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Several integral inequalities and their applications on means

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Abstract

In this paper we prove several sharp inequalities that are new versions and extensions of Jensen and $H-H$ inequalities. Then we apply them on means.

Keywords: Jensen's inequality, Hermite-Hadamard inequality, Integral inequality, Identric mean, Logarithmic mean.
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1. Introduction

Let μ be a positive measure on X such that $\mu(X)=1$. If f is a real-valued function in $L^1(\mu)$, $a<f(x)<b$ for all $x\in X$, and φ is convex on (a,b), then

$$\varphi\left(\int_X f\,d\mu\right)\leqslant\int_X(\varphi\circ f)\,d\mu \tag{1.1}$$

especially

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\leqslant\frac{1}{b-a}\int_a^b(\varphi\circ f)(x)dx$$

The inequality (1.1) is known as Jensen's inequality.
Let $f:[a,b]\longrightarrow\mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2} \tag{1.2}$$

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is known in the literature as Hermite-Hadamard inequality ($H-H$ inequality). It is well known that Jensen and $H-H$ inequalities play an important role in non-linear analysis. In recent years there have bean many extensions, generalizations and refinements of these inequalities; see [9, 12, 13, 14] and the references therein.
In this paper we prove several sharp integral inequalities that are new versions of (1.1)and (1.2) inequalities. Then we apply these inequalities on means. For the statement of the main results we introduce some notations and terminologies.

Definition 1.1. *A function* $M:(0,\infty)\times(0,\infty)\longrightarrow(0,\infty)$ *called a mean if*

1. $M(x,y)=M(y,x)$
2. $M(x,x)=x$
3. $x<M(x,y)<y$, *whenever* $x<y$
4. $M(\alpha x,\alpha y)=\alpha M(x,y)$ *for all* $\alpha>0$.

Example 1.2.

1. *The Arithmetic mean* $A(a,b)=\dfrac{a+b}{2}$,
2. *The Geometric mean* $G(a,b)=\sqrt{ab}$,
3. *The logarithmic mean* $L(a,b)=\dfrac{a-b}{\ln a-\ln b}(a\neq b)$,
4. *The generalized logarithmic mean*
$$L_n(a,b)=\left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}}(a\neq b),$$
5. *The Identric mean* $I(a,b)=\dfrac{1}{e}\left(\dfrac{b^a}{a^a}\right)^{\frac{1}{b-a}}(a\neq b)$,

It is well- known that

$$\min\{a,b\}\leqslant G(a,b)\leqslant L(a,b)\leqslant I(a,b)\leqslant A(a,b)\leqslant\max\{a,b\}$$

Recently, the identric mean $I(a,b)$ has been the subject of intensive research. In particular, many remarkable inequalities for $L(a,b)$ and $I(a,b)$ can be found in the literature, see [4, 5, 6, 7, 8, 10, 11, 15]. We define

$$Q_n(a,b)=\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{(n+1)(b^{\frac{1}{n}}-a^{\frac{1}{n}})}=\frac{1}{n+1}\sum_{k=0}^{n}\left(b^ka^{n-k}\right)^{\frac{1}{n}}\qquad(a\neq b),$$
$$P_n(a,b)=\left(\frac{n}{n+1}\right)^n\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b-a}\right)^n\qquad(a\neq b),$$
$$T_n(a,b)=\left(\frac{b^n-a^n}{n(\ln b-\ln a)}\right)^{\frac{1}{n}}\qquad(a\neq b)$$

By easy calculations we can see that Q_n, P_n and T_n are means.
In this paper we prove that

$$G(a,b) \leqslant L(a,b) \leqslant I(a,b) \leqslant P_n(a,b) \leqslant A(a,b) \leqslant T_n(a,b) \leqslant L_n(a,b)$$

and

$$L(a,b) \leqslant Q_n(a,b) \leqslant P_n(a,b)$$

2. Main results

Theorem 2.1. *Let μ be a positive measure on X such that $\mu(X) < \infty$. If f is a non-negative function in $L^{\alpha(\alpha-1)}(\mu)(\alpha > 1)$, then*

$$\begin{aligned}\frac{1}{(\mu(X))^{\alpha(\alpha-2)}}\left(\int_X f d\mu\right)^{\alpha(\alpha-1)} &\leqslant \left(\int_X f^{\alpha-1} d\mu\right)^{\alpha} \leqslant \mu(X)\left(\int_X f^{\alpha} d\mu\right)^{\alpha-1} \\ &\leqslant (\mu(X))^{\alpha-1}\int_X f^{\alpha(\alpha-1)} d\mu \end{aligned} \tag{2.1}$$

