



The existence of uniqueness non standard equilibrium problems

Dhuha M. Abbas^a, Ayed E. Hashoosh^{a,*}, Wijdan S. Abed^b

^aDepartment of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq

^bDepartment of Mathematics, Faculty of Science, Arak University, Arak 38156-8-8349, Iran

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, the concept of $\eta\xi$ -monotonous operator is explored using KKM mapping. The existence results and uniqueness defined on its bounded and unbounded domains are discussed. Our findings improve and develop some well-known solutions in literature.

Keywords: Monotonicity, Equilibrium problem set-valued mapping, Hemicontinuity, KKM-mapping, Semi continuous, Convex function

1. Introduction

Equilibrium problems have shown to be significant mathematical models in a variety of real world problems. This forms play vital roles in applied math and economics, ranging from price balance and market to general economic or traffic network equilibrium, just to name a few cases of a plethora of equation matter whose presence derives from a long history of active and rich research field in frugality, industry or other branches of applied sciences. It is worth mentioning that the equilibrium issue is defined as follows:

(EP) Find $n \in K$ such that $\Psi(n, m) \geq 0$, for all $m \in K$.

Here K is a nonempty subset of a topological vector space and $\Psi : K \times K \rightarrow \mathbb{R}$ is a real valued equilibrium of bifunction, that is, $\Psi(n, n) = 0$ for all $n \in K$. This formula was first used by N.Isoda [1] in 1955 to characterize Nash equilibrium. In 1972, Ky Fan [2] reexamined the question (EP) and established some existence results to solve under the strong assumption that the K is a compact

*Corresponding author

convex subset of a true Hausdorff topological vector space. Since then, many authors have weakened the compression of K by using an appropriate form of coercion condition. (EP) has further been studied along the years 1975 – 1979. For instance, Moscow [3] and J.Mosco [4] presented a general formula for equilibrium problems called Implicit Variational Problems which include fixed point matters, variable and semi-variable inequalities and Nash equilibrium as special cases of poise issue. This general formulation of (EP) has been described by the sum of two binary functions, simply, to begin what was later called a mixed equilibrium problem.

Inspired by the concept of monotony operator in the sense of Minty, both Moscow and J.Mosco managed to introduce the notion of monotone binary operations and thus obtained some existence results for solving this type of equilibrium case under an assumption weaker than those in Fan's results [2].

It is interesting, the term equilibrium problem appeared first time in the paper by Blum and Oettli [3] in which they offered some evidence on the unifying aspect of (EP) and provided some basic concepts and findings. These problems were subsequently studied extensively by various authors in the context of Hilbert and Banach spaces under different conditions.

Later on, further papers have been conducted to study special cases of generalized monotone [2,5,6,7 and 8]. All of these papers have been based on bifunction equilibrium problems. In 1990 Karamardian and Schaible [9] introduced various types of generalized monochromatic designations. therefore, several researchers [2, 10] extended Karamardian and Schaible [9]'s idea of binary functions to study (EP).

Fang and Huang in 2003 considered special forms of the equilibrium problem by [11] by using new skill of fixed-point theorem and the notion of relaxed $\eta\alpha$ semimonotone mapping. In 2018 , Malek Abbasi and Mahboubeh Rezaei [12] studied some of the returns of generalized vector equilibrium problems and vector general differential inequalities using the famous KKM mapping.

Equally important is the field of monotone operator. Monotone operator related to equilibrium problem were emerged as a powerful tool to investigate and study a wide class of unrelated problems arising in industrial, regional, physical, pure, and applied sciences in a unified and general framework. Monotone operator have been extended and generalized in several directions using novel and new techniques(see [13],[14],[15])

In 2020, Chadll and G. Pany [16] considered a new class of (MEP), under a weak type of α - monotone in the setting of Banach spaces. This class of matters broadens and generalizes some of the basic findings related to mixed variational like inequalities, mixed variational and other classic problems as special cases of (EP) and thereby the existence and uniqueness of solutions are established.

This paper is divide into four sections including the introduction. In section two, some apposite definitions that are important to our work is recalled. In section three, new class of monotonicity, called $\eta\xi$ - monotone is gave. Moreover, the existence results and uniqueness defined on its bounded and unbounded domains are investigated. In the last section of this work, we come up with the most significant conclusions of this paper.

