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Sequential bipolar metric space and well-posedness of fixed point problems

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Abstract

In this paper, we introduce the concept of sequential bipolar metric spaces which is a generalization of bipolar metric spaces and bipolar $b-$metric spaces and in view of this concept we prove some fixed point theorems for a class of covariant and contravariant contractive mappings over such spaces. Supporting example have been cited in order to validity of the underlying space. Moreover, our fixed point results are applied to well-posedness of fixed point problems.

Keywords: Fixed point, Sequential bipolar metric space, Well-posedness of fixed point problem.
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1. Introduction and Preliminaries

In recent years fixed point theory is one of the important research area in Mathematics, particularly in functional analysis. To investigate fixed points of mappings, researchers find interest to work in finding solutions of natural problems in and around the globe with also development of fixed point theory over a variety of topological structured metric spaces. Starting from the late nineties much work has been progressed in the development of fixed point theory either by considering (i) generalization of the underlying spaces or (ii) relaxing or generalizing the type of mappings or (iii) by the combination of both (i) and (ii) (See [2]-[5], [7], [8], [10], [11], [14], [15], [17], [18]). In surveying the literatures of fixed point theory one can find its applications in different areas of Mathematics namely boundary value problems, nonlinear differential and integral equations, nonlinear functional equations, nonlinear matrix equations, homotopy theory etc.

In the year 2016, Mutlu and Gürdal have introduced the concept of bipolar metric spaces which is given below and proved some contractive fixed point theorems and coupled fixed point theorems on such spaces (See [12],[13]).

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Definition 1.1. *[12] Let E and F be two nonempty sets. Suppose that a function $d : E \times F \to [0, \infty)$ satisfies the following conditions:*

(B_1) $d(\xi, \eta) = 0$ if and only if $\xi = \eta$;

(B_2) $d(\xi, \eta) = d(\eta, \xi)$ for all $\xi, \eta \in E \cap F$;

(B_3) $d(\xi_1, \eta_2) \leq d(\xi_1, \eta_1) + d(\xi_2, \eta_1) + d(\xi_2, \eta_2)$ for all $(\xi_1, \eta_1), (\xi_2, \eta_2) \in E \times F$.

The function d is called a bipolar metric on (E, F) and the triplet (E, F, d) is called a bipolar-metric space.

Recently Roy and Saha [16] have generalized bipolar metric spaces by introducing the concept of bipolar cone$_{tvs}$ $b-$metric space and by exhibiting some topological properties on such spaces. In the same article Roy and Saha had been able to prove Cantor's intersection theorem and also proved several fixed point theorems. The definition of bipolar cone$_{tvs}$ $b-$metric is given below:

Definition 1.2. *[16] Let E be a real Hausdorff topological vector space with a solid cone K and $\preceq$ be the partial ordering on E induced by K. Also let P and Q be two nonempty sets and $d_b : P \times Q \to K$ be a function, satisfies the following properties:*

i) $d_b(\xi, \eta) = \theta_E$ if and only if $\xi = \eta$;

ii) $d_b(\xi, \eta) = d_b(\eta, \xi)$ for all $\xi, \eta \in P \cap Q$;

iii) $d_b(\xi_1, \eta_2) \preceq s[d_b(\xi_1, \eta_1) + d_b(\xi_2, \eta_1) + d_b(\xi_2, \eta_2)]$ for all $\xi_1, \xi_2 \in P$ and $\eta_1, \eta_2 \in Q$, where the coefficient $s \geq 1$.

Then d_b is called a bipolar cone$_{tvs}$ $b-$metric on (P, Q) and the triplet (P, Q, d_b) is called a bipolar cone$_{tvs}$ $b-$metric space.

Recently Bajović et al. (See [1]) have modified the fixed point theorems proved by Roy et al. [16].

Remark 1.3. *If we take $E = \mathbb{R}$ with the usual cone $K = [0, \infty)$ then (P, Q, d_b) gives us a bipolar $b-$metric space.*

M. Jleli and B. Samet have given in their article [9] the following definition of generalized metric space. The sequences on such spaces play a vital role and triangle inequality is not needed to define such a metric structure.

Let A be a non-empty set and $d_{JS} : A \times A \to [0, \infty]$ be a mapping. For any $\xi \in A$, let us define the set

$$C(d_{JS}, A, \xi) = \{\{\xi_n\} \subset A : \lim_{n \to \infty} d_{JS}(\xi_n, \xi) = 0\}. \tag{1.1}$$

Definition 1.4. *[9] Let $d_{JS} : A \times A \to [0, \infty]$ be a mapping which satisfies the following conditions:*

(i) $d_{JS}(\xi, \eta) = 0$ implies $\xi = \eta$ for all $\xi, \eta \in A$;

(ii) for every $\xi, \eta \in A$, we have $d_{JS}(\xi, \eta) = d_{JS}(\eta, \xi)$;

(iii) if $(\xi, \eta) \in A \times A$ and $\{\xi_n\} \in C(d_{JS}, A, \xi)$ then $d_{JS}(\xi, \eta) \leq p \limsup_{n \to \infty} d_{JS}(\xi_n, \eta)$, for some $p > 0$.

The pair (A, d_{JS}) is called a generalized metric space, usually known as $JS-$metric space.

2. Introduction to sequential bipolar metric space

In this section we introduce the notion of sequential bipolar metric spaces and prove some fixed point theorems in such spaces.

Let X and Y be two non-empty sets and $D_{sb} : X \times Y \to [0, \infty]$ be a function. For $\xi \in X$ and $\eta \in Y$ let us define the following sets

$$S_L(X, D_{sb}, \eta) = \{\{\xi_n\} \subset X : \lim_{n\to\infty} D_{sb}(\xi_n, \eta) = 0\};$$
$$S_R(Y, D_{sb}, \xi) = \{\{\eta_n\} \subset Y : \lim_{n\to\infty} D_{sb}(\xi, \eta_n) = 0\}.$$

Definition 2.1. *Let X and Y be two nonempty sets and $D_{sb} : X \times Y \to [0, \infty]$ satisfies the following conditions:*

$(D_{sb}1)$ $D_{sb}(\xi, \eta) = 0$ *implies* $\xi = \eta \in X \cap Y$*;*
$(D_{sb}2)$ $D_{sb}(\xi, \eta) = D_{sb}(\eta, \xi)$ *for all* $\xi \in X$ *and* $\eta \in Y$*;*
$(D_{sb}3)$ *there exists some* $k > 0$ *such that for all* $\xi_1, \xi_2 \in X$ *and* $\eta_1, \eta_2 \in Y$ *we have*

$$D_{sb}(\xi_1, \eta_2) \le k \limsup_{n\to\infty} [D_{sb}(\xi_1, \eta_1) + D_{sb}(\xi_n, \eta_1)] \text{ for any } \{\xi_n\} \in S_L(X, D_{sb}, \eta_2) \text{ and}$$
$$D_{sb}(\xi_1, \eta_2) \le k \limsup_{n\to\infty} [D_{sb}(\xi_2, \eta_n) + D_{sb}(\xi_2, \eta_2)] \text{ for any } \{\eta_n\} \in S_R(Y, D_{sb}, \xi_1).$$

Then D_{sb} is called sequential bipolar metric and the triplet (X, Y, D_{sb}) is called sequential bipolar metric space. If $X \cap Y \neq \emptyset$ then the space is called joint otherwise it is called disjoint. The sets X and Y are said to be the left pole and the right pole of (X, Y, D_{sb}) respectively.

