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A simple proof for the algorithms of relaxed (u, v)-cocoercive mappings and α-inverse strongly monotone mappings

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Abstract

In this paper, a simple proof is presented for the convergence of the algorithms for the class of relaxed (u, v)-cocoercive mappings and α-inverse strongly monotone mappings. Based on α-expansive maps, for example, a simple proof of the convergence of the recent iterative algorithms by relaxed (u, v)-cocoercive mappings due to Kumam-Jaiboon is provided. Also a simple proof for the convergence of the iterative algorithms by inverse-strongly monotone mappings due to Iiduka-Takahashi in a special case is provided. These results are an improvement as well as a refinement of previously known results.

Keywords: Inverse-strongly monotone mappings, Strongly monotone mappings, Relaxed (u, v)-cocoercive mappings.
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1. Introduction

In this paper, some results for the class of relaxed (u, v)-cocoercive mappings and α-inverse strongly monotone mappings (in a spacial case) are presented. There are many papers in the literature on iterative algorithms which are used for example in optimization problems, variational inequality

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problems, fixed point problems, equilibrium problems, Nash equilibrium problems, game theory, saddle point problems, minimization problem, feasibility problems, complementarity problems; see [15] and the references therein. In this paper, a simple proof for the convergence of recent iterative algorithms is presented which improve and refine the proof of known results in the literature.

2. Preliminaries

In this paper, it is assumed that C is a nonempty closed convex subset of a real Hilbert space H with inner product $\langle ., . \rangle$ and norm $\|.\|$. Recall the following well known concepts:

(1) a mapping $A : C \to H$ is said to be inverse-strongly monotone [6] [28], if there exist $\alpha > 0$ such that
$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2,$$
for all $x, y \in C$,

(2) a mapping $A : C \to H$ is said to be strongly monotone [30] [1, §1, p. 200], if there exists a constant $\alpha > 0$ such that
$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \tag{2.1}$$

(3) a mapping $B : C \to H$ is said to be relaxed (u, v)-cocoercive [6], if there exist two constants $u, v > 0$ such that
$$\langle Bx - By, x - y \rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2,$$
for all $x, y \in C$. For $u = 0$, B is v-strongly monotone [30]. Clearly, every v-strongly monotone map is a relaxed (u, v)-cocoercive map,

(4) for a map $B : C \to H$ the classical variational problem is to find a $u \in C$ such that $\langle Bu, v - u \rangle \geq 0$, $\forall v \in C$. We denote by $VI(C, B)$ the set of solutions of the variational inequality problem,

(5) let C be a nonempty closed convex subset of a real Hilbert space H. Let B be a self mapping on C. Suppose that there exists a positive integer α such that
$$\|Bx - By\| \geq \alpha \|x - y\|,$$
for all $x, y \in C$, then B is said to be α-expansive.

In this paper, using relaxed (u, v)-cocoercive mappings, a new proof of some recent iterative algorithms is presented. A similar comment applies for inverse-strongly monotone mappings.

3. Relaxed (u, v)-cocoercive mappings

G. Cai and S. Bu [2], W. Chantarangs, C. Jaiboon, and P. Kumam [3], J.S. Jung [11], P. Kumam and C. Jaiboon [15], X. Qin, M. Shang and H. Zhou [22], X. Qin, M. Shang and Y. Su [21], X. Qin, M. Shang and Y. Su [20], considered some iterative algorithms for finding a common element of the set of fixed points of nonexpansive mappings and the set of solutions of a variational inequality $VI(C, A)$, where A is a relaxed (u, v)-cocoercive mapping of C into H.

