Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 415-423
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5051

Feeble regular and feeble normal spaces in α-topological spaces using graph

Balqees K. Mahmoud[a,*], Yousif Y. Yousif[a]

[a]Department of Mathematics, College of Education for Pure Sciences (Ibn Al-Haitham), University of Baghdad, Iraq

(Communicated by Madjid Eshaghi Gordji)

Abstract

This paper introduces some properties of separation axioms called α -feeble regular and α -feeble normal spaces (which are weaker than the usual axioms) by using elements of graph which are the essential parts of our α -topological spaces that we study them. Also, it presents some dependent concepts and studies their properties and some relationships between them.

Keywords: α-feebly regular, α-quasiregular, α-feebly normal
2010 MSC: Primary 90C33; Secondary 26B25.

1. Introduction

The concepts of α -open sets (α -closed sets) introduced by many researchers [11, 1] and the generalization of them are studied in many references [5, 13]. Separation axioms and their properties are an important terms which studied in different topological spaces [13, 3, 17]. In addition, many mathematicians used the generalized closed set to define some types of weaker separation axioms [2, 6, 15].
Combinatorial applications appeared before many years and became an important part in multiple aspects, also many sciences are interested by the uses of graphs in several fields [4, 12]. Some concepts in graphs are defined in topologized graph and in α -topological spaces [16, 7, 8,, 9] like surrounding, hyperedges, pre-hyperedges, ... etc.

The concept of feebly open sets (its complement feebly closed sets) introduced in topological spaces which are precisely associated with semi-open sets. Also, feebly regular and feebly normal

*Corresponding author
Email addresses: balqees.khaleel1003a@ihcoedu.uobaghdad.edu.iq (Balqees K. Mahmoud), yousif.y.y@ihcoedu.uobaghdad.edu.iq (Yousif Y. Yousif)

Received: February 2021 *Accepted:* April 2021

spaces are defined in [14, 10] and some relationships are studied.

In recent paper, we introduce the concept of α -feeble regular and α -feeble normal spaces in $\alpha-$ topological spaces and its application in topologized graph. Furthermore, we study some specific properties and relationships between them.

2. Preliminaries and Basic definitions

Through this section, we consider the non-empty set $X = V_G \cup E_G$, where V_G, E_G are the set of vertices, the set of edges of any graph G and define a topology on X. Also, $\text{Int}(A)$ and $Cl(A)$ denoted the interior and the closure respectively of any subset A of X. In addition, A is called α open if it satisfied that $A \subseteq \text{Int}(Cl(\text{Int}(A)))$ for all $A \subseteq X$ (clearly its complement is α-closed), and the set of all α-open set formed a topology on X called α-topology. Generally, any open set is α -open (its complement is α -closed), and $\alpha - \text{Int}(A), \alpha - Cl(A)$ denoted the interior and the closure in α -topological space X respectively.
Separation axioms $(T_0, T_1, T_2, T_3, T_4)$ are special properties for some spaces which satisfied there conditions. Sequentially, the relationship between open sets and α -open sets resulted α -separation axioms $(\alpha - T_0, \alpha - T_1, \alpha - T_2, \alpha - T_3, \alpha - T_4)$, so every T_i -space (where $\text{i} = 0, 1, 2, 3, 4$) is $\alpha - T_i$ space. Regularity and normality are specific properties of T_3 and T_4 which are refer to the stronger or weaker axioms between the others.
Now we recall some basic definitions and theorems to complete this research.

Definition 2.1. [7]*Let X be an α -topological space ,and $x \in X$, the intersection of all α -open sets contained x is called α -surrounding set of x, and denoted by $x^{\alpha\diamond}$*

Definition 2.2. [9] *If X is α -topological space, then X is α -feebly Hausdorff if for any two points $x, y, x \neq y$, there exist α -open sets U_x, U_y such that $x \in U_x, y \in U_y$ and $U_x \cap U_y \subseteq x^{\alpha_\diamond}$.*

Definition 2.3. [16] : *If X is α -topological space, then X is feebly regular if for any point x and any closed set C such that $x \notin C$, there exist two open sets U_x and U_C such that $x \in U_x, y \in U_C$ and $(U_x \cap U_y) \subseteq (x^\diamond \cap C^\diamond)$.*