Example 2.2. *Let $X = U_n(\mathbb{R})$ and $Y = L_n(\mathbb{R})$ be the space of all upper triangular matrix of order n and the space of all lower triangular matrix of order n. Let $D_{sb} : X \times Y \to \mathbb{R}_+$ be defined by*

$$D_{sb}(U, V) = \sum_{i,j=1}^{n} (|u_{i,j}| + |v_{i,j}|)^2 \text{ for all } U = (u_{i,j})_{n\times n} \in X \text{ and } V = (v_{i,j})_{n\times n} \in Y.$$

Clearly conditions $(D_{sb}1)$ and $(D_{sb}2)$ are satisfied. Now let $A = (a_{i,j})_{n\times n}, U = (u_{i,j})_{n\times n} \in X$ and $B = (b_{i,j})_{n\times n}, V = (v_{i,j})_{n\times n} \in Y$. If $V \neq O_{n\times n}$ then $S_L(X, D_{sb}, V) = \emptyset$. If $V = O_{n\times n}$ and $\{U_k = (u^{(k)}_{i,j})_{n\times n}\} \in S_L(X, D_{sb}, O_{n\times n})$ then

$$D_{sb}(A, B) + D_{sb}(U_k, B) = \sum_{i,j=1}^{n} (|a_{i,j}| + |b_{i,j}|)^2 + \sum_{i,j=1}^{n} (|u^{(k)}_{i,j}| + |b_{i,j}|)^2 \text{ for all } k \in \mathbb{N}. \tag{2.1}$$

Thus $\limsup_{k\to\infty} [D_{sb}(A, B) + D_{sb}(U_k, B)] = \sum_{i,j=1}^{n} (|a_{i,j}| + |b_{i,j}|)^2 + \sum_{i,j=1}^{n} |b_{i,j}|^2 \ge \sum_{i,j=1}^{n} |a_{i,j}|^2 = D_{sb}(A, O_{n\times n})$.

Also if $A \neq O_{n\times n}$ then $S_R(Y, D_{sb}, A) = \emptyset$. If $A = O_{n\times n}$ and $\{V_k = (v^{(k)}_{i,j})_{n\times n}\} \in S_R(Y, D_{sb}, O_{n\times n})$ then similar as above we can show that

$$D_{sb}(O_{n\times n}, V) \le \limsup_{k\to\infty} [D_{sb}(U, V_k) + D_{sb}(U, V)].$$

Hence (X, Y, D_{sb}) is a sequential bipolar metric space.

Remark 2.3. *(i) Any bipolar metric space (X, Y, d) is also sequential bipolar metric space.*

verification: Clearly conditions $(D_{sb}1)$ and $(D_{sb}2)$ are satisfied. Now if $(\xi_1, \eta_1), (\xi_2, \eta_2) \in X \times Y$ and $\{\xi_n\} \in S_L(X, d, \eta_2)$, $\{\eta_n\} \in S_R(Y, d, \xi_1)$ then

$$d(\xi_1, \eta_2) \leq d(\xi_1, \eta_n) + d(\xi_2, \eta_n) + d(\xi_2, \eta_2) \text{ and}$$
$$d(\xi_1, \eta_2) \leq d(\xi_1, \eta_1) + d(\xi_n, \eta_1) + d(\xi_n, \eta_2) \text{ for all } n \geq 1.$$

By taking $n \to \infty$ it is seen that $(D_{sb}3)$ is satisfied for $k = 1$.

(ii) Any bipolar b-metric space (X, Y, d_b) with coefficient $s \geq 1$ is also sequential bipolar metric space.

verification: The conditions $(D_{sb}1)$ and $(D_{sb}2)$ are trivially satisfied. Now if (ξ_1, η_1), $(\xi_2, \eta_2) \in X \times Y$ and $\{\xi_n\} \in S_L(X, d_b, \eta_2)$, $\{\eta_n\} \in S_R(Y, d_b, \xi_1)$ then

$$d_b(\xi_1, \eta_2) \leq s[d_b(\xi_1, \eta_n) + d_b(\xi_2, \eta_n) + d_b(\xi_2, \eta_2)] \text{ and}$$
$$d_b(\xi_1, \eta_2) \leq s[d_b(\xi_1, \eta_1) + d_b(\xi_n, \eta_1) + d_b(\xi_n, \eta_2)] \text{ for all } n \in \mathbb{N}.$$

Taking $n \to \infty$ it is verified that $(D_{sb}3)$ is satisfied for $k = s$.

Any sequential bipolar metric space may not be always bipolar metric or bipolar $b-$metric space. The following example supports our contention.

Example 2.4. *Let $X = \mathbb{Z}^+ \cup \{0\}$, $Y = \mathbb{Z}^- \cup \{0\}$ and $D_{sb} : X \times Y \to [0, \infty]$ be defined by $D_{sb}(0, 0) = 0$, $D_{sb}(0, -n) = \frac{1}{n+1}$, $D_{sb}(n, 0) = \frac{1}{n}$, $D_{sb}(n, -n) = 1$ for all $n \geq 1$ and $D_{sb}(m, -n) = 10$ for all $m, n (m \neq n) \geq 1$. Then $S_L(X, D_{sb}, -n) = \emptyset$ and $S_R(Y, D_{sb}, n) = \emptyset$ for any $n \in \mathbb{N}$. The conditions $(D_{sb}1)$ and $(D_{sb}2)$ hold trivially. Now*

Case-I: *If $\xi_1 = 0 = \eta_2$ then the condition $(D_{sb}3)$ is clearly satisfied for any $\xi_2 \in X$ and $\eta_1 \in Y$.*

Case-II: *Let $\xi_1 = k$ for $k \in \mathbb{N}$, $\eta_2 = 0$ and $\xi_2 \in X$ be arbitrary. Let $\{\xi_n\} \in S_L(X, D_{sb}, 0)$ be taken as arbitrary. Then two sub cases arise:*

Subcase-I: *$\eta_1 = 0$. Then we see that*

$$\begin{aligned} \limsup_{n\to\infty}[D_{sb}(\xi_1, \eta_1) + D_{sb}(\xi_n, \eta_1)] &= \limsup_{n\to\infty}[D_{sb}(k, 0) + D_{sb}(\xi_n, 0)] \\ &\geq D_{sb}(k, 0) = D_{sb}(\xi_1, \eta_2). \end{aligned} \tag{2.2}$$

Subcase-II: *$\eta_1 = -l$, $l \geq 1$. Then we see that*

$$\begin{aligned} \limsup_{n\to\infty}[D_{sb}(\xi_1, \eta_1) + D_{sb}(\xi_n, \eta_1)] &= \limsup_{n\to\infty}[D_{sb}(k, -l) + D_{sb}(\xi_n, -l)] \\ &\geq 1 > \frac{1}{k} = D_{sb}(\xi_1, \eta_2). \end{aligned} \tag{2.3}$$