2. Preliminaries

We will introduce new concept of monotonicity is called $\eta\xi$ -monotone operator.

Definition 2.1. *Let E be a nonempty subset of real reflexive Banach space X . $F, \eta : E \times E \rightarrow \mathbb{R}$ real-valued bifunctions, $\xi : X \rightarrow X$ affine functions, then F is said to be $\eta\xi$ -monotone if*

$$F(\xi(a), \xi(b)) + F(\xi(b), \xi(a)) + \eta(\xi(a), \xi(b)) \leq 0; \quad \forall \xi(a), \xi(b) \in E. \quad (2.1)$$

Remark 2.2. Let X be a Banach space and E a nonempty convex and subset of X . $\eta, \gamma : E \times E \rightarrow R \cup [+\infty]$ two real bifunctions, then:

$$\lim_{\lambda \rightarrow 0} \frac{\eta(\xi(c), \xi(c\lambda))}{\lambda} = 0$$

$$\eta(\xi(c), \xi(d)) \leq \lim_{\lambda \rightarrow +0} \frac{\lambda - 1}{\lambda} [\gamma(\xi(c), \xi(c)) + \eta(\xi(c), \xi(c))]. \tag{2.2}$$

Definition 2.3. [17] A bifunction $F : E \times E \rightarrow R$ then F is said to be monotone if and only if

$$F(e, d) + F(d, e) \leq 0; \quad \forall e, d \in E. \tag{2.3}$$

Definition 2.4. [17] Let E be a nonempty convex and subset of areal reflexive Banach space X . $F : E \times E \rightarrow R$ is real-valued bifunction, then F is said to be equilibrium function if

$$F(k, k) = 0 \quad \forall k \in E. \tag{2.4}$$

Now, We consider the following generalized equilibrium problem (for short EP γ) is to find $\xi(h) \in E$ such that

$$F(\xi(h), \xi(d)) + \gamma(\xi(h), \xi(d)) \geq 0; \quad \forall \xi(d) \in E. \tag{2.5}$$

In order to high light the generality of a problem (6), We recall some Special Cases,

- I . If $\gamma \equiv 0, \xi \equiv I$ then Problem (2.5) is reduced to the Classical equilibrium problem; [3] (for short, EP), which is to find $h \in E$ such that $F(h, d) \geq 0 \quad \forall d \in E$.
- II . If $\gamma(\xi(h), \xi(d)) = \gamma(\xi(d)) - \gamma(\xi(h)), \xi \equiv I$ then problem (2.5) is reduced to the mixed equilibrium problem [18] (for short MEP).
- III. If $\xi \equiv I$ the identity operator, the extended general equilibrium problem; [17]: find $h \in E$ such that

$$\begin{aligned} &F(\xi(h), \xi(d)) + \gamma(\xi(h), \xi(d)) \\ &= F(h, d) + \gamma(h, d) \geq 0 \quad \forall d \in E. \end{aligned}$$

Definition 2.5. [19] A bifunction $F : E \times E \rightarrow R$, then F is called vector 0 -diagonally Convex if, for any finite subset $\{a_1, a_2, \dots, a_n\}$ of E and $\alpha_i \geq 0 \quad (i = 1, 2, \dots, n)$ with $u = \sum_{i=1}^n \alpha_i a_i$ and $\sum_{i=1}^n \alpha_i = 1$ one has

$$\sum_{i=1}^n \alpha_i F(a, a_i) \geq 0. \tag{2.6}$$

Definition 2.6. [20] Let E be a nonempty subset of a Hausdorff topological vector space A . A mapping $\alpha : E \rightarrow 2^A$ is said to be a KKM mapping if, for any finite subset $\{v_1, v_2, \dots, v_n\}$ of E , We have $co \{v_1, v_2, \dots, v_n\} \subset \bigcap_{i=1}^n \alpha(v_i)$ where $co \{v_1, v_2, \dots, v_n\}$ denotes the convex hull of $\{v_1, \dots, v_n\}$

Definition 2.7. [21] A real-valued function, defined on a convex subset E of X , is said to be hemicontinuous, if

$$\lim_{r \rightarrow 0^+} \wedge(ru + (1 - r)v) = \wedge(v); \quad \forall u, v \in E. \tag{2.7}$$

Lemma 2.8. [22] Let E be a nonempty subset of a Hausdorff topological vector space A and let $\beta : E \rightarrow 2^A$ be a KKM mapping. If $\beta(v)$ is closed in A for every $v \in E$ and compact for some $v_0 \in E$, then $\bigcap_{v \in E} \beta(v) \neq \emptyset$.

