Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 399-413
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.22279.2344

A nonunique common fixed point theorem of Rhoades type in b-metric spaces with applications

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The aim of this paper is to prove a nonunique common fixed point theorem of Rhoades type for two self-mappings in complete b-metric spaces. This theorem extends the results of [16] and [46]. Examples are furnished to illustrate the validity of our results. We apply our theorem to establish the existence of common solutions of a system of two nonlinear integral equations and a system of two functional equations arising in dynamic programming.

Keywords: b-metric space, Common fixed point, Picard sequence, Nonlinear integral equations, Dynamic programming.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

In 1989, Bakhtin [8] introduced the concept of b-metric spaces under the name quasi-metric spaces and showed a contraction principle in these spaces. Czerwik [14] utilized such spaces for $s = 2$ to prove some generalizations of Banach's fixed point theorem [9] and for an arbitrary $s \geq 1$ in [15]. Berinde [11] extended a result of [8] to a class of φ-contractions with φ a comparison function.

In 2010, Khamsi and Hussain [34] reintroduced the notion of a b-metric and called it a metric-type. Khamsi [33] also introduced another definition of a metric type. It is easy to see that every metric is a metric type and every metric type is a b-metric, meanwhile there exists a metric type which is not a metric and there exists a b-metric which is not a metric type, see Dung et al. [17].

Afterwards, several authors proved fixed and common fixed point theorems for single-valued and multivalued mappings in b-metric spaces, see [1], [5, 6], [13], [16], [22], [24, 25, 26, 27], [36], [40, 41, 42],

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Received: January 2021 *Accepted:* March 2021

[47, 48, 49, 50, 51] and [53]. The authors in [2, 3], [19], [28, 29], [44] and [52] gave remarks and shorter proofs of some results in b-metric spaces.

Khojasteh et al. [35] categorized the theorems which ensure the existence of a fixed point of a mapping T into the following four types.

(T_1): T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a Picard operator.

(T_2): Unnamed type: T has a unique fixed point and $\{T^n x\}$ does not necessarily converge to the fixed point.

(T_3): T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point for all $x \in X$. Such a mapping is called a weakly Picard operator.

(T_4): T may have more than one fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point.

For more details, we refer the reader to [35]. Inspired by the above, Khojasteh et al. [35] established two new types of fixed point theorems of single-valued and multivalued mappings which belong to (T_3).

Definition 1.1. *[8] Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}_+$ is called a b-metric on X if for all $x, y, z \in X$, the following conditions are satisfied.*

(bm-1) $d(x, y) = 0 \iff x = y$,

(bm-2) $d(x, y) = d(y, x)$,

(bm-3) $d(x, y) \leq s(d(x, z) + d(z, y))$.

The pair (X, d) is called a b-metric space with a coefficient s.

Every metric space is a b-metric space with $s = 1$, but the converse is not true in general as it is shown by the following example.

Example 1.2. *[48] Let $X = \{0, 1, 2\}$ and $d : X \times X \to \mathbb{R}_+$ defined by:*

$$\begin{aligned} d(0,0) &= d(1,1) = d(2,2) = 0, \\ d(1,0) &= d(0,1) = d(2,1) = d(1,2) = 1, \\ d(0,2) &= d(2,0) = m, \end{aligned}$$

where m is given real number such that $m \geq 2$. It is easy to check that for all $x, y, z \in X$

$$d(x, y) \leq \frac{m}{2}(d(x, z) + d(z, y)).$$

Therefore, (X, d) is a b-metric space with a coefficient $s = \frac{m}{2}$. The ordinary triangle inequality does not hold if $m > 2$ and so (X, d) is not a metric space.

Example 1.3. *[26] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.*

Proposition 1.4. *[52] Let (X, d) be a metric space. Suppose that there exists a convex and nondecreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(2) \geq 2$ and $\varphi(0) = 0$ such that $\varphi\left(\frac{x}{2}\right) \geq \frac{\varphi(x)}{\varphi(2)}$ for all $x \in [0, +\infty)$.*

Then D defined by $D(x, y) = \varphi(d(x, y))$ is a b-metric with $b = \frac{\varphi(2)}{2}$ and so (X, D) is a b-metric space.

In addition, the b-metric D is continuous at each $(x, y) \in X \times X$.

For other examples of a b-metric, see [8].

Definition 1.5. *[15] Let (X, d) be a b-metric space and $\{x_n\}$ a sequence in X. The sequence $\{x_n\}$ is said to be*

(i) Convergent to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) (X, d) is complete if every Cauchy sequence in X is convergent.