Lemma 3.1. *Let A be a relaxed (m, v)-cocoercive mapping and ϵ-Lipschitz continuous such that $v - m\epsilon^2 > 0$ and $VI(C, A) \neq \emptyset$. Then A is a $(v - m\epsilon^2)$-expansive mapping and $VI(C, A)$ is a singleton set.*

Proof . To see that $VI(C,A)$ is a singleton set, one can see [23, Proposition 2]. Next, since A is (m,v)-cocoercive and ϵ-Lipschitz continuous, for each $x,y\in C$, it is concluded that

$$\begin{aligned}\langle Ax-Ay,x-y\rangle &\geq (-m)\|Ax-Ay\|^2+v\|x-y\|^2\\ &\geq (-m\epsilon^2)\|x-y\|^2+v\|x-y\|^2\\ &=(v-m\epsilon^2)\|x-y\|^2\geq 0,\end{aligned}$$

hence, we have

$$\|Ax-Ay\|\geq (v-m\epsilon^2)\|x-y\|,$$

so A is $(v-m\epsilon^2)$-expansive. □

To see an example, Theorem 3.1 from P. Kumam and C. Jaiboon [15] is considered. To solve the mixed equilibrium problem for an equilibrium function $\Theta: E\times E\longrightarrow\mathbb{R}$, assume [15, §2, p. 512] that Θ satisfies the following conditions:

(H1) Θ is monotone, *i.e.*, $\Theta(x,y)+\Theta(y,x)\leq 0, \forall x,y\in E$.

(H2) For each fixed $y\in E, x\mapsto\Theta(x,y)$ is convex and upper semicontinuous.

(H3) For each $x\in E, x\mapsto\Theta(x,y)$ is convex.

Theorem 3.2. (*i.e.*, Theorem 3.1, from [15, §3, p.515]) *Let E be a nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from E to $\mathbb{R}$. Let Θ be a bifunction from $E\times E$ to $\mathbb{R}$ satisfying $(H1)-(H3)$, let $\{T_n\}$ be an infinite family of nonexpansive mappings of E into itself and let B be a ξ-Lipschitz continuous and relaxed (m,v)-cocoercive map of E into H such that*

$$\Gamma:=\bigcap_{n=1}^{\infty}F\left(T_n\right)\cap\Omega\cap VI(E,B)\neq\emptyset;$$

here Ω is the set of solutions of the mixed variational probem (i.e. the set of x's in E with $\Theta(x,y)+\varphi(y)-\varphi(x)\geq 0,\ \forall y\in E$) and $F(T_n)$ is the set of fixed points of T_n). Let $\mu>0,\gamma>0,r>0$, be three constants. Let f be a contraction of E onto itself with $\alpha\in(0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{(1+\mu)\bar{\gamma}}{\alpha}$. For $x_1\in H$ arbitrarily and fixed $u\in H$, suppose $\{x_n\},\{y_n\}$ and $\{z_n\}$ are generated iteratively by

$$\begin{cases}\Theta(z_n,x)+\varphi(x)-\varphi(z_n)+\frac{1}{r}\langle K'(z_n)-K'(x_n),\eta(x,z_n)\rangle\geq 0,\\ y_n=\alpha_n z_n+(1-\alpha_n)W_nP_E\left(z_n-\lambda_nBz_n\right),\\ x_{n+1}=\epsilon_n\left(u+\gamma f(W_nx_n)\right)+\beta_nx_n+\left((1-\beta_n)I-\epsilon_n(I+\mu A)\right)W_nP_E(y_n-\tau_nBy_n),\end{cases}$$

for all $n\in\mathbb{N}$ and $x\in E$, where W_n is the W-mapping defined by (2.3) in [15, §2, p. 514] and $\{\epsilon_n\},\{\alpha_n\}$ and $\{\beta_n\}$ are three sequences in (0,1). Assume the following conditions are satisfied:

(C1) $\eta: E\times E\longrightarrow H$ is Lipschitz continuous with constant $\lambda>0$ such that:

- *(a) $\eta(x,y)+\eta(y,x)=0,\forall x,y\in E$,*
- *(b) $\eta(\cdot,\cdot)$ is affine in the first variable,*
- *(c) for each fixed $y\in E, x\mapsto\eta(y,x)$ is sequentially continuous from the weak topology to the weak topology;*