Case-III: *Let $\xi_1 = 0$, $\eta_2 = -k$ for $k \in \mathbb{N}$ and $\eta_1 \in X$ be arbitrary. Let $\{\eta_n\} \in S_R(Y, D_{sb}, 0)$ be taken as arbitrary. Then two sub cases arise:*

Subcase-I: *$\xi_2 = 0$. Then we see that*

$$\begin{aligned} \limsup_{n\to\infty}[D_{sb}(\xi_2, \eta_n) + D_{sb}(\xi_2, \eta_2)] &= \limsup_{n\to\infty}[D_{sb}(0, \eta_n) + D_{sb}(0, -k)] \\ &\geq D_{sb}(0, -k) = D_{sb}(\xi_1, \eta_2). \end{aligned} \tag{2.4}$$

Subcase-II: $\xi_2 = l$, $l \geq 1$. *Then we see that*

$$\begin{aligned}\limsup_{n\to\infty}[D_{sb}(\xi_2,\eta_n)+D_{sb}(\xi_2,\eta_2)] &= \limsup_{n\to\infty}[D_{sb}(l,\eta_n)+D_{sb}(l,-k)]\\ &\geq 1 > \frac{1}{k+1} = D_{sb}(\xi_1,\eta_2). \end{aligned} \tag{2.5}$$

Case-IV: *If $\xi_1 = k$, $\eta_2 = -l$ for $k, l \geq 1$ then the condition $(D_{sb}3)$ is clearly satisfied for any $\xi_2 \in X$ and $\eta_1 \in Y$ since $S_L(X, D_{sb}, -l) = \emptyset$ and $S_R(Y, D_{sb}, k) = \emptyset$.*

Therefore it follows that (X, Y, D_{sb}) is a sequential bipolar metric space. But it is not a bipolar $b-$metric space for any $k \geq 1$. Clearly if we choose $\xi_1 = m$, $\eta_2 = -n$ for $m, n(m \neq n) \geq 1$ and $\xi_2 = 0 = \eta_1$ then

$$D_{sb}(\xi_1,\eta_1)+D_{sb}(\xi_2,\eta_1)+D_{sb}(\xi_2,\eta_2) = \frac{1}{m}+\frac{1}{n+1} \to 0 \text{ as } m,n\to\infty. \tag{2.6}$$

Also $D_{sb}(\xi_1, \eta_2) = 10$ for any $m, n(m \neq n) \geq 1$ and hence (X, Y, D_{sb}) is not a bipolar $b-$metric space for any $k \geq 1$.

Definition 2.5. *i) The opposite of a sequential bipolar metric space (X, Y, D_{sb}) is defined as the sequential bipolar metric space $(Y, X, \bar{D}_{sb})$, where the function*
$\bar{D}_{sb} : Y \times X \to [0, \infty]$ is defined as $\bar{D}_{sb}(y, x) = \bar{D}_{sb}(x, y)$.
ii) Let (X_1, Y_1) and (X_2, Y_2) be two pairs of sets.

The function $G : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a covariant mapping if $G(X_1) \subset X_2$ and $G(Y_1) \subset Y_2$ and we denote this as $G : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$.

The function $G : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a contravariant mapping if $G(X_1) \subset Y_2$ and $G(Y_1) \subset X_2$ and we denote this as $G : (X_1, Y_1) \rightleftharpoons (X_2, Y_2)$.

If (X_1, Y_1, D^1_{sb}) and (X_2, Y_2, D^2_{sb}) are two sequential bipolar metric spaces then we use the notations $G : (X_1, Y_1, D^1_{sb}) \rightrightarrows (X_2, Y_2, D^2_{sb})$ and $G : (X_1, Y_1, D^1_{sb}) \rightleftharpoons (X_2, Y_2, D^2_{sb})$ for covariant mappings and contravariant mappings respectively.

Definition 2.6. *Let (X, Y, D_{sb}) be a sequential bipolar metric space. A point $\zeta \in X \cup Y$ is said to be a left point if $\zeta \in X$, a right point if $\zeta \in Y$ and a central point if both hold.*

A sequence $\{\xi_n\} \subset X$ is called a left sequence and a sequence $\{\eta_n\} \subset Y$ is called a right sequence.

A sequence $\{\nu_n\} \subset X \cup Y$ is said to converge to a point ν if and only if $\{\nu_n\}$ is a left sequence, ν is a right point and $D_{sb}(\nu_n, \nu) \to 0$ as $n \to \infty$ or $\{\nu_n\}$ is a right sequence, ν is a left point and $D_{sb}(\nu, \nu_n) \to 0$ as $n \to \infty$.

Definition 2.7. *A sequence $\{(\xi_n, \eta_n)\} \subset X \times Y$ is called a bisequence. If the sequences $\{\xi_n\}$ and $\{\eta_n\}$ both converge then the bisequence $\{(\xi_n, \eta_n)\}$ is called convergent in $X \times Y$.*

If $\{\xi_n\}$ and $\{\eta_n\}$ both converge to a point $\nu \in X \cap Y$ then the bisequence $\{(\xi_n, \eta_n)\}$ is called biconvergent.

A sequence $\{(\xi_n, \eta_n)\}$ is a Cauchy bisequence if $D_{sb}(\xi_n, \eta_m) \to 0$ whenever $n, m \to \infty$.

A sequential bipolar metric space is said to be complete if every Cauchy bisequence is convergent.

Definition 2.8. *Let (X_1, Y_1, D^1_{sb}) and (X_2, Y_2, D^2_{sb}) be two sequential bipolar metric spaces:*

i) The mapping $G : (X_1, Y_1, D^1_{sb}) \rightrightarrows (X_2, Y_2, D^2_{sb})$ is called left-continuous at a point $\xi_0 \in X_1$ if for every sequence $\{\eta_n\} \subset Y_1$ with $\eta_n \to \xi_0$ we have $G(\eta_n) \to G(\xi_0)$ in (X_2, Y_2, D^2_{sb}).

ii) The mapping $G : (X_1, Y_1, D^1_{sb}) \rightrightarrows (X_2, Y_2, D^2_{sb})$ is called right-continuous at a point $\eta_0 \in Y_1$ if for every sequence $\{\xi_n\} \subset X_1$ with $\xi_n \to \eta_0$ we have $G(\xi_n) \to G(\eta_0)$ in (X_2, Y_2, D^2_{sb}).

iii) The mapping $G : (X_1, Y_1, D^1_{sb}) \rightrightarrows (X_2, Y_2, D^2_{sb})$ is said to be continuous, if it is left-continuous at each point $\xi \in X_1$ and right-continuous at each point $\eta \in Y_1$.

iv) A contravariant mapping $G : (X_1, Y_1, D^1_{sb}) \rightleftharpoons (X_2, Y_2, D^2_{sb})$ is continuous if and only if it is continuous as a covariant map $G : (X_1, Y_1, D^1_{sb}) \rightrightarrows (Y_2, X_2, \bar{D}^2_{sb})$.