Remark 1.6. *As in the metric space, a b-metric space can be endowed with the topology induced by its convergence. Consequently, most of the notions which are true for metric spaces can be extended in the setting of b-metric spaces, see [4].*

Remark 1.7. *In general, a b-metric need not be continuous in each variable, see [4], [18], [26] and [40].*

The following lemma was established by [25], [41], [50] and [51].

Lemma 1.8. *Let (X, d) be a b-metric space with a coefficient $s \geq 1$ and $\{x_n\}$ a sequence in X such that*

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n),\ \ n = 1, 2, ...,$$

where $0 \leq \lambda < 1$. Then $\{x_n\}$ is a Cauchy sequence.

Khojasteh et al. [35] established two new types of fixed point theorems of single-valued and multivalued mappings which belong to (T_3). Rhoades [46] extended Theorem 1 of [35] for two self-mappings. M. Demma and P. Vetro [16] generalized Theorem 1of [35] from metric spaces to b-metric spaces.

It is our purpose in this paper to prove a nonunique common fixed point theorem for two self-mappings in the framework of b-metric spaces. This theorem extends Theorem 11 of [16], Theorem 1 of [35] and Theorem 2.1 of [46]. Examples are provided to illustrate the validity of our results. We apply our Theorem 2.1 to realize the existence of common solutions of a system of two nonlinear integral equations and a system of two functional equations arising in dynamic programming.

Before stating and showing our Theorem 2.1, we start by the following lemma which shorten the proof of this theorem.

Lemma 1.9. *Let (X, d) be a complete b-metric space and S, T self-mappings of X such that,*

$$s^{\alpha} d(Sx, Ty) \leq N(x, y) M(x, y), \tag{1.1}$$

holds for all $x, y \in X$, with $\alpha \geq 1$, where

$$N(x, y) = \frac{\max\{d(x, y), d(x, Sx) + d(y, Ty), d(x, Ty) + d(y, Sx)\}}{d(x, Sx) + d(y, Ty) + 1} \tag{1.2}$$

and

$$M(x, y) = \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2s}\right\}. \tag{1.3}$$

Then, each fixed point of S is a fixed point of T and conversely.

Proof . *Let $u \in F(S)$ where*

$$F(S) = \{x \in X : x = Sx\}$$

and assume that $u \notin F(T)$*. We have*

$$N(u,u) = \frac{d(u,Tu)}{d(u,Tu)+1} < 1 \quad \textit{and} \quad M(u,u) = d(u,Tu).$$

By replacing in (1.1)*, we get*

$$s^{\alpha} d(u,Tu) \leq \frac{d^2(u,Tu)}{d(u,Tu)+1} < d(u,Tu),$$

which is a contradiction. Hence, $u \in F(T)$*.*
Similarly, it can be shown that if $v \in F(T)$*, then* $v \in F(S)$*.* □

2. Main results

Now, we prove our main result.

Theorem 2.1. *Let* (X,d) *be a complete b-metric space and* S, T *self-mappings of* X *verifying* (1.1), (1.2) *and* (1.3)*. If one of the following conditions is fulfilled*
1) d *is discontinuous,* $\alpha \geq 2$ *and* $s > 1$*,*
2) d *is continuous,* $\alpha \geq 1$ *and* $s \geq 1$*,*
3) d *is discontinuous and either* S *or* T *is continuous in* X*,* $1 \leq \alpha < 2$*,* $s > 1$*,*
then
(a) S *and* T *have at least one common fixed point* $z \in X$*.*
(b) For n *even,* $\left\{(ST)^{n/2}x\right\}$ *and* $\left\{T(ST)^{n/2}x\right\}$ *converge to a common fixed point for each* $x \in X$*.*
(c) If z *and* w *are distinct common fixed points of* S *and* T*, therefore* $d(z,w) \geq s^{\alpha}/2$*.*

Proof . Let $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by:

$$x_{2n+1} = Sx_{2n} \ \text{ and } \ x_{2n+2} = Tx_{2n+1} \text{ for all } n \in \mathbb{N}.$$