3. THE MAIN RESULTS

In this paper some results existence for equilibrium ($EP\gamma$) are recalled and through the results of this section we confirmed the existence of a solution to the problem ($EP\gamma$) without supposing boundedness of the set E then we were can to reach the unique of solution.

Theorem 3.1. *Let E be a nonempty subset of a real reflexive Banach space X . Assume that*

i) $F : E \times E \rightarrow R$ is $\eta\xi$ -monotone bifunction, hemicontinuous in 1.argument, and convex in 2.argument, where $F(\xi(c), \xi(c)) = 0$ for all $\xi(c) \in E$.

ii) $\eta, \gamma : E \times E \rightarrow R, \xi : X \rightarrow X, \eta, \gamma$ be convex in 2.argument, ξ affine function.

Then generalized equilibrium problem ($EP\gamma$) is equivalent to the following Problem: Find $\xi(c) \in E$ such that

$$F(\xi(d), \xi(c)) + \eta(\xi(c), \xi(d)) \leq \gamma(\xi(c), \xi(d)); \forall \xi(d) \in E. \quad (3.1)$$

Proof . Suppose that ($EP\gamma$) has a solution. So there exists $\xi(c) \in E$ such that

$$F(\xi(c), \xi(d)) + \gamma(\xi(c), \xi(d)) \geq 0 \quad \forall \xi(d) \in E. \quad (3.2)$$

Since F is $\eta\xi$ -monotone bifunction, we have

$$F(\xi(d), \xi(c)) + F(\xi(c), \xi(d)) + \eta(\xi(c), \xi(d)) \leq 0 \quad \forall \xi(c), \xi(d) \in E. \quad (3.3)$$

Then

$$F(\xi(d), \xi(c)) + \eta(\xi(c), \xi(d)) \leq -F(\xi(c), \xi(d)) \leq \gamma(\xi(c), \xi(d)). \quad (3.4)$$

Therefore, $\xi(c) \in E$ is a solution of problem (3.1).

Conversely, Assume that $\xi(c) \in E$ is a solution of problem (3.1) and fixed $\xi(c) \in E$. For $\lambda \in [0, 1]$, we let $\xi(c\lambda) = \lambda\xi(d) + (1 - \lambda)\xi(c)$. Then $\xi(c\lambda) \in E$, Since E is convex any ξ is affine, we have

$$F(\xi(c\lambda), \xi(c)) + \eta(\xi(c), \xi(c\lambda)) \leq \gamma(\xi(c), \xi(c\lambda)). \quad (3.5)$$

Then

$$F(\xi(c\lambda), \xi(c)) - \gamma(\xi(c), \xi(c\lambda)) \leq -\eta(\xi(c), \xi(c\lambda)). \quad (3.6)$$

By the convexity of F in the 2 .argument any ξ is affine, we have

$$\begin{aligned} 0 &= F(\xi(c\lambda), \xi(c\lambda)) \leq \lambda F(\xi(c\lambda), \xi(d)) + (1 - \lambda)F(\xi(c\lambda), \xi(c)) \\ 0 &= F(\xi(c\lambda), \xi(c\lambda)) \leq \lambda F(\xi(c\lambda), \xi(d)) + F(\xi(c\lambda), \xi(c)) - \lambda F(\xi(c\lambda), \xi(c)) \end{aligned} \quad (3.7)$$

So,

$$\lambda[F(\xi(c\lambda), \xi(c)) - F(\xi(c\lambda), \xi(c))] \leq F(\xi(c\lambda), \xi(c)). \quad (3.8)$$