(C2) $K: E\longrightarrow\mathbb{R}$ is η-strongly convex with constant $\sigma>0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu>0$ such that $\sigma>\lambda\nu$;

(C3) for each $x \in E$, there exists a bounded subset $D_x \subset E$ and $z_x \in E$ such that, for any $y \in E \setminus D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0;$$

(C4) $lim_{\eta\to\infty}\alpha_n = 0, lim_{\eta\to\infty}\epsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\epsilon_n = \infty$*;*
(C5) $0 < lim\ inf_{\eta\to\infty}\beta_n \leq lim\ sup_{\eta\to\infty}\beta_n < 1$*;*
(C6) $lim_{\eta\to\infty}|\lambda_{n+1} - \lambda_n| = lim_{\eta\to\infty}|\tau_{n+1} - \tau_n| = 0$*;*
(C7) $\{\tau_n\}, \{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 \leq a \leq b \leq \frac{2(v - m\xi^2)}{\xi^2}$.
Then $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* $z \in \Gamma := \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \cap VI(E, B)$*, provided that* S_r *(here* S_r *is given in [15, §2, p. 513]) is firmly nonexpansive, which solves the following optimization problem:*

$$OP : \min_{x\in\Gamma} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x);$$

here h *is a potential function for* γf.

Next, in the following remark, a simple proof for some similar results is presented:

Remark 3.3. ***A simple Proof:*** *(i). Consider Theorem 3.2 and the* ξ*-Lipschitz continuous and relaxed* (m, v)*-cocoercive mapping* B *in Theorem 3.2. From condition (C7) we may assume that* $(v - m\xi^2) > 0$*, and hence from Lemma 3.1,* B *is* $(v - m\epsilon^2)$*-expansive, i.e,*

$$\|Bx - By\| \geq (v - m\xi^2)\|x - y\|, \tag{3.1}$$

and $VI(E, B)$ *is singleton i.e. there exists an element* $p \in E$ *such that* $VI(E, B) = \{p\}$*, hence* $\Gamma = \{p\}$ *in Theorem 3.2. The authors prove (see (3.25) in [15, p. 521]) that*

$$\lim_n \|Bz_n - Bp\| = 0. \tag{3.2}$$

Now, put $x = z_n$ *and* $y = p$ *in* (3.1)*, and from* (3.1) *and* (3.2)*, we have*

$$\lim_n \|z_n - p\| = 0.$$

Hence, $z_n \to p$*. As a result one of the main claims of Theorem 3.2 is established (note* $\Gamma = VI(E, B) = \{p\}$*). Note the proof in Theorem 3.2 can be simplified further by using this remark (for example Step 5 in [15, p. 526-528] is not needed since one can deduce it from (3.33) in [15, p. 524] and the fact that* $z_n \to p$*).*

(ii). A similar remark applies to the main results in [2, 3, 22, 20, 21].

4. Inverse strongly monotone mappings

J. Chen, L. Zhang, T. Fan [4], S. Takahashi and W. Takahashi [27], H. Iiduka and W.Takahashi [5], K. R. Kazmi, Rehan Ali and Mohd Furkan [13], X. Qin, M. Shang and Y. Su [21], M. Zhang [30], S. Shan and N. Huang [25], T. Jitpeera and P. Kumam [7], S. Peathanom and W. Phuengrattana[18], Piri [19], X. Qin, M. Shang and Y. Su [20] and M. Lashkarizadeh Bami and E. Soori [16] considered some iterative methods for finding a common element of a set of fixed points of nonexpansive mapping and the set of solutions of a variational inequality $VI(C, A)$, where A is an α-inverse strongly monotone mapping of C into H. In this section, a spacial case similar to cocoercive mappings is studied.