Definition 2.4. [16] *If X is topological space, then X is feebly normal if for any two closed sets $C, D, C \neq D$ in X, there exist two open sets U_C, U_D such that $C \subseteq U_C, D \subseteq U_D$ and $(U_C \cap U_D)$ $\subseteq C^\diamond$.*

Proposition 2.5. [16] *Let* S *be any subspace of any topological space X. If X is feebly Hausdorff (respecively feebly regular) then so is S. If X is feebly normal and S is closed, then S is feebly normal.*

Theorem 2.6. [8] *A hypergraph $H = (V, E, f)$ with an α -topology τ on V such that, $f(e)$ is α-closed, $\forall e \in E$, then the collection of sets $\widehat{\tau} = \left\{W^\square \cup F : W \in \tau, F \subseteq E\right\}$ is the finest $\alpha-$ topology on $V \cup E$ compatible with H which have the property that τ is the relative α -topology on V.*

Definition 2.7. [8] *: Let $H = (V, E, f)$ be an α -hypergraph with τ an α-topology on V, and $\hat{\tau}$ the α-topology combinatorially induced by τ. Then the combinatorial extension of V dependent on f is the α-hyperedge space $H = (V, E, \partial)$, where ∂ is the α-boundary operator dependent on $\hat{\tau}$.*

3. Feeble Regular Spaces in Alpha - Topological Spaces

In this section, we present the definition of α -feebly regular in α -space and study its relationship with another concepts.

Definition 3.1. *If X is α -topological space, then X is α -feebly regular if for any point x and any closed set C such that $x \notin C$, there exist two α -open sets U_x, U_C such that $x \in U_x, y \in U_C$ and $(U_x \cap U_y) \subseteq (x^{\alpha\diamond} \cap C^{\alpha\diamond})$*

Lemma 3.2. *Let X be an α -topological space, $A \subseteq X$, then $A^{\alpha\diamond} = \bigcup_{a\in A} a^{\alpha\diamond}$.*

Proof . Assume that $x \in A^{\alpha\diamond}$, then every α -open set contains A and x. If $x \notin a^{\alpha\diamond}$, for all $a \in A$, then there exists an α -open set U_a containing a but not x, and the union $\bigcup_{a\in A} U_a$ is α -open and contains A but not x, that is a contradiction.
The converse, if $x \in \bigcup_{a\in A} a^{\alpha\diamond}$, then $x \in a^{\alpha\diamond}$ for all $a \in A$, means every α -open set U_a contains a and x. Hence $x \in A^{\alpha\diamond}$. □

The next corollary shows that α -feebly regular is a hereditary property.

Proposition 3.3. *Let X be an α -topological space, S be any subspace of X, if X is α -feebly regular, then S is α -feebly regular.*

Proof . let $x \in S, C \subseteq S$ be α -closed, since S is a subspace of X, then $x \in X, C \subseteq X$ be α -closed. Since X is α -feebly regular, then there exist an α -open sets U_x, U_C of x, C in X such that $(U_x \cap U_C) \subseteq (x^{\alpha\diamond} \cap C^{\alpha\diamond})$. So $S \cap (U_x \cap U_C) \subseteq S \cap (x^{\alpha\diamond} \cap C^{\alpha\diamond})$, hence $(S \cap U_x) \cap (S \cap U_C) \subseteq (x^{\alpha\diamond} \cap (S \cap C^{\alpha\diamond})$. Then there exist an α -open sets V_x, V_C in S such that $(V_x \cap V_C) \subseteq (x^{\alpha\diamond} \cap C^{\alpha\diamond})$ where $V_x = (S \cap U_x), V_C = (S \cap U_C)$, that implies S is α -feebly regular. □
We introduce the next property that is weaker than α -feebly regular.