Proof . For proving the first part, considering the convexity of $t^{\alpha-1}(\alpha > 1)$ on $[0,\infty)$ and Jensen's inequality, we have

$$\begin{aligned} \left(\frac{1}{\mu(X)}\int_X f d\mu\right)^{\alpha-1} &\leqslant \frac{1}{\mu(X)}\int_X f^{\alpha-1} d\mu \\ \Rightarrow \left(\frac{1}{\mu(X)}\int_X f d\mu\right)^{\alpha(\alpha-1)} &\leqslant \frac{1}{\mu^{\alpha}(X)}\left(\int_X f^{\alpha-1} d\mu\right)^{\alpha} \\ \Rightarrow \frac{1}{(\mu(X))^{\alpha(\alpha-2)}}\left(\int_X f d\mu\right)^{\alpha(\alpha-1)} &\leqslant \left(\int_X f^{\alpha-1} d\mu\right)^{\alpha} \end{aligned}$$

For the proof of second part let $\frac{1}{p}+\frac{1}{q} = 1$, $(p,q > 1)$. By Holder's inequality we have

$$\begin{aligned} \int_X f^{\alpha-1} d\mu &\leqslant \left(\int_X (f^{\alpha-1})^p d\mu\right)^{\frac{1}{p}}\left(\int_X 1^q d\mu\right)^{\frac{1}{q}} \\ &= \left(\int_X f^{p(\alpha-1)} d\mu\right)^{\frac{1}{p}}(\mu(X))^{\frac{1}{q}} \end{aligned}$$

So

$$\left(\int_X f^{\alpha-1} d\mu\right)^{\alpha} \leqslant \left(\int_X f^{p(\alpha-1)} d\mu\right)^{\frac{\alpha}{p}}(\mu(X))^{\frac{\alpha}{q}}$$

Put $q = \alpha > 1$, then $p = \frac{\alpha}{\alpha-1} > 1$. Therefore

$$\left(\int_X f^{\alpha-1} d\mu\right)^{\alpha} \leqslant \left(\int_X f^{\alpha} d\mu\right)^{\alpha-1}\mu(X)$$

The proof of third part is similar to the proof of first part and therefore omitted. □

Corollary 2.2. *Let $\alpha > 1$, then*

(i) $C_1 \leqslant C_{\alpha-1} \leqslant C_\alpha \leqslant C_{\alpha(\alpha-1)}$, *where*

$$C_\alpha = \left(\frac{a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}$$

is the mean power of positive numbers $a_1, a_2, \ldots, a_n$ of the order α.

(ii) $(a_1 + a_2 + \ldots + a_n)^\alpha \leqslant n^{\alpha-1}\left[a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha\right]$

Proof . *(i)* Let $X = \{x_1, x_2, \ldots, x_n\}$, $\mu(\{x_i\}) = \dfrac{1}{n}$ and $f(x_i) = a_i$. Then (2.1) becomes

$$\begin{aligned}
\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^{\alpha(\alpha-1)} &\leqslant \left(\frac{a_1^{\alpha-1} + a_2^{\alpha-1} + \ldots + a_n^{\alpha-1}}{n}\right)^{\alpha} \\
&\leqslant \left(\frac{a_1^{\alpha} + a_2^{\alpha} + \ldots + a_n^{\alpha}}{n}\right)^{\alpha-1} \\
&\leqslant \left(\frac{a_1^{\alpha(\alpha-1)} + a_2^{\alpha(\alpha-1)} + \ldots + a_n^{\alpha(\alpha-1)}}{n}\right)^{\alpha} \\
\Rightarrow \frac{a_1 + a_2 + \ldots + a_n}{n} &\leqslant \left(\frac{a_1^{\alpha-1} + a_2^{\alpha-1} + \ldots + a_n^{\alpha-1}}{n}\right)^{\frac{1}{\alpha-1}} \\
&\leqslant \left(\frac{a_1^{\alpha} + a_2^{\alpha} + \ldots + a_n^{\alpha}}{n}\right)^{\frac{1}{\alpha}} \leqslant \left(\frac{a_1^{\alpha(\alpha-1)} + \ldots + a_n^{\alpha(\alpha-1)}}{n}\right)^{\frac{1}{\alpha(\alpha-1)}}
\end{aligned}$$