Proposition 2.9. *Let (X, Y, D_{sb}) be a sequential bipolar metric space. If a central point ζ is a limit of a sequence such that $D_{sb}(\zeta, \zeta) = 0$, then it is the unique limit of this sequence.*

Proof . Let $\{\xi_n\}$ be a left sequence in (X, Y, D_{sb}) which converges to some $\zeta \in X \cap Y$ with $D_{sb}(\zeta, \zeta) = 0$. If $\eta \in Y$ be a limit of this sequence then we get

$$D_{sb}(\zeta, \eta) \leq k \limsup_{n\to\infty}[D_{sb}(\zeta, \zeta) + D_{sb}(\xi_n, \zeta)] = 0. \tag{2.7}$$

Thus (2.7) shows that $\zeta = \eta$. Therefore ζ is the unique limit of $\{\xi_n\}$. In a similar way if $\{\eta_n\}$ is a right sequence in (X, Y, D_{sb}) which converges to $\zeta \in X \cap Y$ with $D_{sb}(\zeta, \zeta) = 0$ then also ζ is the unique limit of $\{\eta_n\}$. □

Proposition 2.10. *In a sequential bipolar metric space (X, Y, D_{sb}) every convergent Cauchy bisequence is biconvergent.*

Proof . Let $\{(\xi_n, \eta_n)\}$ be a Cauchy bisequence converges to $(\xi, \eta) \in X \times Y$ that is $\xi_n \to \eta$ and $\eta_n \to \xi$ as $n \to \infty$. Then

$$D_{sb}(\xi, \eta) \leq k \limsup_{n\to\infty}[D_{sb}(\xi, \eta_m) + D_{sb}(\xi_n, \eta_m)] \text{ for all } m \in \mathbb{N}. \tag{2.8}$$

Taking $m \to \infty$ in the right hand side of (2.8) we get $D_{sb}(\xi, \eta) = 0$ and therefore $\xi = \eta \in X \cap Y$. Hence the bisequence $\{(\xi_n, \eta_n)\}$ is biconvergent. □

Remark 2.11. *Proposition 2.10 shows that if a Cauchy bisequence biconverges to some $\zeta \in X \cap Y$ then $D_{sb}(\zeta, \zeta) = 0$.*

Proposition 2.12. *In a sequential bipolar metric space (X, Y, D_{sb}) if a Cauchy bisequence has a convergent bisubsequence then it is also convergent.*

Proof . Let $\{(\xi_n, \eta_n)\}$ be a Cauchy bisequence which has a convergent bisubsequence $\{(\xi_{n_p}, \eta_{n_p})\}$ converging to $(\xi, \eta) \in X \times Y$. Then we have

$$D_{sb}(\xi_m, \eta) \leq k \limsup_{p\to\infty}[D_{sb}(\xi_m, \eta_{n_r}) + D_{sb}(\xi_{n_p}, \eta_{n_r})] \text{ for all } m, r \in \mathbb{N}. \tag{2.9}$$

Taking $m, r \to \infty$ from (2.9) we see that $\xi_m \to \eta$. Similarly we can show that $\eta_m \to \xi$ as $m \to \infty$. Hence our proposition. □

3. Some fixed point theorems

In this section some fixed point theorems have been proved in the context of a sequential bipolar metric space.

Theorem 3.1. *Let (X, Y, D_{sb}) be a complete sequential bipolar metric space and $T : (X, Y, D_{sb}) \rightrightarrows (X, Y, D_{sb})$ be a mapping satisfying*

$$D_{sb}(T\xi, T\eta) \leq a\, D_{sb}(\xi, \eta) \tag{3.1}$$

for all $(\xi, \eta) \in X \times Y$ and for some $a \in [0, 1)$. If for some $(\xi_0, \eta_0) \in X \times Y$, $\delta(D_{sb}, T, (\xi_0, \eta_0)) = \sup\{D_{sb}(T^i\xi_0, T^j\eta_0) : i, j \geq 1\} < \infty$ then the function $T : X \cup Y \to X \cup Y$ has a fixed point $\zeta \in X \cap Y$. Moreover if for some $\nu \in X$ or $\nu \in Y$, $D_{sb}(\nu, \zeta) < \infty$ or $D_{sb}(\zeta, \nu) < \infty$ then $\nu = \zeta$.

Proof . Let us denote $\xi_n = T^n\xi_0$ and $\eta_n = T^n\eta_0$ and $\delta(D_{sb}, T^{p+1}, (\xi_0, \eta_0)) = \sup\{D_{sb}(T^{p+i}\xi_0, T^{p+j}\eta_0) : i, j \geq 1\}$ for any $p = 0, 1, 2, \ldots$. Then for all $i, j \in \mathbb{N}$ we have

$$\begin{aligned} D_{sb}(T^{n+i}\xi_0, T^{n+j}\eta_0) &\leq a\, D_{sb}(T^{n-1+i}\xi_0, T^{n-1+j}\eta_0) \\ &\leq a\, \delta(D_{sb}, T^n, (\xi_0, \eta_0)) \text{ for all } n \geq 1. \end{aligned} \tag{3.2}$$

Since $\delta(D_{sb}, T^{p+1}, (\xi_0, \eta_0)) \leq \delta(D_{sb}, T, (\xi_0, \eta_0)) < \infty$ for all $p \geq 0$ then from (3.2) it follows that $\delta(D_{sb}, T^{n+1}, (\xi_0, \eta_0)) \leq a\, \delta(D_{sb}, T^n, (\xi_0, \eta_0))$ for all $n \geq 1$. Therefore

$$\begin{aligned} \delta\left(D_{sb}, T^{n+1}, (\xi_0, \eta_0)\right) &\leq a\, \delta(D_{sb}, T^n, (\xi_0, \eta_0)) \\ &\leq a^2\, \delta\left(D_{sb}, T^{n-1}, (\xi_0, \eta_0)\right) \\ &\cdots \\ &\leq a^n\, \delta(D_{sb}, T, (\xi_0, \eta_0)) \text{ for any } n \geq 1. \end{aligned} \tag{3.3}$$

Thus for any $1 \leq n < m$ we have,

$$D_{sb}(\xi_n, \eta_m) = D_{sb}(T^n\xi_0, T^m\eta_0) \leq \delta(D_{sb}, T^n, (\xi_0, \eta_0)) \to 0 \text{ as } n \to \infty. \tag{3.4}$$

Therefore $\{(\xi_n, \eta_n)\}$ is a Cauchy bisequence. Since (X, Y, D_{sb}) is complete, this sequence converges and thus by Proposition 2.10 biconverges to some $\zeta \in X \cap Y$ such that $D_{sb}(\zeta, \zeta) = 0$. Now,

$$D_{sb}(\xi_{n+1}, T\zeta) \leq a\, D_{sb}(\xi_n, \zeta) \to 0 \text{ as } n \to \infty. \tag{3.5}$$

So $\xi_{n+1} \to T\zeta$ as $n \to \infty$. Since $\{\xi_n\}$ converges to the central limit $\zeta \in X \cap Y$ with $D_{sb}(\zeta, \zeta) = 0$ then by Proposition 2.9 we get $T\zeta = \zeta$ and ζ is a fixed point of T.