If $x_{2n+1} = x_{2n+2}$, therefore

$$Sx_{2n} = Tx_{2n+1} = x_{2n+1}.$$

So $x_{2n+1} \in F(T)$, by using lemma 1.9, $x_{2n+1} \in F(S)$ and (a) is satisfied.
Similarly, if there exists a value of n for which $x_{2n} = x_{2n+1}$, then

$$x_{2n} \in F(S) \cap F(T),$$

and again (a) is verified, hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. If $x_{2n+1} \neq x_{2n+2}$, applying the inequality (1.1), we obtain

$$s^{\alpha} d(x_{2n+1}, x_{2n+2}) = s^{\alpha} d(Sx_{2n}, Tx_{2n+1}) \leq N(x_{2n}, x_{2n+1}) M(x_{2n}, x_{2n+1}), \tag{2.1}$$

where

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \frac{\max\left\{\begin{array}{c} d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1}), \\ d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n}) \end{array}\right\}}{d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1}) + 1} \\ &= \frac{\max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2})\right\}}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + 1} \\ &\leq \frac{\max\left\{\begin{array}{c} d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), \\ s\left(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\right) \end{array}\right\}}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + 1}, \end{aligned}$$

which yields

$$N(x_{2n}, x_{2n+1}) \leq \frac{s\left(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\right)}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + 1} = s\beta_{2n+1}, \tag{2.2}$$

where

$$0 < \beta_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} < 1,\ n \geq 1$$

and

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\left\{ \begin{array}{c} d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), \\ \dfrac{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})}{2s} \end{array} \right\} \\ &= \max\left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2s} \right\} \\ &\leq \max\left\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2} \right\} \\ &= \max\left\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \right\}, \end{aligned}$$

which gives

$$M(x_{2n}, x_{2n+1}) \leq \max\left\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \right\}. \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1), we get

$$\begin{aligned} s^{\alpha} d(x_{2n+1}, x_{2n+2}) &\leq s\beta_{2n+1} \max\left\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \right\} \\ &= s\beta_{2n+1} d(x_{2n}, x_{2n+1}). \end{aligned}$$

Thus

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq \frac{\beta_{2n+1}}{s^{\alpha-1}} d(x_{2n}, x_{2n+1}) \\ &\leq \beta_{2n+1} d(x_{2n}, x_{2n+1}). \end{aligned}$$

In a similar manner, we find

$$d(x_{2n}, x_{2n+1}) \leq \beta_{2n} d(x_{2n-1}, x_{2n}).$$

Hence, for all $n \geq 1$

$$d(x_n, x_{n+1}) \leq \beta_n d(x_{n-1}, x_n). \tag{2.4}$$

As in the proof of lemma 2.3 of Rhoades [46], the sequence $\{\beta_n\}$ is strictly nonincreasing, we have for all $n \geq 1$

$$d(x_n, x_{n+1}) < \beta_1 d(x_{n-1}, x_n).$$

Applying lemma 1.8, it follows that $\{x_n\}$ is a Cauchy sequence with (X, d) is complete, there exists a point $z \in X$ such that $\lim\limits_{n\to\infty} x_n = z$.

Case 1: d is discontinuous $\alpha \geq 2$ and $s > 1$. If $z \notin F(T)$, from the triangle inequality and (1.1) we obtain

$$\begin{aligned} d(z, Tz) &\leq sd(z, x_{2n+1}) + sd(x_{2n+1}, Tz) \\ &= sd(z, x_{2n+1}) + sd(Sx_{2n}, Tz) \\ &\leq sd(z, x_{2n+1}) + \frac{1}{s^{\alpha-1}} N(x_{2n}, z) M(x_{2n}, z), \end{aligned} \tag{2.5}$$

where

$$
\begin{aligned}
N(x_{2n},z) &= \frac{\max\{d(x_{2n},z), d(x_{2n},x_{2n+1})+d(z,Tz), d(x_{2n},Tz)+d(z,x_{2n+1})\}}{d(x_{2n},x_{2+1})+d(z,Tz)+1} \\
&\le \frac{\max\{d(x_{2n},z), d(x_{2n},x_{2n+1})+d(z,Tz), sd(x_{2n},z)+sd(z,Tz)+d(z,x_{2n+1})\}}{d(x_{2n},x_{2+1})+d(z,Tz)+1}, \\
M(x_{2n},z) &= \max\left\{d(x_{2n},z), d(x_{2n},x_{2n+1}), d(z,Tz), \frac{d(x_{2n},Tz)+d(z,x_{2n+1})}{2s}\right\}. \\
&\le \max\left\{d(x_{2n},z), d(x_{2n},x_{2n+1}), d(z,Tz), \frac{sd(x_{2n},z)+sd(z,Tz)+d(z,x_{2n+1})}{2s}\right\}.
\end{aligned}
$$

Letting $n\to\infty$ in (2.5), we obtain

$$
\begin{aligned}
d(z,Tz) &\le \frac{1}{s^{\alpha-1}}\frac{sd(z,Tz)}{d(z,Tz)+1}d(z,Tz) \\
&= \frac{1}{s^{\alpha-2}}\frac{d(z,Tz)}{d(z,Tz)+1}d(z,Tz) \\
&< d(z,Tz)
\end{aligned}
$$

which implies that $z\in F(T)$. By lemma 1.9, $z\in F(S)$ and (a) is verified.