Lemma 4.1. *Let A be an α-inverse strongly monotone mapping and $VI(C, A) \neq \emptyset$. Suppose that A is also an γ-expansive mapping. Then $VI(C, A)$ is a singleton.*

Proof . Since A is an α-inverse strongly monotone mapping, it is implied that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \geq 0, \tag{4.1}$$

(so A is monotone). Since A is γ-expansive, it is concluded that

$$\|Ax - Ay\| \geq \gamma \|x - y\|. \tag{4.2}$$

Therefore, A is one to one, because if $Ax = Ay$, then from (4.2), $\|x - y\| = 0$, and hence $x = y$. Let $x_1, x_2 \in VI(C, A)$. Then

$$\langle Ax_1, y - x_1 \rangle \geq 0, \tag{4.3}$$

for each $y \in C$, and

$$\langle Ax_2, y - x_2 \rangle \geq 0, \tag{4.4}$$

for each $y \in C$. Substitute x_1 in (4.4) and x_2 in (4.3), then $\langle Ax_1, x_2 - x_1 \rangle \geq 0$ and $\langle Ax_2, x_1 - x_2 \rangle \geq 0$. Adding them, it is implied that

$$\langle Ax_2 - Ax_1, x_2 - x_1 \rangle \leq 0. \tag{4.5}$$

Since A is monotone, then $\langle Ax_2 - Ax_1, x_2 - x_1 \rangle \geq 0$, and hence from (4.5), it is concluded that

$$\langle Ax_2 - Ax_1, x_2 - x_1 \rangle = 0. \tag{4.6}$$

Then from (4.1), $Ax_2 = Ax_1$ and since A is one to one, it is gotten that $x_2 = x_1$. Then $VI(C, A)$ is singleton. $\square$

Remark 4.2. *Note in Lemma 4.1 we can replace (a): A being a γ-expansive mapping with A being one to one, and (b): A being an α-inverse strongly monotone with A being monotone.*

Remark 4.3. ***A simple proof in a spacial case:*** *(i). Consider Theorem 3.1 in [5, §3, p. 343]; note A is an α-inverse strongly monotone mapping there. If we consider an extra condition of γ-expansiveness (or A being one to one) in Theorem 3.1 in [5] then from Lemma 4.1, we have that $VI(C, A)$ is a singleton and hence in Theorem 3.1 in [5, §3, p. 343] $F(S) \cap VI(C, A)$ is a singleton, i.e, $F(S) \cap VI(C, A) = VI(C, A) = \{u\}$ for an element $u \in C$. The authors prove (see line 8 from below in [5, p. 345]) that*

$$\lim_n \|Ax_n - Au\| = 0. \tag{4.7}$$

Now, put $x = x_n$ and $y = u$ and then from (4.7) and (5), we have

$$\lim_n \|x_n - u\| = 0.$$

Hence, $x_n \to u$. Therefore we have the main claim of Theorem 3.1 (note $F(S) \cap VI(C, A) = VI(E, B) = \{u\}$). As a result in this situation one can remove everything in the proof after line 8 from below in [5, p. 345].

(ii). A similar comment applies to the main results in [4, 5, 7, 9, 8, 10, 11, 12, 14, 13, 16, 18, 19, 20, 21, 24, 25, 26, 27, 17, 29, 30] is similar to [5].

Remark 4.4. *Note that the proof of these algorithms for inverse strongly monotone mappings can not be simplified like in the proof of algorithms for cocoercive mappings in general. Indeed in Remark 4.3, a simple proof is just introduced for a special case of these algorithms.*

5. Discussion

In this paper, a simple proof for the convergence of the algorithms of relaxed (u, v)-cocoercive mappings and α-inverse strongly monotone mappings are presented. Indeed, a refinement of the proofs of some well known results is presented.

6. Conclusion

In this paper, some refinements of the proofs of some well known results are given. Indeed, the proofs of these results are made shorter than the original ones for the class of relaxed (u, v)-cocoercive mappings and α-inverse strongly monotone mappings.

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