Hence $\quad C_1 \leqslant C_{\alpha-1} \leqslant C_\alpha \leqslant C_{\alpha(\alpha-1)}$.

(ii) $C_1 \leqslant C_\alpha \Rightarrow C_1^\alpha \leqslant C_2^\alpha$. So

$$\begin{aligned}
\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^{\alpha} &\leqslant \frac{a_1^{\alpha} + a_2^{\alpha} + \ldots + a_n^{\alpha}}{n} \\
\Rightarrow (a_1 + a_2 + \ldots + a_n)^\alpha &\leqslant n^{\alpha-1}\left(a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha\right).
\end{aligned}$$

□

Remark 2.3. *In general case we can prove that for any $\alpha, \beta \in \mathbb{R}$*

$$\alpha < \beta \Longrightarrow C_\alpha \leqslant C_\beta$$

For more detils see ([13], [14]) and corollary 2.12.

Theorem 2.4. *Let $b > a > 0$ and $n \in \mathbb{N}$, then*

$$Q_n(a,b) \leqslant P_n(a,b) \tag{2.2}$$

Proof . In middle part of inequalities(2.1), put $f(x) = x$, $X = [c,d]$ and $\alpha = n$. Hence

$$\begin{aligned}
\left(\int_c^d x^{n-1}dx\right)^n &\leqslant (d-c)\left(\int_c^d x^n dx\right)^{n-1} \\
\Rightarrow \left(\frac{d^n - c^n}{n}\right)^n &\leqslant (d-c)\left(\frac{d^{n+1} - c^{n+1}}{n+1}\right)^{n-1} \\
\Rightarrow \frac{1}{(n+1)(d-c)} &\leqslant \frac{n^n}{(n+1)^n}\frac{\left(d^{n+1} - c^{n+1}\right)^{n-1}}{(d^n - c^n)^n}
\end{aligned}$$

Multiplying both sides of inequality by $d^{n+1}-c^{n+1}$, we get

$$\frac{1}{n+1}\frac{d^{n+1}-c^{n+1}}{d-c}\leqslant\left(\frac{n}{n+1}\right)^n\left(\frac{d^{n+1}-c^{n+1}}{d^n-c^n}\right)^n$$

Now let $b^{\frac{1}{n}}=d$ and $a^{\frac{1}{n}}=c$. Thus

$$\frac{1}{n+1}\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b^{\frac{1}{n}}-a^{\frac{1}{n}}}\leqslant\left(\frac{n}{n+1}\right)^n\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b-a}\right)^n$$

and $Q_n(a,b)\leqslant P_n(a,b)$. □

Theorem 2.5. *Let $b>a>0$ and $n\in\mathbb{N}$, Then the following inequalities hold:*

(*i*) $G(a,b)\leqslant I(a,b)\leqslant P_n(a,b)$

(*ii*) $\lim_{n\to\infty}P_n(a,b)=I(a,b)$

Proof . By the concavity of $\varphi(x)=\ln x$ in $(0,\infty)$ and Jensen's inequality we have

$$\ln\left(\frac{1}{b-a}\int_a^b x^{\frac{1}{n}}dx\right)\geqslant\frac{1}{b-a}\int_a^b\ln x^{\frac{1}{n}}dx=\frac{1}{n(b-a)}\int_a^b\ln x dx$$

By easy calculations we see that

$$\ln\left(\frac{1}{b-a}\int_a^b x^{\frac{1}{n}}dx\right)=\ln\frac{n}{n+1}\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b-a}\right)$$

and

$$\frac{1}{n(b-a)}\int_a^b\ln x dx=\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{n(b-a)}}-\ln e^{\frac{1}{n}}.$$