Now let $\nu \in X$ be a fixed point of T such that $D_{sb}(\nu, \zeta) < \infty$. Then by contractive condition (3.1) we see that

$$D_{sb}(\nu, \zeta) = D_{sb}(T\nu, T\zeta) \leq a\, D_{sb}(\nu, \zeta), \tag{3.6}$$

which implies that $D_{sb}(\nu, \zeta) = 0$ that is $\nu = \zeta$. Similar conclusion holds whenever $\nu \in Y$. □

Theorem 3.2. *Let (X, Y, D_{sb}) be a complete sequential bipolar metric space and $T : (X, Y, D_{sb}) \rightleftharpoons (X, Y, D_{sb})$ be a mapping satisfying*

$$D_{sb}(T\eta, T\xi) \leq a\, D_{sb}(\xi, \eta) + b\, D_{sb}(\xi, T\xi) + c\, D_{sb}(T\eta, \eta) \tag{3.7}$$

for all $(\xi, \eta) \in X \times Y$ and for $a, b, c \in [0, 1)$ with $b < \frac{1}{k}$ and $a + b + c < 1$. If for some $\xi_0 \in X$, $\delta(D_{sb}, T, (\xi_0, T\xi_0)) = \sup\{D_{sb}(\xi_i, \eta_j); i, j \geq 1 : \eta_n = T\xi_n$ and $\xi_{n+1} = T\eta_n$ for all $n \geq 0\} < \infty$ then $\{(\xi_n, \eta_n)\}$ biconverges to some $\zeta \in X \cap Y$ with $D_{sb}(\zeta, \zeta) = 0$. If $D_{sb}(\zeta, T\zeta) < \infty$ then ζ will be a fixed point of T. Moreover if ζ and ν are two fixed points of T such that $D_{sb}(\zeta, \nu) < \infty$ and $D_{sb}(\nu, \nu) < \infty$ then $\zeta = \nu$.

Proof . Let us denote $\delta\left(D_{sb}, T^{p+1}, (\xi_0, T\xi_0)\right) = \sup\{D_{sb}(\xi_{p+i}, \eta_{p+j}) : i, j \geq 1\}$ for any $p \geq 0$. Then for all $i, j \geq 1$ we get

$$\begin{aligned} D_{sb}(\xi_{n+i}, \eta_{n+j}) &= D_{sb}(T\eta_{n-1+i}, T\xi_{n+j}) \\ &\leq a\, D_{sb}(\xi_{n+j}, \eta_{n-1+i}) + b\, D_{sb}(\xi_{n+j}, T\xi_{n+j}) + c\, D_{sb}(T\eta_{n-1+i}, \eta_{n-1+i}) \\ &= a\, D_{sb}(\xi_{n+j}, \eta_{n-1+i}) + b\, D_{sb}(\xi_{n+j}, \eta_{n+j}) + c\, D_{sb}(\xi_{n+i}, \eta_{n-1+i}) \\ &\leq (a+b+c)\, \delta\left(D_{sb}, T^n, (\xi_0, T\xi_0)\right) \text{ for any } n \in \mathbb{N}. \end{aligned} \tag{3.8}$$

Since $\delta\left(D_{sb}, T^{p+1}, (\xi_0, T\xi_0)\right) \leq \delta\left(D_{sb}, T, (\xi_0, T\xi_0)\right) < \infty$ for all $p \geq 0$ then from (3.8) it follows that $\delta\left(D_{sb}, T^{n+1}, (\xi_0, T\xi_0)\right) \leq (a+b+c)\, \delta\left(D_{sb}, T^n, (\xi_0, T\xi_0)\right)$ for all $n \geq 1$. So by routine calculation we have $\{(\xi_n, \eta_n)\}$ is a Cauchy bisequence. As (X, Y, D_{sb}) is complete, this bisequence converges and thus by Proposition 2.10 biconverges to some $\zeta \in X \cap Y$ such that $D_{sb}(\zeta, \zeta) = 0$. Now if $D_{sb}(\zeta, T\zeta) < \infty$ then,

$$\begin{aligned} D_{sb}(\xi_{n+1}, T\zeta) &= D_{sb}(T\eta_n, T\zeta) \\ &\leq a\, D_{sb}(\zeta, \eta_n) + b\, D_{sb}(\zeta, T\zeta) + c\, D_{sb}(T\eta_n, \eta_n) \\ &= a\, D_{sb}(\zeta, \eta_n) + b\, D_{sb}(\zeta, T\zeta) + c\, D_{sb}(\xi_{n+1}, \eta_n) \text{ for all } n \geq 0. \end{aligned} \tag{3.9}$$

Taking $n \to \infty$ in (3.9) we get $\limsup_{n\to\infty} D_{sb}(\xi_{n+1}, T\zeta) \leq b\, D_{sb}(\zeta, T\zeta)$. Also,

$$D_{sb}(\zeta, T\zeta) \leq k \limsup_{n\to\infty} [D_{sb}(\xi_{m+1}, \eta_n) + D_{sb}(\xi_{m+1}, T\zeta)] \text{ for all } m \geq 0. \tag{3.10}$$

Letting $m \to \infty$ we see that $D_{sb}(\zeta, T\zeta) \leq kbD_{sb}(\zeta, T\zeta)$. Hence $T\zeta = \zeta$.

Now if ζ and ν are two fixed points of T with $D_{sb}(\zeta, \nu) < \infty$ then

$$\begin{aligned} D_{sb}(\zeta, \nu) &= D_{sb}(T\zeta, T\nu) \\ &\leq a\, D_{sb}(\nu, \zeta) + b\, D_{sb}(\zeta, T\zeta) + c\, D_{sb}(T\nu, \nu) \\ &= a\, D_{sb}(\nu, \zeta)\ [\because D_{sb}(\zeta, T\zeta) = 0 \text{ and } D_{sb}(T\nu, \nu) = 0]. \end{aligned} \tag{3.11}$$

Therefore $D_{sb}(\zeta, \nu) = 0$ that is $\zeta = \nu$. $\square$

Theorem 3.3. *Let (X, Y, D_{sb}) be a complete sequential bipolar metric space and $T : (X, Y, D_{sb}) \rightrightarrows (X, Y, D_{sb})$ be a mapping satisfying*

$$D_{sb}(T\xi, T\eta) \leq a\, D_{sb}(\xi, \eta) + b\, D_{sb}(T\xi, \eta) + c\, D_{sb}(\xi, T\eta) \tag{3.12}$$

for all $(\xi, \eta) \in X \times Y$ and for $a, b, c \in [0, 1)$ with $a + b + c < 1$. If for some $(\xi_0, \eta_0) \in X \times Y$, $\delta\left(D_{sb}, T, (\xi_0, \eta_0)\right) = \sup\{D_{sb}(T^i\xi_0, T^j\eta_0) : i, j \geq 1\} < \infty$ then $\{(T^n\xi_0, T^n\eta_0)\}$ biconverges to some $\zeta \in X \cap Y$ with $D_{sb}(\zeta, \zeta) = 0$. If $\limsup_{n\to\infty} D_{sb}(\xi_n, T\zeta) < \infty$ then ζ will be a fixed point of T. Moreover if for some $\nu \in X$ or $\nu \in Y$, $D_{sb}(\nu, \zeta) < \infty$ or $D_{sb}(\zeta, \nu) < \infty$ then $\nu = \zeta$.