To show (b), using the fact that x_0 is arbitrary, we can write $x_{2n+1}=(ST)^{n/2}x$ and $x_{2n+2}=T(ST)^{n/2}x$.

To establish (c), suppose that $z,w\in F(S)\cap F(T)$ with $z\neq w$. Applying (1.1), we find $N(z,w)=2d(z,w)$ and $M(z,w)=d(z,w)$.

So, (1.1) becomes $d(z,w)\le \frac{2}{s^{\alpha}}d^2(z,w)$ which gives (c).

Case 2: d is continuous, $\alpha\ge 1$ and $s\ge 1$. If $z\notin F(T)$, putting $x=x_{2n}$, $y=z$ in the inequality (1.1) and taking $n\to\infty$ we have

$$
\begin{aligned}
d(z,Tz) &\le \frac{1}{s^{\alpha}}\frac{d(z,Tz)}{d(z,Tz)+1}d(z,Tz) \\
&< d(z,Tz)
\end{aligned}
$$

which yields $z\in F(T)$, so by lemma 1.9, $z\in F(S)$ and (a) is verified.

Case 3: d is discontinuous and either S or T is continuous in X, $1\le\alpha<2$ and $s>1$.

Assume that S is continuous in X. So

$$Sz=\lim_{n\to\infty} Sx_{2n}=z,$$

that is, $z\in F(S)$. and by lemma 1.9, $z\in F(T)$ and (a) is satisfied.

The conclusions (b) and (c) follow as in the case 1, , which complete the proof. □

The following example supports our theorem 2.1.

Example 2.2. *Let $X=\{0,1,\frac{3}{2}\}$ and $d: X\times X\to\mathbb{R}_+$ defined by:*

$$
\begin{aligned}
d(1,0) &= d(0,1)=d(\tfrac{3}{2},1)=d(1,\tfrac{3}{2})=1, \\
d(0,\tfrac{3}{2}) &= d(\tfrac{3}{2},0)=\tfrac{5}{2}, \\
d(0,0) &= d(\tfrac{3}{2},\tfrac{3}{2})=d(1,1)=0.
\end{aligned}
$$

(X,d) is a complete b-metric space with $s=\frac{5}{4}$. Define $S,T: X\rightarrow X$ by:

$$\begin{aligned} S(0) &= 0,\ S(1)=1,\ S(\frac{3}{2})=0,\\ T(0) &= 0,\ T(1)=1,\ T(\frac{3}{2})=1. \end{aligned}$$

1) The cases $(x,y)\in\left\{(0,0),(0,1),(1,0),(1,1),\left(1,\frac{3}{2}\right),\left(\frac{3}{2},0\right)\right\}$ are clear.
2) For the case $(x,y)=\left(0,\frac{3}{2}\right)$, we have

$$sd(S(0),T\left(\frac{3}{2}\right))=\frac{5}{4}.$$

where

$$N(0,\frac{3}{2})=\frac{7}{4} \text{ and } M(0,\frac{3}{2})=\frac{5}{2}$$

Therefore, $1<\frac{7}{2}$ which is verified.
3) For the case $(x,y)=\left(\frac{3}{2},1\right)$, we get

$$sd(S(\frac{3}{2}),T(1))=\frac{5}{4}.$$

where

$$N(\frac{3}{2},1)=\frac{5}{7} \text{ and } M(\frac{3}{2},1)=\frac{5}{2}$$

Thus, $\frac{1}{2}<\frac{5}{7}$, which is verified.
4) For the case $(x,y)=\left(\frac{3}{2},\frac{3}{2}\right)$, we obtain

$$sd(S(\frac{3}{2}),T\left(\frac{3}{2}\right))=\frac{5}{4}.$$

where

$$N(\frac{3}{2},\frac{3}{2})=\frac{7}{9} \text{ and } M(\frac{3}{2},\frac{3}{2})=\frac{5}{2}$$

So, $\frac{1}{2}<\frac{7}{9}$ which is satisfied. Hence, S and T verify all the conditions of theorem 2.1 with $\alpha=1$ and S and T have two distinct common fixed points 0 and 1. Furthermore, $d(0,1)=1>\frac{s}{2}=\frac{5}{8}$.