On the other hand by the right side of $H-H$ inequality we have

$$\frac{1}{n(b-a)}\int_a^b\ln x dx\geqslant\frac{\ln a+\ln b}{2n}$$

So

$$\ln\frac{n}{n+1}\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b-a}\right)\geqslant\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{n(b-a)}}-\ln e^{\frac{1}{n}}\geqslant\ln(ab)^{\frac{1}{2n}}$$

$$\Rightarrow\frac{n}{n+1}\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b-a}\right)\geqslant\left[\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}\right]^{\frac{1}{n}}\geqslant(ab)^{\frac{1}{2n}}$$

$$\Rightarrow\sqrt{ab}\leqslant\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}\leqslant\left(\frac{n}{n+1}\right)^n\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b-a}\right)^n$$

$$\Rightarrow G(a,b)\leqslant I(a,b)\leqslant P_n(a,b)$$

(ii) we have

$$\lim_{n\to\infty}\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b-a}\right)^n = e^{\lim_{n\to\infty} n\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}-b+a}{b-a}\right)}$$

By easy calculations and L'Hopital's rule, we get

$$\begin{aligned}\lim_{n\to\infty} n\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}-a+b}{(b-a)} &= \frac{1}{b-a}\lim_{t\to 0}\frac{b^{1+t}-a^{1+t}-b+a}{t}\\ &= \frac{1}{b-a}\lim_{t\to 0}\frac{b^{1+t}\ln b-a^{1+t}\ln a}{1} = \frac{1}{b-a}\ln\frac{b^b}{a^a}\\ &= \ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.\end{aligned}$$

So

$$\begin{aligned}\lim_{n\to\infty} P_n(a,b) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\left(\frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{b-a}\right)^n &= \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}\\ &= I(a,b).\end{aligned}$$

□

Theorem 2.6. *Let f be convex function on $[a,b]$ and $n\in\mathbb{N}$. Then the following inequalities hold:*

$$f\left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)\leqslant\frac{1}{b-a}\int_a^b f(x^n)dx\leqslant\frac{1}{n+1}\sum_{k=0}^{n} f\left(b^k a^{n-k}\right) \tag{2.3}$$

The inequalities hold in reversed order if f is concave on $[a,b]$.

Proof . By Jensen's inequality we have

$$f\left(\frac{1}{b-a}\int_a^b x^n dx\right)\leqslant\frac{1}{b-a}\int_a^b f(x^n)dx$$

So

$$f\left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)\leqslant\frac{1}{b-a}\int_a^b f(x^n)dx$$

For the proof of right side, by change of variable $x=tb+(1-t)a$, we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x^n)dx &= \int_0^1 f\left((tb+(1-t)a)^n\right)dt\\ &= \int_0^1 f\left(\sum_{k=0}^{n}\binom{n}{k}t^k b^k(1-t)^{n-k}a^{n-k}\right)dt\end{aligned}$$

Since $\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}=1$, by the convexity of f and considering

$$\int_0^1 t^k(1-t)^{n-k} = B(k+1,n-k+1) = \frac{k!(n-k)!}{(n+1)!}$$

$$
\begin{aligned}
&\leqslant \int_0^1 \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k} f(b^k a^{n-k})dt \\
&= \sum_{k=0}^{n} \binom{n}{k} f(b^k a^{n-k}) \int_0^1 t^k(1-t)^{n-k}dt \\
&= \sum_{k=0}^{n} \binom{n}{k} f(b^k a^{n-k}) \frac{k!(n-k)!}{(n+1)!} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} f(b^k a^{n-k}).
\end{aligned}
$$

□

We can write inequalities (2.3) in the following form

$$
f\left(\frac{1}{n+1}\sum_{k=0}^{n} b^k a^{n-k}\right) \leqslant \frac{1}{b-a}\int_a^b f(x^n)dx \leqslant \frac{1}{n+1}\sum_{k=0}^{n} f\left(b^k \cdot a^{n-k}\right)
$$

It shows that inequality (2.3) is an extension of $H-H$ inequality.