Proof . Let us denote $\xi_n = T^n\xi_0$ and $\eta_n = T^n\eta_0$ and $\delta\left(D_{sb}, T^{p+1}, (\xi_0, \eta_0)\right) = \sup\{D_{sb}(T^{p+i}\xi_0, T^{p+j}\eta_0) : i, j \geq 1\}$ for any $p = 0, 1, 2, \ldots$. Then for all $i, j \in \mathbb{N}$ we have

$$\begin{aligned} D_{sb}(T^{n+i}\xi_0, T^{n+j}\eta_0) = D_{sb}(TT^{n-1+i}\xi_0, TT^{n-1+j}\eta_0) &\leq a\, D_{sb}(T^{n-1+i}\xi_0, T^{n-1+j}\eta_0) + \\ b\, D_{sb}(T^{n+i}\xi_0, T^{n-1+j}\eta_0) &+ c\, D_{sb}(T^{n-1+i}\xi_0, T^{n+j}\eta_0) \\ &\leq (a+b+c)\, \delta\left(D_{sb}, T^n, (\xi_0, \eta_0)\right) \text{ for all } n \geq 1. \end{aligned} \tag{3.13}$$

Since $\delta\left(D_{sb}, T^{p+1}, (\xi_0, \eta_0)\right) \leq \delta\left(D_{sb}, T, (\xi_0, \eta_0)\right) < \infty$ for all $p \geq 0$ then from (3.13) it follows that $\delta\left(D_{sb}, T^{n+1}, (\xi_0, \eta_0)\right) \leq (a+b+c)\, \delta\left(D_{sb}, T^n, (\xi_0, \eta_0)\right)$ for all $n \geq 1$. In a similar way as in the previous theorems we get $\{(\xi_n, \eta_n)\}$ is a Cauchy bisequence. As (X, Y, D_{sb}) is complete, the iterative bisequence converges and thus by Proposition 2.10 biconverges to some $\zeta \in X \cap Y$ such that $D_{sb}(\zeta, \zeta) = 0$. So,

$$\begin{aligned} D_{sb}(\xi_{n+1}, T\zeta) &= D_{sb}(T\xi_n, T\zeta) \\ &\leq a\, D_{sb}(\xi_n, \zeta) + b\, D_{sb}(T\xi_n, \zeta) + c\, D_{sb}(\xi_n, T\zeta) \text{ for any } n \geq 0. \end{aligned} \tag{3.14}$$

Therefore from (3.14) and by our assumption we obtain $\limsup_{n\to\infty} D_{sb}(\xi_n, T\zeta) = 0$. Thus

$$D_{sb}(\zeta, T\zeta) \leq \lim_{m\to\infty} \{k \limsup_{n\to\infty} [D_{sb}(\xi_m, \eta_n) + D_{sb}(\xi_m, T\zeta)]\} = 0. \tag{3.15}$$

Hence ζ is a fixed point of T. Now let $\nu \in X$ be a fixed point of T such that $D_{sb}(\nu, \zeta) < \infty$. Then by contractive condition (3.12) we see that

$$\begin{aligned} D_{sb}(\nu, \zeta) &= D_{sb}(T\nu, T\zeta) \\ &\leq a\, D_{sb}(\nu, \zeta) + b\, D_{sb}(T\nu, \zeta) + c\, D_{sb}(\nu, T\zeta) \\ &= (a+b+c)\, D_{sb}(\nu, \zeta), \end{aligned} \tag{3.16}$$

implying that $D_{sb}(\nu, \zeta) = 0$ that is $\nu = \zeta$. Similar conclusion holds if $\nu \in Y$. □

Example 3.4. *Let us consider the sequential bipolar metric space $(U_n(\mathbb{R}), L_n(\mathbb{R}), D_{sb})$ cited in Example 2.2. Then it can be easily checked that $(U_n(\mathbb{R}), L_n(\mathbb{R}), D_{sb})$ is complete. Let us define $T : U_n(\mathbb{R}) \cup L_n(\mathbb{R}) \to U_n(\mathbb{R}) \cup L_n(\mathbb{R})$ by $T((u_{i,j})_{n\times n}) = (\frac{u_{i,j}}{2})_{n\times n}$. Then it can be easily verified that $D_{sb}(T\xi, T\eta) \leq \frac{1}{2} D_{sb}(\xi, \eta)$ for all $\xi \in U_n(\mathbb{R})$ and $\eta \in L_n(\mathbb{R})$. Also we see that for any $(A_0, B_0) = ((a_{i,j})_{n\times n}, (b_{i,j})_{n\times n}) \in U_n(\mathbb{R}) \times L_n(\mathbb{R}))$, $D_{sb}(T^i A_0, T^j B_0) = \sum_{r,s=1}^n \left(\frac{1}{2^i}|a_{r,s}| + \frac{1}{2^j}|b_{r,s}|\right)^2 \leq \sum_{r,s=1}^n \left(|a_{r,s}| + |b_{r,s}|\right)^2 = D_{sb}(A_0, B_0) < \infty$ for any $i, j \geq 1$. Thus $\delta(D_{sb}, T, (A_0, B_0)) \leq D_{sb}(A_0, B_0) < \infty$. Therefore T satisfies all the conditions of Theorem 3.1 and the null matrix $O_{n\times n}$ is the unique fixed point of T.*

Theorem 5.1 and Theorem 5.3 of [16] can be derived from our Theorem 3.1 and Theorem 3.2 when the topological vector space is $\mathbb{R}$ endowed with the usual cone $\mathbb{P} = \{x \in \mathbb{R} : x \geq 0\}$.

Corollary 3.5. *Let (X, Y, d_b) be a complete bipolar $b-$metric space with coefficient $k \geq 1$ and $T : (X, Y, d_b) \rightrightarrows (X, Y, d_b)$ be a mapping which satisfies*

$$d_b(T\xi, T\eta) \leq a\, d_b(\xi, \eta) \tag{3.17}$$

for all $(\xi, \eta) \in X \times Y$ and for some $a \in [0, \frac{1}{k})$. Then the mapping $T : X \cup Y \to X \cup Y$ has a unique fixed point.