Taking $s=1$ in theorem 2.1, we find theorem 2 of Rhoades [46].
If $S=T$ in theorem 2.1, we have the following corollary.

Corollary 2.3. *Let (X,d) be a complete b-metric space and T a self-mapping of X satisfying*

$$s^{\alpha}d(Tx,Ty)\leq N(x,y)M(x,y)$$

for all $x,y\in X$, where $\alpha\geq 1$,

$$N(x,y)=\frac{\max\{d(x,y),d(x,Tx)+d(y,Ty),d(x,Ty)+d(y,Tx)\}}{d(x,Tx)+d(y,Ty)+1}$$

and

$$M(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2s}\right\}.$$

If one of the following conditions is fulfilled
1) d is discontinuous, $\alpha\geq 2$ and $s>1$,
2) d is continuous, $\alpha\geq 1$ and $s\geq 1$,
3) d is discontinuous and T is continuous in X, $1\leq\alpha<2$, $s>1$,
then
(a) T has at least one fixed point $z\in X$.
(b) $\{T^n x\}$ converges to a fixed point for each $x\in X$ for all $n\in\mathbb{N}$.
(c) If z and w are distinct fixed points of T, therefore $d(z,w)\geq s^{\alpha}/2$.

The next example illustrates our corollary 2.3.

Example 2.4. *Let $X=\{0,1,\frac{1}{2},...,\frac{1}{n},...\}$ and $d:X\times X\to\mathbb{R}_+$ defined by:*

$$d(x,y)=\begin{cases} 0 & \text{if} \quad x=y,\\ |x-y| & \text{if} \quad x,y\in\left\{0,\frac{1}{2},\frac{1}{4},...,\frac{1}{2n},...\right\},\\ 4 & \text{if} \quad x,y\in\{0,1\},\\ 5 & \text{otherwise.}\end{cases}.$$

(X,d) is a a complete b-metric space with $s=\frac{5}{4}$.. Define $T:X\to X$ by:

$$T(0)=0,\ T(1)=1,\ T(\frac{1}{2n})=0,\ T(\frac{1}{2n+1})=1,\ n\geq 1.$$

1) The cases $x=y$ and $(x,y)\in\left\{\begin{array}{c}(0,1),\left(0,\frac{1}{2n}\right),\left(1,\frac{1}{2n+1}\right),\left(\frac{1}{2n},\frac{1}{2m}\right),\left(0,\frac{1}{2n+1}\right),\\ \left(\frac{1}{2n},1\right),\left(\frac{1}{2n+1},\frac{1}{2m+1}\right),m>n\geq 1\end{array}\right\}$ are obvious.

2) For the case $(x,y)=\left(\frac{1}{2n},\frac{1}{2n+1}\right)$, we obtain

$$s^2d(T(\frac{1}{2n}),T\left(\frac{1}{2n+1}\right))=\frac{25}{4}.$$

$$N(\frac{1}{2n},\frac{1}{2n+1})=\frac{20n}{12n+1}\ \text{and}\ M(\frac{1}{2n},\frac{1}{2n+1})=5$$

So, $1<4n$ which is verified.

3) The case $(x,y)=\left(\frac{1}{2n},\frac{1}{2m+1}\right)$, $m>n\geq 1$ *is similar to the case 4. Hence* T *satisfies all the conditions of corollary* 2.3 *with* $\alpha=2$. *Consequently,* T *has two distinct fixed points* 0 *and* 1. *Furthermore,* $d(0,1)=4>\frac{s^2}{2}=\frac{25}{32}$.

Example 2.5. *Let* (X,d) *defined as in example* 2.2. *Define* $T:X\to X$ *by:*

$$T(0)=0,\ T(1)=1,\ T(\frac{3}{2})=0.$$

1) The cases $x=y$, $(x,y)=(0,1)$ *and* $(x,y)=\left(0,\frac{3}{2}\right)$ *are obvious.*