Corollary 2.7. *Let $b > a > 0$ and $n \in \mathbb{N}$, then*

$$
G(a,b) \leqslant I(a,b) \leqslant L_n(a,b) \tag{2.4}
$$

Proof . $f(x) = \ln x$ is cocave on $(0, \infty)$. So by (2.3) we have

$$
\begin{aligned}
&\frac{1}{n+1}\sum_{k=0}^{n} \ln(b^k a^{n-k}) \leqslant \frac{1}{b-a}\int_a^b \ln x^n dx \leqslant \ln\left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right) \\
&\Rightarrow \ln\left(\prod_{k=0}^{n} b^k a^{n-k}\right)^{\frac{1}{n+1}} \leqslant \frac{n}{b-a}\left(x\ln x - x\Big|_a^b\right) \\
&\qquad \leqslant \ln\left(\frac{1}{n+1}\sum_{k=0}^{n} b^k a^{n-k}\right)
\end{aligned}
$$

Since $\prod_{k=0}^{n} b^k a^{n-k} = (ab)^{\frac{n(n+1)}{2}}$ and

$$
\frac{n}{b-a}\left(x\ln x - x\Big|_a^b\right) = \frac{1}{b-a}\ln\left(\frac{b^b}{a^a}\right)^n - n
$$

we obtain

$$
\ln(ab)^{\frac{n}{2}} \leqslant \ln\left(\frac{b^b}{a^a}\right)^{\frac{n}{b-a}} + \ln e^{-n} \leqslant \ln \frac{1}{n+1}\sum_{k=0}^{n} b^k a^{n-k}
$$

Finally since e^x is increasing, we get

$$
\begin{aligned}
&(ab)^{\frac{n}{2}} \leqslant \left(\frac{b^b}{a^a}\right)^{\frac{n}{b-a}} e^{-n} \leqslant \frac{1}{n+1}\sum_{k=0}^{n} b^k a^{n-k} \\
&\Rightarrow \sqrt{ab} \leqslant \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \cdot \frac{1}{e} \leqslant \left(\frac{1}{n+1}\sum_{k=0}^{n} b^k a^{n-k}\right)^{\frac{1}{n}} \\
&\Rightarrow G(a,b) \leqslant I(a,b) \leqslant L_n(a,b).
\end{aligned}
$$

□

Corollary 2.8. *Let $b > a > 0$ and $n \in \mathbb{N}$, Then*

$$Q_n(a,b) \leqslant A(a,b) \leqslant L_n(a,b) \tag{2.5}$$

Proof . By the concavity of $f(x) = x^{\frac{1}{n}}$ on $[0,\infty)$ and theorem 2.6 we have

$$\left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}} \geqslant \frac{1}{b-a}\int_a^b x dx \geqslant \frac{1}{n+1}\sum_{k=0}^{n}\left(b^k a^{n-k}\right)^{\frac{1}{n}}$$
$$\Rightarrow \frac{1}{n+1}\sum_{k=0}^{n}\left(b^k a^{n-k}\right)^{\frac{1}{n}} \leqslant \frac{a+b}{2} \leqslant \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}}$$
$$\Rightarrow Q_n(a,b) \leqslant A(a,b) \leqslant L_n(a,b).$$

□

In the following theorem we obtain an another extension of $H-H$ inequality.

Theorem 2.9. *Let f be a convex function on $[a,b]$, then*

$$f^n\left(\frac{a+b}{2}\right) \leqslant \left(\frac{1}{b-a}\int_a^b f(x)dx\right)^n \leqslant \frac{1}{b-a}\int_a^b f^n(x)dx \leqslant \frac{1}{n+1}\sum_{k=0}^{n} f^{n-k}(b)f^k(a) \tag{2.6}$$

Proof . By the convexity of f, $\varphi(t) = t^n$ and Jensen's inequality we have

$$f^n\left(\frac{a+b}{2}\right) = f^n\left(\frac{1}{b-a}\int_a^b x dx\right) \leqslant \left(\frac{1}{b-a}\int_a^b f(x)dx\right)^n \leqslant \frac{1}{b-a}\int_a^b f^n(x)dx$$