Proof . Let us choose $(\xi_0, \eta_0) \in X \times Y$ and construct an iterative bisequence $\{(\xi_n, \eta_n)\}$, where $\xi_n = T^n\xi_0$ and $\eta_n = T^n\eta_0$ for all $n \in \mathbb{N}$. Using the contractive condition (3.17) it can be shown that

$$\begin{aligned} d_b(\xi_n, \eta_m) &\leq \frac{(ka)^n}{1-ka}[d_b(\xi_0, \eta_0) + d_b(\xi_1, \eta_0)] \leq \frac{[d_b(\xi_0, \eta_0) + d_b(\xi_1, \eta_0)]}{1-ka} \text{ for } 1 \leq n < m, \\ d_b(\xi_n, \eta_m) &\leq \frac{(ka)^m}{1-ka}[d_b(\xi_0, \eta_0) + d_b(\xi_0, \eta_1)] \leq \frac{[d_b(\xi_0, \eta_0) + d_b(\xi_0, \eta_1)]}{1-ka} \text{ for } 1 \leq m < n. \end{aligned} \tag{3.18}$$

Thus from (3.18) we get $\delta\left(d_b, T, (\xi_0, \eta_0)\right) = \sup\{d_b(T^i\xi_0, T^j\eta_0) : i, j \geq 1\} \leq L < \infty$, where $L = \max\{\frac{[d_b(\xi_0,\eta_0)+d_b(\xi_1,\eta_0)]}{1-ka}, \frac{[d_b(\xi_0,\eta_0)+d_b(\xi_0,\eta_1)]}{1-ka}\}$. Therefore by using Theorem 3.1 it follows that T has a unique fixed point in $X \cup Y$. □

Corollary 3.6. *Let $T : (X, Y, d_b) \rightleftharpoons (X, Y, d_b)$ be a mapping, where (X, Y, d_b) is a complete bipolar $b-$metric space with coefficient $k \geq 1$, satisfying*

$$d_b(T\eta, T\xi) \leq a\, d_b(\xi, \eta) + b\, d_b(\xi, T\xi) + c\, d_b(T\eta, \eta) \tag{3.19}$$

for all $\xi \in X, \eta \in Y$, where $0 \leq a, b < 1$, $0 \leq c < \frac{1}{k+1}$ and $0 \leq ka + kb + c < 1$. Then the function $T : X \cup Y \to X \cup Y$ has a unique fixed point.

Proof . Let ξ_0 be arbitrarily taken. We construct a bisequence $\{(\xi_n, \eta_n)\}$, where $\eta_n = T\xi_n$ and $\xi_{n+1} = T\eta_n$ for all $n \geq 0$. Then one can check that

$$\begin{aligned} d_b(\xi_n, \eta_m) &\leq \frac{(ke)^n}{1-ke}\left(1 + \frac{a+b}{1-c}\right) d_b(\xi_0, \eta_0) \leq \left(1 + \frac{a+b}{1-c}\right)\frac{d_b(\xi_0, \eta_0)}{1-ke} \text{ for } 1 \leq n < m, \\ d_b(\xi_n, \eta_m) &\leq \frac{(ke)^m}{1-ke}\left(e + \frac{a+b}{1-c}\right) d_b(\xi_0, \eta_0) \leq \left(e + \frac{a+b}{1-c}\right)\frac{d_b(\xi_0, \eta_0)}{1-ke} \text{ for } 1 \leq m < n, \end{aligned} \tag{3.20}$$

where $e = \frac{(a+b)(a+c)}{(1-c)(1-b)}$. Therefore $\delta\left(d_b, T, (\xi_0, T\xi_0)\right) = \sup\{d_b(\xi_i, \eta_j) : i, j \geq 1\} \leq M < \infty$, where $M = \max\{\left(1 + \frac{a+b}{1-c}\right)\frac{d_b(\xi_0,\eta_0)}{1-ke}, \left(e + \frac{a+b}{1-c}\right)\frac{d_b(\xi_0,\eta_0)}{1-ke}\}$. Hence due to Theorem 3.2 T has a unique fixed point in $X \cup Y$. $\square$

4. Well-posedness of fixed point problem

Well-posedness of fixed point problem is an interesting study in fixed point theory. The definition of well posedness of fixed point problem over metric spaces is as follows:

Definition 4.1. *[6] Let (X, d) be a metric space and $S : (X, d) \to (X, d)$ be a mapping. The fixed point problem of S is said to be well-posed if (i) S has a unique fixed point $z \in X$, (ii) for any sequence $\{x_n\}$ in X with $d(x_n, S(x_n)) \to 0$ as $n \to \infty$ we have $d(z, x_n) \to 0$ as $n \to \infty$.*

Now we give the definitions of well-posedness of fixed point problems in the setting of bipolar $b-$metric spaces which are running as follows:

Definition 4.2. *Let (X, Y, d_b) be a bipolar b-metric space and $F : (X, Y, d_b) \rightrightarrows (X, Y, d_b)$ be a mapping. The fixed point problem of F is said to be well posed if*

(i) F has a unique fixed point $\zeta \in X \cap Y$;

(ii) for any sequence $\{(\xi_n, \eta_n)\}$ in (X, Y) with $d_b(\xi_n, F\eta_n) \to 0$ and $d_b(F\xi_n, \eta_n) \to 0$ as $n \to \infty$ we have $d_b(\xi_n, \zeta) \to 0$ and $d_b(\zeta, \eta_n) \to 0$ as $n \to \infty$.

Definition 4.3. *Let (X, Y, d_b) be a bipolar b-metric space and $F : (X, Y, d_b) \rightleftharpoons (X, Y, d_b)$ be a mapping. The fixed point problem of F is said to be well posed if*

(i) F has a unique fixed point $\zeta \in X \cap Y$;

(ii) for any sequence $\{(\xi_n, \eta_n)\}$ in (X, Y) with $d_b(\xi_n, F\xi_n) \to 0$ and $d_b(F\eta_n, \eta_n) \to 0$ as $n \to \infty$ we have $d_b(\xi_n, \zeta) \to 0$ and $d_b(\zeta, \eta_n) \to 0$ as $n \to \infty$.

Theorem 4.4. *If (X, Y, d_b) is a complete bipolar $b-$metric space with coefficient $k \geq 1$ and $T : (X, Y, d_b) \rightrightarrows (X, Y, d_b)$ a mapping which satisfies*

$$d_b(T\xi, T\eta) \leq a\, d_b(\xi, \eta) \tag{4.1}$$

for all $\xi \in X$, $\eta \in Y$ and for some $a \in [0, \frac{1}{k})$ then the fixed point problem of T is well-posed.

Proof . From Corollary 3.5 we see that T has a unique fixed point ζ(say)$\in X \cap Y$. Let $\{(\xi_n, \eta_n)\}$ be a bisequence in (X, Y) satisfying $d_b(\xi_n, T\eta_n) \to 0$ and $d_b(T\xi_n, \eta_n) \to 0$ as $n \to \infty$. Then

$$\begin{aligned} d_b(\xi_n, \zeta) &\leq k[d_b(\xi_n, T\eta_n) + d_b(\zeta, T\eta_n) + d_b(\zeta, \zeta)] \\ &= k[d_b(\xi_n, T\eta_n) + d_b(T\zeta, T\eta_n)] \\ &\leq k[d_b(\xi_n, T\eta_n) + a\, d_b(\zeta, \eta_n)] \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{4.2}$$

Also

$$\begin{aligned} d_b(\zeta, \eta_n) &\leq k[d_b(\zeta, \zeta) + d_b(T\xi_n, \zeta) + d_b(T\xi_n, \eta_n)] \\ &= k[d_b(T\xi_n, T\zeta) + d_b(T\xi_n, \eta_n)] \\ &\leq k[a\, d_b(\xi_n, \zeta) + d_b(T\xi_n, \eta_n)] \text{ for all } n \geq 1. \end{aligned} \tag{4.3}$$

