqrt{b}(\frac{-4}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \quad (5.5)$$

When $b < 0$,

$$u_3(x,t) = -\frac{bm^2}{3} - \frac{m^2}{2}\Big(\sqrt{-b}\tanh(\sqrt{-b}(\frac{-4}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \quad (5.6)$$

$$u_4(x,t) = -\frac{bm^2}{3} - \frac{m^2}{2}\Big(\sqrt{-b}\coth(\sqrt{-b}(\frac{-4}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}))\Big)^2. \quad (5.7)$$

When $b = 0$,

$$u_5(x,t) = -\frac{1}{\frac{-4}{3}b^3m^7\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + D}. \quad (5.8)$$

Fig.3 shows 3D plot of the traveling wave solution $u_1(x,t)$ in Eq.(5.1) for $\alpha = 0.75$, $\beta = 1$, $m = 1$, $b = 0.05$. Note that all of the solutions which are given by Eqs.(5.4)-(5.8) satisfy the conformable space-time fractional Kaup-Kupershmidt equation Eq.(5.1). This has been seen by substituting the obtained solutions into the Eq.(5.1) and using the symbolic toolbox of MATLAB.

6. Conclusion

In this paper, we apply to the extended tanh method to the conformable space-time fractional KdV equations: conformable space-time fractional Sawada-Kotera-Ito equation, conformable space-time fractional Lax equation and conformable space-time fractional Kaup-Kupershmidt equation. The obtained traveling wave solutions are expressed by the hyperbolic, trigonometric, exponential and rational functions. These solutions are new and not found elsewhere. The effect of the fractional order derivative on some of these solutions are represented graphically for special values of the parameters. Furthermore, it has been checked that all of the obtained solutions verify the related equations. This means that all of the obtained solutions are exact solutions. The extended tanh method can be also applied to the another conformable nonlinear partial differential equations with constant coefficients and the system of the conformable nonlinear partial differential equations with constant coefficients.

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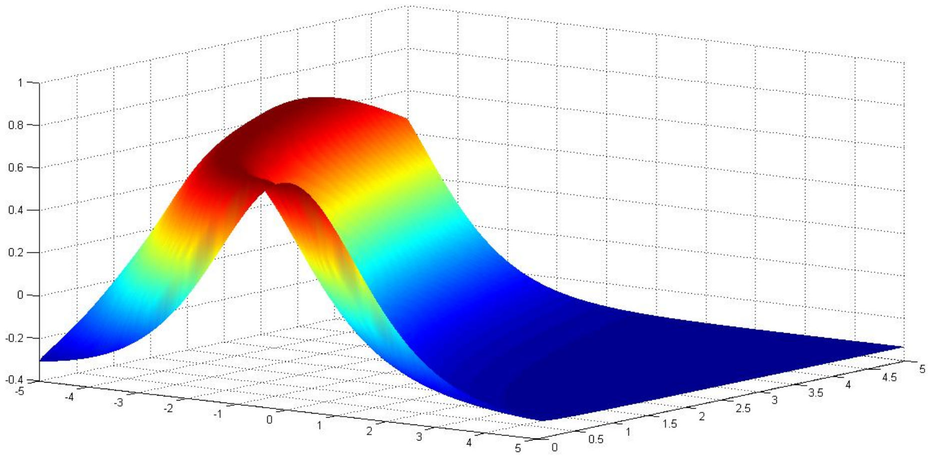

Figure 1: 3D plot of the obtained traveling wave solution $u_3(x,t)$ of Eq.(3.1).

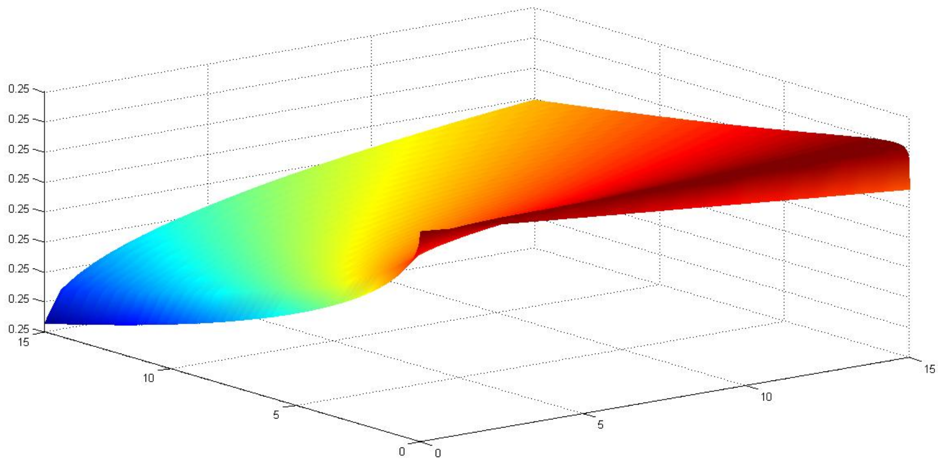

Figure 2: 3D plot of the obtained traveling wave solution $u_3(x,t)$ of Eq.(4.1).

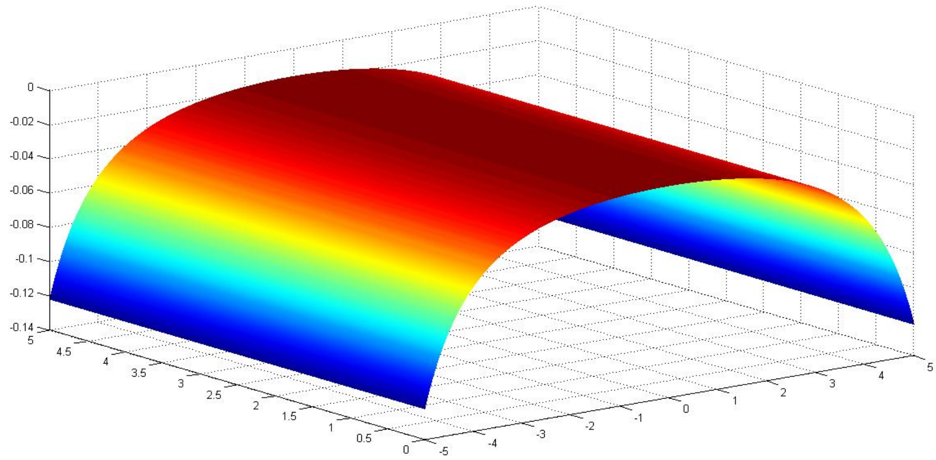

Figure 3: 3D plot of the obtained traveling wave solution $u_1(x,t)$ of Eq.(5.1).