By convexity of γ and η , we get

$$\begin{aligned} \eta(\xi(c), \xi(c\lambda)) &\leq \lambda\eta(\xi(c), \xi(d)) + (1 - \lambda)\eta(\xi(c), \xi(c)) \\ \gamma(\xi(c), \xi(c\lambda)) &\leq \lambda\gamma(\xi(c), \xi(d)) + (1 - \lambda)\gamma(\xi(c), \xi(c)) \end{aligned}$$

$$\begin{aligned} \eta(\xi(c), \xi(c\lambda)) &\leq \lambda\eta(\xi(c), \xi(d)) + \eta(\xi(c), \xi(c)) - \lambda\eta(\xi(c), \xi(c)) \\ \gamma(\xi(c), \xi(c\lambda)) &\leq \lambda\gamma(\xi(c), \xi(d)) + \gamma(\xi(c), \xi(c)) - \lambda\gamma(\xi(c), \xi(c)). \end{aligned}$$

Then

$$\begin{aligned} \lambda[\eta(\xi(c), \xi(c)) - \eta(\xi(c), \xi(d))] &\leq \eta(\xi(c), \xi(c)) - \eta(\xi(c), \xi(c\lambda)) \\ \lambda[\gamma(\xi(c), \xi(c)) - \gamma(\xi(c), \xi(d))] &\leq \gamma(\xi(c), \xi(c)) - \gamma(\xi(c), \xi(c\lambda)). \end{aligned} \tag{3.9}$$

From (3.5),(3.6) and (3.9)

$$\begin{aligned} \lambda[F(\xi(c\lambda), \xi(c)) - F(\xi(c\lambda), \xi(d)) + \eta(\xi(c), \xi(c\lambda)) - \eta(\xi(c), \xi(d)) + \gamma(\xi(c), \xi(c)) - \gamma(\xi(c), \xi(d))] \\ \leq F(\xi(c\lambda), \xi(c)) - \eta(\xi(c), \xi(c\lambda)) - \gamma(\xi(c), \xi(c\lambda)) + \eta(\xi(c), \xi(c)) \\ \leq -2\eta(\xi(c), \xi(c\lambda)) + \eta(\xi(c), \xi(c)) + \gamma(\xi(c), \xi(c)). \end{aligned} \tag{3.10}$$

From hemicontinuous in 1.argument

$$\begin{aligned} \lambda[-F(\xi(c), \xi(d)) - \gamma(\xi(c), \xi(d))] &\leq -2\eta(\xi(c), \xi(c\lambda)) + \lambda\eta(\xi(c), \xi(d)) \\ &\quad + (1 - \lambda)[\gamma(\xi(c), \xi(c)) + \eta(\xi(c), \xi(c\lambda))] \end{aligned} \tag{3.11}$$

$$\begin{aligned} F(\xi(c\lambda), \xi(d)) - \gamma(\xi(c), \xi(d)) &\leq \frac{2\eta(\xi(c), \xi(c\lambda))}{\lambda} - \eta(\xi(c), \xi(d)) \\ &\quad + \frac{1 - \lambda}{\lambda} [\gamma(\xi(c), \xi(c)) + \eta(\xi(c), \xi(c))]. \end{aligned} \tag{3.12}$$

From Remark 2.2. one can get

$$F(\xi(c), \xi(d)) + \gamma(\xi(c), \xi(d)) \geq 0 \quad \forall \xi(d) \in E. \tag{3.13}$$

Therefore, $(EP\gamma)$ has a solution. \square

Theorem 3.2. *Let E be a nonempty closed bounded convex subset of a real reflexive Banach space X . $\xi : X \rightarrow X$ affine furcation. Assume that*

1. $F : E \times E \rightarrow R$ is $\eta\xi$ - monotone bifunction, 0-diagonal convex, hemicontinuous in 1.argument and lsc, convex in 2.argument;
2. $\gamma : E \times E \rightarrow R$ is convex in 2.argument, usc in 1.argument, and 0-diagonal convex;
3. $\eta : E \times E \rightarrow R$ is lsc in 1.argument and convex in 2 argument;

Then $(EP\gamma)$ admit a solution.

Proof . Defined two set valued mappings $\Lambda, \alpha : E \rightarrow 2^E$ as following:

$$\Lambda(\xi(t)) = \{\xi(r) \in E : F(\xi(t), \xi(r)) + \gamma(\xi(t), \xi(r)) \geq 0 \quad \forall \xi(r) \in E\} \tag{3.14}$$

$$\alpha(\xi(r)) = \{\xi(t) \in E : F(\xi(r), \xi(t)) + \eta(\xi(t), \xi(r)) \leq \gamma(\xi(t), \xi(r)) \quad \forall \xi(r) \in E\}. \tag{3.15}$$

Then, the problem $(EP\gamma)$ has a solution iff

$\bigcap_{\xi(r) \in E} \Lambda(\xi(r)) \neq \phi$. Claim 1 \wedge is a KKM mapping.