2) The case $(x,y)=\left(1,\frac{3}{2}\right)$ *follows as example* 2.2. *Hence,* T *satisfies all the conditions of corollary* 2.3 *with* $\alpha=1$ *and* T *has two distinct fixed points* 0 *and* 1. *Furthermore,* $d(0,1)=1>\frac{s}{2}=\frac{5}{8}$. *Since* $sd(T(1),T\left(\frac{3}{2}\right))=\frac{5}{4}$ *and* $\frac{d(1,T(\frac{3}{2})+d(\frac{3}{2},T(1)))}{d(1,T(1))+d(\frac{3}{2},T(\frac{3}{2}))}=\frac{4}{7}$, *we get* $\frac{5}{4}>\frac{4}{7}$ *which is verified. Accordingly, theorem 3 of [16] cannot be applicable.*

3. Application to nonlinear integral equations

Let $X=C[a,b]$ be a set of all real valued continuous functions on $[a,b]$, where $[a,b]$ is a closed and bounded interval in $\mathbb{R}$. For $p>1$ a real number, define $d:X\times X\to\mathbb{R}_+$ by:

$$d(x,y)=\max_{t\in[a,b]}|x(t)-y(t)|^p$$

for all $x,y\in X$. Therefore, (X,d) is a complete b-metric space with $s=2^{p-1}$, see example 2. In this section, we apply our theorem 2.1 to establish the existence of common solutions of a system of two nonlinear integral equations of Fredholm type defined by

$$\begin{aligned} x(t) &= f(t)+\lambda\int_a^b K_1(t,\tau,x(\tau))d\tau, \qquad (3.1)\\ x(t) &= f(t)+\lambda\int_a^b K_2(t,\tau,x(\tau))d\tau, \end{aligned}$$

where $x\in C[a,b]$ is the unknown function, $\lambda\in\mathbb{R}, t,\tau\in[a,b]$, $K_1,K_2:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ are given continuous functions.

Theorem 3.1. *Assume that the following conditions are fulfilled*

(i) There exists a continuous function $\psi:[a,b]\times[a,b]\to\mathbb{R}_+$ *such that for all* $x,y\in X$, $\lambda\in\mathbb{R}$ *and* $t,\tau\in[a,b]$, *we get*

$$|K_1(t,\tau,x(\tau))-K_2(t,\tau,y(\tau))|^p\leq\psi(t,\tau)N(x(\tau),y(\tau)).M(x(\tau),y(\tau)),$$

where

$$N(x(\tau),y(\tau))=\frac{\max\left\{\begin{array}{c}|x(\tau)-y(\tau)|^p,|x(\tau)-Sx(\tau)|^p+|y(\tau)-Ty(\tau)|^p,\\|x(\tau)-Ty(\tau)|^p+|y(\tau)-Sx(\tau)|^p\end{array}\right\}}{d(x,Sx)+d(y,Ty)+1}$$

and

$$M(x(\tau),y(\tau))=\max\left\{\begin{array}{c}|x(\tau)-y(\tau)|^p,|x(\tau)-Sx(\tau)|^p,|y(\tau)-Ty(\tau)|^p,\\\dfrac{|x(\tau)-Ty(\tau)|^p+|y(\tau)-Sx(\tau)|^p}{2s}\end{array}\right\},$$

(ii) $|\lambda|\leq 1$,
(iii)

$$\max_{\tau\in[a,b]}\int_a^b\psi(t,\tau)d\tau\leq\frac{1}{s^\alpha(b-a)^{p-1}},$$

where $s=\dfrac{1}{2^{p-1}}$, $\alpha\geq 1$ *and* $s\geq 1$. *Then, the system* (3.1) *has at least one common solution* $z\in C[a,b]$. *Moreover, If* z *and* w *are two distinct solutions of* (3.1), *therefore* $d(z,w)\geq\dfrac{s^\alpha}{2}$.

Proof . Define two mapping $S,T:X\to X$ by:

$$\begin{aligned}Sx(t)&=f(t)+\lambda\int_a^b K_1(t,\tau,x(\tau))d\tau,\\Tx(t)&=f(t)+\lambda\int_a^b K_2(t,\tau,x(\tau))d\tau,\end{aligned}$$

for all $t\in[a,b]$. So, the existence of a solution of (3.1) is equivalent to the existence of a common fixed point of S and T. Let $q\in\mathbb{R}$ such that $\dfrac{1}{p}+\dfrac{1}{q}=1$. Using the Holder inequality, (i), (ii) and (iii), we have