Lemma 3.4. *If X is α -feebly regular topological space, for any $x \in X$ and S, Z are two subsets have the property that for all $s \in S, \alpha - Cl(s) \cap Z \neq \emptyset$ with $x \notin \alpha - Cl(s)$, then $x \in \alpha - Cl(Z)$ where $x \in S$.*

Proof . By contradiction, if $x \notin \alpha - Cl(Z)$, then there are two α -open sets U_x and U_Z contain x and $\alpha - Cl(Z)$ such that $(U_x \cap U_Z) \subseteq x^{\alpha\diamond} \cap (Cl(Z))^{\alpha_\diamond}$. For any $s \in S$ there exists $z \in Z$ with $s \in z^{\alpha\diamond} \subseteq U_Z$, so $S \subseteq U_Z$. Since $x \in S$, then the α -neighbourhood of x contains some point from S. So there exists $s \in (S \cap U_x \cap U_z) = (S \cap U_x)$, that implies $s \in x^{\alpha\diamond}$ and $x \in \alpha - Cl(s)$ which is contradiction. □
The next definition is related with definition 4.6 and 4.8 in the next section.

Definition 3.5. *Let X be an α -hyperedge space, H be a subset of α -hyperedges, then H is satisfying the α -hyperedge convergence property if for every subset $F \subseteq H$, every subset Z contains at least one endvertex of each of the α -hyperedges in F, every point $x \in \alpha - Cl(F)$ but not an endvertex of any edge in F, and $x \in \alpha - Cl(Z)$.*

Definition 3.6. *An α -hyperedge space X is α -quasiregular if E_X satisfies the α -hyperedge convergence property.*

Theorem 3.7. *Let $H = (V, E, f)$ be an hypergraph, if X is α -quasiregular hyperedge space with underlying hypergraph H, then X is the combinatorial extension of V_X according to f.*

Proof . We have to show that any $A \subseteq X$ is an α -open set if and only if it has the form $F \cup (U \backslash E)^{\square}$ for some α -open subset $U \subseteq X$ and for some $F \subseteq E$.
Now suppose A is an α -open set, then A has the form $F \cup (U \backslash E)^{\square}$ by theorem 2.6.
Conversely, assume that $A = \hat{A} \cup F$ where $\hat{A} = (U \backslash E)^{\square}$ for some α -open set $U \subseteq X$ and $F \subseteq E$. We must prove that $\hat{A}$ is α -open. Since every endvertex belongs to the closure of its incident edges, and U is α -open, then $N_x \subseteq U$ for any $x \in U \cap V$ means that $\hat{A} \subseteq U$. Moreover, the set $S = U \backslash \hat{A}$ contains edges only, and $e \cap U = \{e\}$ for any egde $e \in S$. Since any singleton edge is α-open, then for every $e \in S$, take $x_e \in \partial(e) \backslash U$ and put the set $Z = \{x_e \mid e \in S\}$.
If $\hat{A}$ is not α -open, then there exists some point $x \in \hat{A}$ such that every neighbourhood of x has non-empty intersection with $U \backslash \hat{A}$, which means $x \in \alpha - Cl(S)$ implies that $x \in \alpha - Cl(Z)$ by lemma 3.4. But for all $s \in S, x \in U$, we have $x \notin \alpha - Cl(s)$ implies $s \notin x^{\alpha\diamond}$. That is a contradiction, since $Z \subseteq X \backslash U$ which is α -closed set. So that U which contains x is different from $\alpha - Cl(Z)$. Hence $\hat{A}$ is α -open implies that $A = \hat{A} \cup F$ is α -open. $\square$
The next corollary obtains from the last theorem.

Corollary 3.8. *Let X be an α -quasiregular hyperedge space with α -open subset $A \subseteq V_X$, then $A^{\square}$ is α -open in X.*

Proof . Assume that $X = V_X \cup E_X$, Since $A \subseteq V_X$ is α -open, then $A \cup E_X$ is α -open. Also $A \cup F_X$ is α -open for $F_X \subseteq E_X$ hyperedge. Hence $A^{\square} = A \cup F_X$ is α -open in X, since it has the same form of theorem 2.6. So $A^{\square}$ is α -open. $\square$

Remark 3.9. *Let x be an α -open point (α -hyperedge) in the α -topological space X, and any subset C such that $x \notin C$, then there exist α -open neighbourhoods of x, C such that their intersection contained in $x^{\alpha\diamond} \cap C^{\alpha\diamond}$. If $x \in C^{\alpha\diamond}$, then any α -open set containing C intersects $\{x\}$ in $C^{\alpha\diamond}$, but if $\alpha - Cl(x)$ is disjoint from C, then its complement was the necessary α -neighbourhood of C which has an empty intersection with $\{x\}$.*

The following theorem shows that α -quasiregular is the missing part between α -feebly regular on V with the same one on X.