If \wedge is not a KKM mapping. Then, there exists a finite subset

$\{\xi(t_1), \xi(t_2), \dots, \xi(t_m)\}$ of E and $\beta_i \geq 0$ ($i = 1, 2, \dots, m$) with $\sum_{i=1}^m \beta_i = 1$, in which $\xi(t_0) = \sum_{i=1}^m \beta_i \xi(t_i) \notin \cup_{i=1}^m \wedge(\xi(t_i))$.

Then

$$\begin{aligned} F(\xi(t_0), \xi(t_i)) + \gamma(\xi(t_0), \xi(t_i)) &< 0 \quad \forall i = 1, 2, \dots, m \\ \sum_{i=1}^m \beta_i [F(\xi(t_0), \xi(t_i)) + \gamma(\xi(t_0), \xi(t_i))] &< 0 \quad \forall i = 1, 2, \dots, m \end{aligned} \quad (3.16)$$

We reach to contrary of 0-diagonal convex property for F and γ .

Therefore, \wedge is a KKM mapping.

Claim 2, $\wedge(\xi(r)) \subseteq \alpha(\xi(r)) \quad \forall \xi(r) \in E$.

Let $\xi(t) \in \wedge(\xi(r))$; then

$$F(\xi(t), \xi(r)) + \gamma(\xi(t), \xi(r)) \geq 0 \quad \forall \xi(t) \in E. \quad (3.17)$$

From definition of $\eta\xi$ -monotone bifunction, one can obtain

$$\begin{aligned} &F(\xi(t), \xi(r)) + \eta(\xi(t), \xi(r)) - \gamma(\xi(t), \xi(r)) \\ &\leq -F(g(t), g(r)) - \eta(g(t), \xi(r)) + \eta(\xi(t), \xi(r)) - \gamma(\xi(t), \xi(r)) \leq 0. \end{aligned} \quad (3.18)$$

denotes that $\xi(t) \in \alpha(\xi(r))$.

Thus $\wedge(\xi(r)) \subset \alpha(\xi(r)) \quad \forall \xi(r) \in E$.

Therefore, $\alpha(\xi(r))$ is a KKM mapping.

Since $\eta(\cdot, \xi(r), F(\xi(r), \cdot))$ are lsc and $\gamma(\cdot, \xi(r))$ is usc.

Then

$$\begin{aligned} F(\xi(r), \xi(t)) + \eta(\xi(t), \xi(r)) &\leq \liminf_m (F(\xi(r), \xi(t_m)) + \eta(\xi(t_m), \xi(r))) \\ &\leq \limsup_m (\xi(t_m), \xi(r)) \\ &\leq \gamma(\xi(t), \xi(r)) \end{aligned} \quad (3.19)$$

Thus, $\alpha(\xi(r))$ is weakly closed $\forall \xi(r) \in E$.

Since E is a nonempty, bounded, closed and convex and X is real reflexive, it follows that E is weakly compact. Therefore, $\alpha(\xi(r))$ is weakly compact $\forall \xi(r) \in E$.

From Lemma 2.8 and Theorem 3.1

$$\bigcap_{\xi(r) \in E} \wedge(\xi(r)) = \bigcap_{\xi(r) \in E} \alpha(\xi(r)) \neq \phi. \quad (3.20)$$

Therefore, the problem $(EP \gamma)$ has a solution. \square

Corollary 3.3. *Assume that $\xi(0) \in E$ is a nonempty convex subset of Banach space X and assumptions (1 – 3) in Theorem 3.2 hold. Moreover,*

1. $\gamma(\xi(a), 0) = 0$.
2. $\lambda_1 = \{\xi(a) \in E : \eta(\xi(a), 0) \leq 0\}$ and $\lambda_2 = \{\xi(a) \in E : F(0, \xi(a)) \leq 0\}$ are relative compact sets.

Then $(EP\gamma)$ admit a solution.