$$\begin{aligned}
d(Tx,Ty) &= \max_{t\in[a,b]} |Tx(t)-Ty(t)|^p \\
&\leq |\lambda|^p \max_{t\in[a,b]} \left(\int_a^b |(K_1(t,\tau,x(\tau)) - K_2(t,\tau,x(\tau)))|^p \, d\tau \right) \\
&\leq \left[\max_{t\in[a,b]} \left(\int_a^b 1^q dz \right)^{\frac{1}{q}} \left(\int_a^b |(K_1(t,\tau,x(\tau)) - K_2(t,\tau,x(\tau)))|^p \, d\tau \right)^{\frac{1}{p}} \right]^p \\
&\leq (b-a)^{\frac{p}{q}} \left[\max_{t\in[a,b]} \left(\int_a^b |(K_1(t,\tau,x(\tau)) - K_2(t,\tau,x(\tau)))|^p \, d\tau \right) \right] \\
&\leq (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) N(x(\tau),y(\tau)).M(x(\tau),y(\tau)) d\tau \right) \\
&\leq (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) d\tau \right) N(x,y).M(x,y) \\
&\leq \frac{1}{s^\alpha} N(x,y).M(x,y).
\end{aligned}$$

Thus

$$s^\alpha d(Sx,Ty) \leq M(x,y).N(x,y).$$

Hence, all the conditions of theorem 2.1 hold. Consequently, the system (3.1) has at least one common solution $z \in C[a,b]$. Further, If z and w are distinct solutions of (3.1), therefore $d(z,w) \geq \frac{s^\alpha}{2}$. □

4. Application to dynamic programming

Let X and Y be Banach spaces, $S \subset X$ be the state space, $D \subset Y$ be the decision space and I_X be the identity mapping on X. $B(S)$ denotes the set of all bounded real valued functions on S and

$$d(f,g) = \sup_{x\in S} |f(x)-g(x)|^p.$$

It is clear that $(B(S),d)$ is a complete $b-$metric space with $s = 2^{p-1}$.

As proposed in Bellman and Lee [10], the basic form of the functional equation in dynamic programming is

$$f(x) = opt_{y\in D} H(x,y,f(T(x,y))),\ x \in S,$$

where x and y denote the state and decision vectors, respectively. T denotes the transformation of the process, $f(x)$ denotes the optimal return function with the initial state x and *opt* represents sup or inf.

Many authors proved the existence and the uniqueness of solutions or common solutions for several classes of functional equations or systems of functional equations arising in dynamic programming by employing various fixed and common fixed point theorems, see Bhakta and Mitra [12], Kalinde et al. [30], Li et al. [37], Liu [38], Liu et al. [39] and Pathak et al. [43].

In this section, applying theorem 2.1, we establish the existence of common solutions of the following system of two functional equations arising in dynamic programming.

$$f_i(x) = opt_{y\in D}\{u(x,y) + H_i(x,y,f_i(T(x,y)))\},\ x \in S,\ i = 1,2, \tag{4.1}$$

where $u : S \times D \to S$, $T : S \times D \to S$ and $H_i : S \times D \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$.

Theorem 4.1. *Suppose that the following conditions are verified*

(a_1) *u and H_i are bounded for $i = 1,2$,*

(a_2) *For all $x, y \in X$*

$$s^{\alpha+1}\left|H_1(x,y,g(t)) - H_2(x,y,h(t))\right|^p \le N(g,h)M(g,h), \tag{4.2}$$

where $s = \dfrac{1}{2^{p-1}}$, $\alpha \ge 1$, $s \ge 1$,

$$\begin{aligned} N(g(t),h(t)) &= \frac{\max\left\{\begin{array}{c} |g(t)-h(t)|^p, |g(t)-A_1g(t)|^p + |h(t)-A_2h(t)|^p, \\ |g(t)-A_2h(t)|^p + |h(t)-A_1g(t)|^p \end{array}\right\}}{d(g,A_1g)+d(h,A_2h)+1}, \\ M(g(t),h(t)) &= \max\left\{\begin{array}{c} |g(t)-h(t)|^p, |g(t)-A_1g(t)|^p + |h(t)-A_2h(t)|^p, \\ \dfrac{|g(t)-A_2h(t)|^p + |h(t)-A_1g(t)|^p}{2s} \end{array}\right\} \end{aligned}$$

and

$$A_ig_i(x) = opt_{y\in D}\{u(x,y) + H_i(x,y,g_i(T(x,y)))\},\ x \in S,\ i = 1,2.$$

Then, the system of functional equations (4.1) *possesses at least one common solution in* $B(S)$. *Furthermore, if z and w are two distinct solutions of* (4.1), *therefore* $d(z,w) \ge \dfrac{s^\alpha}{2}$.