Theorem 3.10. *If X is α -quasiregular hyperedge space, V is α -feebly regular, then X is α -feebly regular.*

Proof . Let $x \in X$ and C an arbitrary closed set in X such that $x \notin C$. We must find α -open neighbourhoods U_x and U_C of x and C such that $(U_x \cap U_C) \subseteq (x^{\alpha\diamond} \cap C^{\alpha\diamond})$ in X.
If x is an α -hyperedge, then the proof is clear by remark 3.9. If not, means, x is a vertex, then there exist α -open neighbourhoods U'_x and U'_C of x and $(V \cap C)$ in V respectively such that $(U'_x \cap U'_C) \subseteq x_V^{\alpha\diamond} \cap (V \cap C)_V^{\alpha\diamond}$, by α -feebly regularity of V. Since $(V \cap C)$ is α -closed and different from x, then $x \in V \backslash C$ and $V \backslash C$ is α -open. So we suppose that U'_x is different from C, and we select that U'_C is different from x, when $\alpha - Cl(x)$ is different from C.
Now suppose that $P \subseteq V$ is an α -open set and the set F of incident α -hyperedges with some vertex in each of P and $V \backslash P$. Also, the set Q of endvertices of α -hyperedges in F which contained in P and α -closed. Or else, $\alpha - Cl(Q) \backslash Q$ contains some point p in P, such that $p \in \alpha$-$Cl(F)$ which is not incident with any α -hyperedge in F. Since X is α -quasiregular, p should be in the α -closure of the set of endvertices of edges of F which are not in P, that is contradiction since $V \backslash P$ is α -closed.

Assume that the set T of incident α -hyperedges with some vertex inside and outside U'_x without x itself, and M_T the set of vertices in U'_x incident with some edge in T. So we have $\alpha - Cl\,(M_T) \cap U'_x = M_T$ is α -closed in U'_x, and $\alpha - Cl\,(M_T)$ is different from x. Put $W_x = U'_x \backslash M_T$, which is α -open in V and contains x.
In the same way, assume that the set S of incident α -hyperedges with some vertex inside and outside U'_C without any vertex in $(V \cap C)$, and M_S the set of vertices in U'_C incident with some α- hyperedge in S. So M_S is α -closed in U'_C, and put $W_C = U'_C \backslash M_S$, which is α -open in V and contains $(V \cap C)$. Now suppose that $U_x = W_x^{\square}$ and $U_C = W_C^{\square}$, then both of $W_x^{\square}$ and $W_C^{\square}$ are α -open by corollary 3.8. Since C is α -closed, then every α -hyperedge $h \in C \backslash V$ is incident with some vertex in $(V \cap C)$. Also, since U_C is α -open, and $(V \cap C) \subseteq U_C$, then $h \in U_C$, and $C \subseteq U_C$. Hence $(W_x \cap W_C) \subseteq (U'_x \cap U'_C) \subseteq (x_V^{\alpha\diamond} \cap (V \cap C)_V^{\alpha\diamond}) \subseteq (x_X^{\alpha\diamond} \cap (V \cap C)_X^{\alpha\diamond}) \subseteq (x_X^{\alpha\diamond} \cap C_X^{\alpha\diamond})$. Also, $(U_x \cap U_C)$ is the union of $(W_x \cap W_C)$ with the set of α -hyperedges F. So we have to prove that $F \subseteq x_X^{\alpha\diamond} \cap C_X^{\alpha\diamond}$.
Now we have that both U_x, U_C contain two types of α -hyperedges; which incident (with x, with some vertex in C) respectively and those which endvertices are all contained in U'_x, U'_C respectively. If the first type is not satisfying, then for any $f \in F$, all its endvertices (at least one vertex) must be in $U'_x \cap U'_C$, and if w is an endvertex, we have $f \in w_X^{\alpha\diamond}$ and $w \in x_V^{\alpha\diamond} \subseteq x_X^{\alpha\diamond}$, that implies $f \in x_X^{\alpha\diamond}$, means, f is incident with x, that is a contradiction. If f is not incident with x, but with a vertex in C, then all its endvertices are contained in U'_x, a contradiction because U'_x is disjoint with C. If f is incident with x but not with any vertex in C, then we have two cases, accordingly if $\alpha - Cl_V(x) \cap (V \cap C) = \emptyset$ or not. In the second case, we have $f \in x_X^{\alpha\diamond}$ and $x \in (V \cap C)_V^{\alpha\diamond} \subseteq (V \cap C)_X^{\alpha\diamond}$, which implies that $f \in (V \cap C)_X^{\alpha\diamond}$, that is a contradiction.
Since we select U'_C to be disjoint from x, and f is not incident with a vertex in C, then all its endvertices are contained in U'_C, that is a contradiction. So f must be incident with both x and some vertex in C. But $f \in F$ is arbitrary, so we deduce that $F \subseteq x_X^{\alpha\diamond} \cap C_X^{\alpha\diamond}$. □