Proof . Note that prove $\alpha(\xi(b))$ is compact for some $\xi(0) \in E$ suffices

$$\alpha(\xi(b)) = \{\xi(a) \in E : F(\xi(b), \xi(a)) + \eta(\xi(a), \xi(b)) \leq \gamma(\xi(a), \xi(b)), \forall \xi(b) \in E\}$$

$$\alpha(\xi(b)) \subset \left[\xi(a) \in E : F(\xi(b), \xi(a)) \leq \frac{\gamma(\xi(a), \xi(b))}{2} \right] \cup \left[\xi(a) \in E : \eta(\xi(a), \xi(b)) \leq \frac{\gamma(\xi(a), \xi(b))}{2} \right]. \tag{3.21}$$

Since $\xi(0) \in E$

$$\alpha(\xi(0)) \subset \left[\xi(a) \in E : F(0, \xi(a)) \leq \frac{\gamma(\xi(a), 0)}{2} \right] \cup \left[\xi(a) \in E : \eta(\xi(a), 0) \leq \frac{\gamma(\xi(a), 0)}{2} \right]. \tag{3.22}$$

From conditions (1 – 2), $\alpha(\xi(0))$ is a subset of relative compact set λ_1 or λ_2 . This implies that $\alpha(\xi(b))$ is compact for some $\xi(b) \in E$. \square

Theorem 3.4. *Let the following assumptions hold.*

- 1- *The mappings F, γ are continuous, 0 -diagonal convex $F, \gamma : E \times E \rightarrow R$ and $\xi : X \rightarrow X$ is affine.*
- 2- *There exists a nonempty compact subset N of E and $\xi(h) \in N$ such that $\forall \xi(d) \in E/N$*

$$F(\xi(h), \xi(d)) + \gamma(\xi(h), \xi(d)) \geq 0.$$

Then the problem $(EP\gamma)$ admit a solution.

Proof . Define the set value mapping $\Lambda : E \rightarrow 2^E$, in which

$$\Lambda(\xi(d)) = \{\xi(h) \in E : F(\xi(h), \xi(d)) + \gamma(\xi(h), \xi(d)) \geq 0 \quad \forall \xi(d) \in E\}$$

then the $(EP\gamma)$ has a solution if

$$\bigcap_{\xi(d) \in E} \Lambda(\xi(d)) \neq \phi.$$

Now, we first prove $\Lambda(\xi(d))$ is a KKM mapping since $\xi(d) \in \Lambda(\xi(d)) \Rightarrow \Lambda(\xi(d)) \neq \phi$ by the contrary, let $\Lambda(\xi(d))$ is not a KKM mapping.

Then, there exists a finite subset $\{\xi(h_1), \xi(h_2), \dots, \xi(h_m)\}$ of E , $\lambda_i \geq 0 \quad i = 1, 2, \dots, m \cdot \sum_{i=1}^m \lambda_i = 1$ such that

$$\xi(h_0) = \sum_{i=1}^m \lambda_i \xi(h_i) \notin \cup_{i=1}^m \Lambda(\xi(h_i)).$$

It means that $\xi(h_0) \notin \Lambda(\xi(h_i)) \forall i = 1, 2, \dots, m$ Then

$$\begin{aligned}
 & F(\xi(h_0), \xi(h_i)) + \gamma(\xi(h_0), \xi(h_i)) < 0 \quad \forall i = 1, 2, \dots, m \\
 & \sum_{i=1}^m \lambda_i [F(\xi(h_0), \xi(h_i)) + \gamma(\xi(h_0), \xi(h_i))] < 0 \quad \forall i = 1, 2, \dots, m
 \end{aligned} \tag{3.23}$$

And this contradicts the 0-diagonal convex property of F and γ .
Then \wedge is a KKM mapping

Let $\{h_m\}$ be a sequence in E such that $\xi(h_m) \rightarrow \xi(h)$ and $h_m \in \wedge(\xi(d))$.
So,

$$F(\xi(h_m), \xi(d)) + \gamma(\xi(h_m), \xi(d)) \geq 0.$$

Since F and γ are continuous

$$\begin{aligned}
 & F(\xi(h_m), \xi(d)) \rightarrow F(\xi(h), \xi(d)) \\
 & \gamma(\xi(h_m), \xi(d)) \rightarrow \gamma(\xi(h), \xi(d))
 \end{aligned}$$

$$F(\xi(h_m), \xi(d)) + \gamma(\xi(h_m), \xi(d)) \rightarrow F(\xi(h), \xi(d)) + \gamma(\xi(h), \xi(d)) \geq 0.$$