Proof . It follows from (a_1) and (a_2) that A_1 and A_2 are self-mappings in $B(S)$. Assume that $opt_{y\in D} = \sup_{y\in D}$. Let $g, h \in B(S)$ and $\epsilon > 0$. Applying (4.2), we deduce that there exists $y, z \in D$ such that

$$A_1g(x) < u(x,y) + H_1(x,y,g(T(x,y))) + \epsilon \tag{4.3}$$

$$A_2h(x) < u(x,z) + H_2(x,z,h(T(x,y))) + \epsilon. \tag{4.4}$$

It is easy to see that

$$A_1g(x) \ge u(x,z) + H_1(x,z,g(T(x,z))) \tag{4.5}$$

$$A_2h(x) \ge u(x,y) + H_2(x,y,h(T(x,y))) \tag{4.6}$$

By virtue of (4.3) and (4.6), we infer that

$$\begin{aligned} A_1g(x) - A_2h(x) &< H_1(x,z,g(T(x,z))) - H_2(x,z,g(T(x,z))) + \epsilon \\ &\le |H_1(x,z,g(T(x,z))) - H_2(x,z,g(T(x,z)))| + \epsilon. \end{aligned} \tag{4.7}$$

From (4.4) and (4.5) we conclude that

$$\begin{aligned} A_1g(x) - A_2h(x) &> H_1(x,y,g(T(x,y))) - H_2(x,y,h(T(x,y))) - \epsilon \\ &\ge -|H_1(x,y,g(T(x,y))) - H_2(x,y,h(T(x,y)))| - \epsilon. \end{aligned} \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$|A_1g(x) - A_2h(x)| \le \max\left\{\left| \begin{array}{c} H_1(x,y,g(T(x,y))) - H_2(x,y,h(T(x,y))), \\ |H_1(x,z,g(T(x,z))) - H_2(x,z,h(T(x,z)))| \end{array} \right|\right\} + \epsilon.$$

Using (a_1) and the above inequality we obtain

$$\begin{aligned}
& |A_1g(x) - A_2h(x)|^p \\
\le\ & \left(\max\left\{ \begin{array}{c} |H_1(x,y,g(T(x,y))) - H_2(x,y,h(T(x,y)))|, \\ |H_1(x,z,g(T(x,z))) - H_2(x,z,h(T(x,z)))| \end{array} \right\} + \epsilon\right)^p \\
\le\ & 2^{p-1}(\left(\max\left\{ \begin{array}{c} |H_1(x,y,g(T(x,y))) - H_2(x,y,h(T(x,y)))|, \\ |H_1(x,z,g(T(x,z))) - H_2(x,z,h(T(x,z)))| \end{array} \right\}\right)^p + \epsilon^p) \\
=\ & \max\left\{ \begin{array}{c} s\,|H_1(x,y,g(T(x,y))) - H_2(x,y,h(T(x,y)))|^p, \\ s\,|H_1(x,z,g(T(x,z))) - H_2(x,z,h(T(x,z)))|^p \end{array} \right\} + s\epsilon^p \\
\le\ & \frac{1}{s^\alpha} N(g(t),h(t))M(g(t),h(t)) + s\epsilon^p \\
\le\ & \frac{1}{s^\alpha} N(g,h)M(g,h) + s\epsilon^p.
\end{aligned}$$

Hence

$$s^\alpha d(A_1g, A_2h) \le N(g,h)M(g,h) + s^{\alpha+1}\epsilon^p \tag{4.9}$$

Similarly, the inequality (4.9) also holds for $opt_{y\in D} = \inf_{y\in D}$. Letting $\epsilon \to 0$ in (4.9) we deduce that

$$s^\alpha d(A_1g, A_2h) \le N(g,h)M(g,h)$$

Due to theorem 2.1, A_1 and A_2 have at least one common fixed point $z \in B(S)$, i.e., z is a common solution of the system of functional equations (4.1). Furthermore, If z and w are distinct solutions of (4.1), therefore $d(z,w) \ge \frac{s^\alpha}{2}$. $\square$

5. Open problems

Recently, some researchers worked on orthogonal fixed point theory and fixed point theorems in R-metric spaces (see [7], [20, 21], [23], [31, 32] and [45]), Among other things in this paper, we suggest some problems

Open problem(I)- Common fixed point theorems of Rhoades type in orthogonal b-metric spaces.

Open problem(II)- Common fixed point theorems of Rhoades type on R-metric spaces.

Acknowledgement

The authors are thankful to the editor and referees for valuable suggestions.

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