For the proof of right side by change of variable $x = tb + (1-t)a$, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f^n(x)dx &= \int_0^1 f^n\left(tb+(1-t)a\right)dt \\
&\leqslant \int_0^1 \left(tf(b)+(1-t)f(a)\right)^n dt \\
&= \int_0^1 \left(\sum_{k=0}^{n}\binom{n}{k} t^{n-k} f^{n-k}(b)(1-t)^k f^k(a)\right) dt \\
&= \sum_{k=0}^{n}\binom{n}{k} f^{n-k}(b) f^k(a) \int_0^1 t^{n-k}(1-t)^k dt \\
&= \sum_{k=0}^{n}\binom{n}{k} f^{n-k}(b) f^k(a) \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}\sum_{k=0}^{n} f^{n-k}(b) f^k(a).
\end{aligned}$$

□

Corollary 2.10. *Let $b > a > 0$ and $n \in \mathbb{N}$, then*

$$G(a,b) \leqslant L(a,b) \leqslant T_n(a,b) \leqslant L_n(a,b) \tag{2.7}$$

Proof . By the convexity of $f(x)=e^x$ on $\mathbb{R}$, for $d>c>0$, $n\in\mathbb{N}$ and Theorem2.9 we have

$$\begin{aligned} e^{\frac{n(c+d)}{2}} &\leqslant \left(\frac{1}{d-c}\int_c^d e^x dx\right)^n \leqslant \frac{1}{d-c}\int_c^d e^{nx}dx \leqslant \frac{1}{n+1}\sum_{k=0}^{n} e^{b(n-k)}e^{ak} \\ \Rightarrow e^{\frac{n(c+d)}{2}} &\leqslant \left(\frac{e^d-e^c}{d-c}\right)^n \leqslant \frac{e^{nd}-e^{nc}}{n(d-c)} \leqslant \frac{1}{n+1}\frac{e^{(n+1)d}-e^{(n+1)c}}{e^d-e^c} \\ \Rightarrow e^{\frac{c+d}{2}} &\leqslant \frac{e^d-e^c}{d-c} \leqslant \left(\frac{e^{nd}-e^{nc}}{n(d-c)}\right)^{\frac{1}{n}} \leqslant \left(\frac{1}{n+1}\frac{e^{(n+1)d}-e^{(n+1)c}}{e^d-e^c}\right)^{\frac{1}{n}}. \end{aligned}$$

Put $e^d=b$ and $e^c=a$, then

$$\begin{aligned} &\sqrt{ab} \leqslant \frac{b-a}{\ln b-\ln a} \leqslant \left(\frac{b^n-a^n}{n(\ln b-\ln a)}\right)^{\frac{1}{n}} \leqslant \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}} \\ &\Rightarrow G(a,b)\leqslant L(a,b)\leqslant T_n(a,b)\leqslant L_n(a,b). \end{aligned}$$

□

Lemma 2.11. *Let μ be a positive measure on X such that $\mu(X)<\infty$. If f is a non-negative function in $L^\alpha(\mu)(\alpha\geqslant 1)$, then*

$$\left(\frac{1}{\mu(X)}\int_X f^{\frac{1}{\alpha}}d\mu\right)^\alpha \leqslant \frac{1}{\mu(X)}\int_X fd\mu \leqslant \left(\frac{1}{\mu(X)}\int_X f^\alpha d\mu\right)^{\frac{1}{\alpha}} \tag{2.8}$$

Proof . Considering the convexity of $\varphi(t)=t^\alpha$, concavity of $\psi(t)=t^{\frac{1}{\alpha}}(\alpha\geqslant 1)$ and Jensen's inequalities the assertion is obvious. □

Corollary 2.12. *With the notations of corollary 2.2 we have $C_{\frac{1}{\alpha}}\leqslant C_1\leqslant C_\alpha$.*

Proof . Let $X=\{x_1,x_2\dots,x_n\}$, $\mu(\{x_i\})=\dfrac{1}{n}$ and $f(x_i)=a_i$. Then(2.8) becomes

$$\begin{aligned} &\left(\frac{a_1^{\frac{1}{\alpha}}+a_2^{\frac{1}{\alpha}}+\ldots+a_n^{\frac{1}{\alpha}}}{n}\right) \leqslant \frac{a_1+a_2+\ldots+a_n}{n} \leqslant \left(\frac{a_1^\alpha+a_2^\alpha+\ldots+a_n^\alpha}{n}\right)^{\frac{1}{\alpha}} \\ &\Rightarrow C_{\frac{1}{\alpha}}\leqslant C_1\leqslant C_\alpha. \end{aligned}$$