Then

$$F(\xi(h), \xi(d)) + \gamma(\xi(h), \xi(d)) \geq 0.$$

So, $\xi(h) \in \wedge(\xi(d))$

$\Rightarrow \wedge(\xi(d))$ is closed for every $\xi(h) \in E$ and compact for some $h \in E$, by lemma 2.8.

$$\bigcap_{\xi(v) \in E} \wedge(\xi(d)) \neq \phi.$$

This completes the proof.

□

Theorem 3.5. *Suppose that the same hypotheses in Theorem 3.2 hold. By adding*

- i) γ is a monotone bifunction.*
- ii) There exists $k > 0$, such that*

$$F(\xi(n), \xi(m)) \leq -k \|\xi(m) - \xi(n)\|^2 \quad \forall \xi(n), \xi(m) \in E$$

Proof . that the existence of solution follow from Theorem 3.1 we need only to prove the uniqueness of the solution. By assuming the contrary that there exists two arbitrary solution, say $\xi(n_1)$ and $g(n_2)$ for the (EP γ).

$$\begin{aligned}
 & F(\xi(n_1), \xi(m)) + \gamma(\xi(n_1), \xi(m)) \geq 0 \quad \forall \xi(n) \in E \\
 & F(\xi(n_2), \xi(m)) + \gamma(\xi(n_2), \xi(m)) \geq 0 \quad \forall \xi(m) \in E.
 \end{aligned}$$

Since $\xi(n_1), \xi(n_2) \in E$, then

$$\begin{aligned}
 & F(\xi(n_1), \xi(n_2)) + \gamma(\xi(n_1), \xi(n_2)) \geq 0 \\
 & F(\xi(n_2), \xi(n_1)) + \gamma(\xi(n_2), \xi(n_1)) \geq 0
 \end{aligned}$$

$$\begin{aligned} F(\xi(n_1), \xi(n_2)) + \gamma(\xi(n_1), \xi(n_2)) + F(\xi(n_2), \xi(n_1)) + \gamma(\xi(n_2), \xi(n_1)) &\geq 0 \\ F(\xi(n_1), \xi(n_2)) + F(\xi(n_2), \xi(n_1)) &\geq -\gamma(\xi(n_1), \xi(n_2)) - \gamma(\xi(n_2), \xi(n_1)). \end{aligned} \quad (3.24)$$

By using γ monotone bifunction then

$$\begin{aligned} \gamma(\xi(n_1), \xi(n_2)) + \gamma(\xi(n_2), \xi(n_1)) &\leq 0 \\ -\gamma(\xi(n_1), \xi(n_2)) - \gamma(\xi(n_2), \xi(n_1)) &\geq 0. \end{aligned} \quad (3.25)$$

From (3.24) and (3.25), we have

$$F(\xi(n_1), \xi(n_2)) + F(\xi(n_2), \xi(n_1)) \geq 0 \quad (3.26)$$

$$\begin{aligned} F(\xi(n_1), \xi(n_2)) &\leq -k \|\xi(n_2) - \xi(n_1)\|^2 \\ F(\xi(n_2), \xi(n_1)) &\leq -k \|\xi(n_2) - \xi(n_1)\|^2 \end{aligned}$$

$$F(\xi(n_1), \xi(n_2)) + F(\xi(n_2), \xi(n_1)) \leq -2k \|\xi(n_2) - \xi(n_1)\|^2 \leq 0. \quad (3.27)$$

From (3.26) and (3.27) we have

$$0 \leq F(\xi(n_1), \xi(n_2)) + F(\xi(n_2), \xi(n_1)) \leq -2k \|\xi(n_2) - \xi(n_1)\|^2 \leq 0.$$

a contradiction, then

$$\|\xi(n_2) - \xi(n_1)\|^2 = 0 \Rightarrow \xi(n_2) = \xi(n_1).$$

Therefore (EP γ) has a unique solution. \square

4. CONCLUSION

In the above discussion, we have been considered with a new class of equilibrium problems involving a bifunction. We have made use of a new technique of monotonicity called $\eta\xi$ -monotone subsets in the setting of Banach space. The results coming out of this whole discussion have provided us with an inclusion that turns out to be useful for generalizing some existence results recently obtained in the literature.