Therefore from (4.2) and (4.3) we get

$$d_b(\xi_n, \zeta) \leq k\, d_b(\xi_n, T\eta_n) + k^2 a[a\, d_b(\xi_n, \zeta) + d_b(T\xi_n, \eta_n)] \text{ for any } n \in \mathbb{N}. \tag{4.4}$$

Which implies that

$$d_b(\xi_n, \zeta) \leq \frac{k}{1-(ka)^2} d_b(\xi_n, T\eta_n) + \frac{k^2 a}{1-(ka)^2} d_b(T\xi_n, \eta_n) \to 0 \text{ as } n \to \infty. \tag{4.5}$$

From (4.3) and (4.5) it follows that $d_b(\zeta, \eta_n) \to 0$ as $n \to \infty$. Hence due to Definition 4.2 we see that the fixed point problem of T is well-posed. □

Theorem 4.5. *Let $T : (X, Y, d_b) \rightleftharpoons (X, Y, d_b)$ be a mapping, where (X, Y, d_b) is a complete bipolar $b-$metric space with coefficient $k \geq 1$, satisfying*

$$d_b(T\eta, T\xi) \leq a\, d_b(\xi, \eta) + b\, d_b(\xi, T\xi) + c\, d_b(T\eta, \eta) \tag{4.6}$$

for all $\xi \in X$, $\eta \in Y$, where $0 \leq a, b < 1$, $0 \leq c < \frac{1}{k+1}$ and $0 \leq ka + kb + c < 1$. Then the fixed point problem of T is well-posed.

Proof . From Corollary 3.6 it follows that T has a unique fixed point ζ(say)$\in X \cap Y$. Let $\{(\xi_n, \eta_n)\}$ be a bisequence in (X, Y) satisfying $d_b(\xi_n, T\xi_n) \to 0$ and $d_b(T\eta_n, \eta_n) \to 0$ as $n \to \infty$. Then

$$\begin{aligned} d_b(\xi_n, \zeta) &\leq k[d_b(\xi_n, T\xi_n) + d_b(\zeta, T\xi_n)] \\ &= k[d_b(\xi_n, T\xi_n) + d_b(T\zeta, T\xi_n)] \\ &\leq k[d_b(\xi_n, T\xi_n) + a\, d_b(\xi_n, \zeta) + b\, d_b(\xi_n, T\xi_n) + c\, d_b(T\zeta, \zeta)] \text{ for any } n \geq 1. \end{aligned} \tag{4.7}$$

(4.7) shows that $d_b(\xi_n, \zeta) \leq \frac{k(1+b)}{1-ka} d_b(\xi_n, T\xi_n) \to 0$ as $n \to \infty$. Also for all $n \in \mathbb{N}$.

$$\begin{aligned} d_b(\zeta, \eta_n) &\leq k[d_b(T\eta_n, \zeta) + d_b(T\eta_n, \eta_n)] \\ &= k[d_b(T\eta_n, T\zeta) + d_b(T\eta_n, \eta_n)] \\ &\leq k[a\, d_b(\zeta, \eta_n) + b\, d_b(\zeta, T\zeta) + c\, d_b(T\eta_n, \eta_n) + d_b(T\eta_n, \eta_n)]. \end{aligned} \tag{4.8}$$

Thus from (4.8) we get $d_b(\zeta, \eta_n) \leq \frac{k(1+c)}{1-ka} d_b(T\eta_n, \eta_n) \to 0$ as $n \to \infty$. Therefore by the Definition 4.3 the fixed point problem of T is well posed. □

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References

[1] D. Bajović, Z. D. Mitrović and M. Saha, *Remark on contraction principle in cone$_{tvs}$ b-metric spaces*, J Anal. 29 (2021) 273-–280.

[2] I. Beg, A. K. Laha and M. Saha, *Coincidence point of isotone mappings in partially ordered metric space,* Rend. Circ. Math. Palermo, (2016), doi:10.1007/S12215016-0232-3.

[3] D. Dey and M. Saha, *An extension of Banach fixed point theorem in Fuzzy metric space,* Bol. Sociedade Paranaense Mat. 32(1) (2014) 299–304.

[4] D. Dey and M. Saha, *Common fixed point theorems in a complete 2-metric space,* Acta Univ. Palackianae Olomucensis, Facultas Rerum Naturalium, Math. 52(1) (2013) 79–87.

[5] D. Dey, K. Roy and M. Saha, *On generalized contraction principles over $S-$metric spaces with application to homotopy,* J. New Theory, 31 (2020) 95–103.

[6] D. Dey, R. Fierro and M. Saha, *Well-posedness of fixed point problems,* J. Fixed Point Theory Appl. doi.org/10.1007/s11784-018-0538-i, Springer (2018).

[7] M. Gangopadhyay, A. P. Baisnab and M. Saha, *Expansive mappings and their fixed points in a vector metric space,* Int. J. Math. Arc. 4(8) (2013) 147–153.

[8] M. Gangopadhyay, M. Saha and A. P. Baisnab, *Some fixed point theorems in Partial metric spaces,* Turkic World Math. Soc. J. App. Eng. Math. 3(2) (2013) 206–213.

[9] M. Jleli and B. Samet, *A generalized metric space and related fixed point theorems,* Fixed Point Theory Appl. (2015), doi:10.1186/s13663-015-0312-7.

[10] A. K. Laha and M. Saha, *Fixed point on $\alpha - \psi$ multivalued contractive mappings in cone metric space,* Acta Comm. Univ. Tartuensis Math. 20(1) (2016) 35–43.

[11] A. K. Laha and M. Saha, *Fixed point for a class of set valued mappings on a metric space endowed with a graph,* ROMAI J. 11(1) (2015) 115–129.

[12] A. Mutlu and U. Gürdal, *Bipolar metric spaces and some fixed point theorems,* J. Nonlinear Sci. Appl. 9 (2016) 5362–5373.

[13] A. Mutlu, K. Özkan and U. Gürdal, *Coupled fixed point theorems on bipolar metric spaces,* European J. Pure Appl. Math. 10(4) (2017) 655–667.

[14] K. Roy, M. Saha and I. Beg, *Fixed point of contractive mappings of integral type over an $S^{JS}-$metric space,* Tamkang J. Math. DOI:10.5556/j.tkjm.52.2021.3298, online published.

[15] K. Roy and M. Saha, *Fixed points of mappings over a locally convex topological vector space and Ulam-Hyers stability of fixed point problems,* Novi Sad J. Math. 50(1) (2020) 99–112.

[16] K. Roy and M. Saha, *Generalized contractions and fixed point theorems over bipolar cone$_{tvs}$ $b-$metric spaces with an application to homotopy theory,* Mat. Vesnik, 72(4) (2020) 281–296.

[17] K. Roy and M. Saha, *On fixed points of Ćirić-type contractive mappings over a C^*-algebra valued metric space and Hyers-Ulam stability of fixed point problems,* J. Adv. Math. Stud. 12(3) (2019) 350-363.

[18] M. Saha and R. Chikkala, *Fixed point theorem over a quasi metric space,* South East Asian J. Math. Math. Sci. 8(2) (2010) 61–67.