□

Corollary 2.13. *Let $b>a>0$ and $n\in\mathbb{N}$, then*

$$P_n(a,b)\leqslant A(a,b)\leqslant L_n(a,b) \tag{2.9}$$

Proof . Put $X=[a,b]$, $f(x)=x$, and $\alpha=n$, then (2.8) becomes

$$\begin{aligned} &\left(\frac{1}{b-a}\int_a^b x^{\frac{1}{n}}dx\right)^n \leqslant \frac{1}{b-a}\int_a^b xdx \leqslant \left(\frac{1}{b-a}\int_a^b x^n dx\right)^{\frac{1}{n}} \\ \Rightarrow &\left[\frac{1}{b-a}\cdot\frac{n}{n+1}\left(b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}\right)\right]^n \leqslant \frac{a+b}{2} \leqslant \left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right)^{\frac{1}{n}} \\ &\Rightarrow P_n(a,b)\leqslant A(a,b)\leqslant L_n(a,b). \end{aligned}$$

□

Now we want to prove that $A(a,b)\leqslant T_n(a,b)$. For this reason we need the following Lemma.

Lemma 2.14. *Let $t \geqslant 1$ and $n > 3$, then*

$$n(1+t)^n \ln t \leqslant 2^n(t^n - 1)$$

Proof . Consider$f : [1, \infty) \longrightarrow \mathbb{R}$ defined by

$$f(t) = n(1+t)^n \ln t - 2^n(t^n - 1)$$

We have $f'(t) = n^2(1+t)^{n-1} \ln t + n\dfrac{(1+t)^n}{t} - n \cdot 2^n t^{n-1}$ and $f'(1) = 0$. By easy calculations, we see that

$$f''(t) = n^2(n-1)(1+t)^{n-2} \ln t + \frac{n\left((2n-1)t - 1\right)(1+t)^{n-1}}{t^2} - n(n-1)2^n t^{n-2}$$

and $f''(1) = 0$. Put $g(t) = \dfrac{t^2}{n} f''(t)$, so

$$\begin{aligned} g(t) &= n(n-1)(1+t)^{n-2}t^2 \ln t + \left((2n-1)t - 1\right)(1+t)^{n-1} - (n-1)2^n t^n \\ g'(t) &= 2n(n-1)t(1+t)^{n-2} \ln t + n(n-1)(n-2)(1+t)^{n-3} \ln t \\ &\quad + n(n-1)(1+t)^{n-2} \cdot \frac{1}{t} + (2n-1)(1+t)^{n-1} \\ &\quad + (n-1)\left((2n-1)t - 1\right)(1+t)^{n-2} - n(n-1)2^n t^{n-1}. \end{aligned}$$

Hence

$$g'(1) = 2^{n-2}[-n^2 + 3n] < 0 \qquad (n > 3)$$

On the other hand

$$g'(t) = \frac{1}{n}\left[2tf'''(t) + \frac{t^2}{2} f''(t)\right]$$

$$\Rightarrow 0 > g'(1) = \frac{1}{n}\left[2f'''(1)\right] \Rightarrow f'''(1) < 0$$

It follows that f'' attains its maximum at $t = 1$.
Thus $f''(t) \leqslant f''(1) = 0$. Again it follows that $f'(t) \leqslant f'(1) = 0$. Finally we deduce that

$$f(t) \leqslant f(1) = 0.$$

This shows that

$$n(1+t)^n \ln t \leqslant 2^n(t^n - 1) \quad (t \geqslant 1, n > 3).$$

□

Corollary 2.15. *Let $b \geqslant a > 0$ and $n > 3$, then*

$$A(a, b) \leqslant T_n(a, b) \tag{2.10}$$

Proof . If we put $t = \dfrac{b}{a} \geqslant 1$ in lemma 2.14, we get

$$\begin{aligned} &n\left(1 + \frac{b}{a}\right)^n \ln \frac{b}{a} \leqslant 2^n \left(\frac{b^n}{a^n} - 1\right) \\ &\Rightarrow n(a+b)^n (\ln b - \ln a) \leqslant 2^n(b^n - a^n) \Rightarrow \\ &\left(\frac{a+b}{2}\right)^n \leqslant \frac{b^n - a^n}{n(\ln b - \ln a)} \Rightarrow A(a, b) \leqslant T_n(a, b). \end{aligned}$$

□

Finally we prove that $L(a, b) \leqslant Q_n(a, b)$. First we prove the following lemma.