References

- [1] H. Nikaido and K. Isoda, *Note on noncooperative convex games*, Pacific J. Math. 5(51) (1955) 807–815.
- [2] M. Bianchi and S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Opt. The. Appl. 90 (1996) 31–43.
- [3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, 63(1) (1994) 123–145.
- [4] J. L. Joly and U. Mosco, *A propos de l'existence et de la r'egularit'e des solutions de certaines in'equations quasi-variationnelles*, J. Funct. Anal. 34(1) (1979) 107–137.
- [5] S. M. Kang, S. Y. Cho and Z. Liu, *Convergence of iterative sequences for generalized equilibrium problems involving inverse-strongly monotone mappings*, J. Ineq. Appl. 118(2) (2010) 417–428.
- [6] N. K. Mahato and C. Nahak, *Equilibrium problems with generalized relaxed monotonicities in Banach spaces*, Opsearch, vol. 51, no. 2,(2014) pp. 257-269.
- [7] M. A. Noor, *Auxiliary principle technique for equilibrium problems*, J. Opt. Theo. Appl. 122 (2004) 371–86.
- [8] H. A. Rizvi, A. Kiliçman and R. Ahmad, *Generalized equilibrium problem with mixed relaxed monotonicity*, Sci. World J. Article ID 807324 (2014) 1–4.

- [9] S. Karamardian and S. Schaible, *Seven kinds of monotone maps*, J. Opt. Theo. Appl. 66(1) (1990) 37–46.
- [10] M. Bianchi and R. Pini, *A note on equilibrium problems with properly quasimonotone bifunctions*, J. Global Opt. 20(1) (2001) 67–76.
- [11] Y. P. Fang and N. J. Huang, *Variational-like inequalities with generalized monotone mappings in Banach spaces*, J. Opt. Theo. Appl. 118(6) (2003) 327–338.
- [12] M. Abbasi and M. Rezaei, *Existence results for generalized-vector equilibrium problems*, Iran. J. Math. Sci. Inf. 13(2) (2018) 29–43.
- [13] A. A. Alwan, H. A. Naser, and A. E. Hashoosh, *The solutions of mixed hemiequilibrium problem with application in Sobolev space*, IOP Conference Series: Materials Science and Engineering, 571(1) (2019) 1–9.
- [14] A. E. Hashoosh, *Existence and uniqueness solutions for a class of hemivariational inequality*, J. Math. Ineq. 1(11) (2017) 565–576.
- [15] M. Alimohammady and A. E. Hashoosh, *Existence theorems for monotone operator of nonstandard hemivariational inequality*, Adv. Math. 10 (2015) 3205–312.
- [16] O. Chadli, G. Pany and R. Mohapatra, *Existence and iterative approximation method for solving mixed equilibrium problem under generalized monotonicity in Banach spaces*, J. Num. Alg. 10(1) (2020) 74–91.
- [17] A. Hashoosh, M. Alimohammady and M. K. Kalleji, *Existence results for some equilibrium problems involving α -monotone bifunction*, Int. J. Math. Math. Sci., 2016 (2016) 1–6.
- [18] S. W. Sintunavarat, *Mixed equilibrium Problems with weakly relaxed a monotone bifunction in Banach spaces*, J. Func. Spac. Appl. 2013 (2013) 1–5.
- [19] O. Chadli, Y. Chiang and S. Huang, *Topological pseudomonotonicity and vector equilibrium problems*, J. Math. Anal. Appl. 270(2) (2002) 435–450.
- [20] B. Knaster, K. Kuratowski, and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n -dimensionale simplexe*, J. Fund. Math. 14(1) (1929) 132–137,
- [21] F. E. Browder, *The solvability of non-linear functional equations*, Duke Math. J. 30 (1963) 557–566, .
- [22] Y. P. Fang and N. J. Huang, *Variational-like inequalities with generalized monotone mappings in Banach spaces*, J. Opt. Theo. Appl. 118(6) (2003) 327–338 .