Lemma 2.16. *Let $t \geqslant 1$ and $n \in \mathbb{N}$, then*

$$(n+1)(t^n-1)(t-1) \leqslant n(t^{n+1}-1)\ln t$$

Proof . Let $f : [1,\infty) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} f(t) &= (n+1)(t^n-1)(t-1) - n(t^{n+1}-1)\ln t \\ &= (n+1)t^{n+1} - (n+1)t^n - (n+1)t + (n+1) - nt^{n+1}\ln t + n\ln t. \end{aligned}$$

By differentiating and easy calculations we obtain

$$\begin{aligned} f'(t) &= (n^2+n+1)t^n - n(n+1)t^{n-1} - (n+1) - n(n+1)t^n\ln t + \frac{n}{t} \\ \Rightarrow f''(t) &= n^3t^{n-1} - n(n^2-1)t^{n-2} - n^2(n+1)t^{n-1}\ln t - \frac{n}{t^2} \\ \Rightarrow f'''(t) &= (n^4-2n^3-n^2)t^{n-2} - n(n^2-1)(n-2)t^{n-3} - n^2(n^2-1)t^{n-2}\ln t + \frac{2n}{t^3}. \end{aligned}$$

It follows that $f'(1) = f''(1) = f'''(1) = 0$. We have

$$\begin{aligned} f^{(4)}(t) &= (n^5-5n^4+3n^3+3n^2)t^{n-3} - (n^5-5n^4+5n^3+5n^2-6n)t^{n-4} \\ &\quad - n^2(n^2-1)(n-2)t^{n-3}\ln t - \frac{6n}{t^4} \\ \Rightarrow f^{(4)}(1) &= -2n^3-2n^2 < 0 \end{aligned}$$

It follows that f''' attains its maximum at $t=1$.
Thus $f'''(t) \leqslant f'''(1) = 0$. It follows that f'' is decreasing on $[1,\infty)$. Hence $f''(t) \leqslant f''(1) = 0$ and f' is decreasing on $[1,\infty)$, that is $f'(t) \leqslant f'(1) = 0$. Finally we deduce that f is decreasing and $f(t) \leqslant f(1) = 0$ The proof is complete. □

Corollary 2.17. *Let $b \geqslant a > 0$ and $n \in \mathbb{N}$ then*

$$L(a,b) \leqslant Q_n(a,b). \tag{2.11}$$

Proof . If we put $t = \left(\frac{b}{a}\right)^{\frac{1}{n}} \geqslant 1$ in lemma2.16, we obtain

$$\begin{aligned} (n+1)\left(\frac{b}{a}-1\right)\left(\frac{b^{\frac{1}{n}}}{a^{\frac{1}{n}}}-1\right) &\leqslant n\left(\frac{b^{\frac{n+1}{n}}}{a^{\frac{n+1}{n}}}-1\right)\ln\left(\frac{b}{a}\right)^{\frac{1}{n}} \\ (n+1)\left(\frac{b-a}{a}\right)\left(\frac{b^{\frac{1}{n}}-a^{\frac{1}{n}}}{a^{\frac{1}{n}}}\right) &\leqslant \frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{a^{\frac{n+1}{n}}}(\ln b - \ln a) \\ \frac{b-a}{\ln b - \ln a} &\leqslant \frac{b^{\frac{n+1}{n}}-a^{\frac{n+1}{n}}}{(b^{\frac{1}{n}}-a^{\frac{1}{n}})(n+1)} \Rightarrow L(a,b) \leqslant Q_n(a,b). \end{aligned}$$

□

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