The next corollaries immediately result from the above theorem.

Corollary 3.11. *Let H be an α -hyperedge space, H is α-feebly regular if and only if H is α-quasiregular and V is α-feebly regular.*

Proof . Let $H = V_H \cup E_H$ be an α -hyperedge space. If $V_H \cup E_H$ is α -feebly regular, then V_H is $\alpha-$ feebly regular by proposition 3.3. Also, V_H is α -closed implies that H is α -quasiregular by lemma 3.4. The converse directly deduces by theorem 3.10. □

Corollary 3.12. *An α-topologized hypergraph X is α -feebly regular if and only if it is α -quasiregular and V is α -regular.*

Proof . By corollary 3.11 X is α -quasiregular, and since the vertex set of α -topologized hypergraph is α -closed, then X is $\alpha - T_1$ space. That means α -feeble regularity and α -regularity are the same in $\alpha - T_1$ space. So V is α -regular. □

Corollary 3.13. *Let X be an α -topological space, then:*

1- Every α -regular space is α -almost regular space.

2- Every α -almost regulae space is α -weakly regular space.

3- Every α -weakly regular space is α -feebly regular space .

Proof . this proof is immediate from their definitions. □

4. Feeble Normal Spaces in Alpha-topological Spaces

In this section, we present the definition of α -feebly normal in α -space and study its relationship with another concepts.

Definition 4.1. *Let X be an α -topological space and D a decomposition of X, a subset S is saturated if it is the union of parts of D. The decomposition D is upper semi α -continuous, if for every part $P \in D$ and every α -open set U contains P, there exists a saturated α -open set V such that $P \subseteq V \subseteq U$.*

Lemma 4.2. *Let $p : X \to Y$ be a quotient map induced by a decomposition D of X, p is an α- closed map if and only if D is upper semi α -continuous.*

Proof . Let p be an α -closed map, $P \in D$ and U_P be an α -open set contains P, then $(X - U_P)$ is $\alpha-$ closed set. But $p\left(X - U_P\right)$ is α -closed in Y, so $p^{-1}\left(p\left(X - U_P\right)\right)$ is α -closed in X (which is a union of parts of D). Hence $V_P = X - p^{-1}\left(p\left(X - U_P\right)\right)$ is a saturated α -open set in X such that $P \subseteq V_P \subseteq U_P$, then D is upper semi α -continuous.
Conversely, assume that D is upper semi α -continuous, $P \in D$ and $F \subseteq X$ is an α -closed, then $(X - F)$ is an α -open and contains P. By upper semi α -continuity of D, there is a saturated α -open set V such that $P \subseteq V \subseteq (X - F)$. Since p is a quotient map induced by D, then $P \in p(V) \subseteq p(X - F) \subseteq D - p(F)$, means that $D - p(F)$ is an α -open and $p(F)$ is an α -closed. □

Definition 4.3. *Let X be an α -topological space, an α -open connected subset is a pre α hyperedge if it is not α -closed, a pre α -edge if its boundary contains at most two points, and a proper pre α -hyperedge(edge) if it has exactly two boundary points.*
A pre α -hyperedge(edge) selection $\{U_i\}_{i \in I}$ is a set of disjoint pre- α -hyperedges(edge). Let $\mathcal{U}$ be a pre α -hyperedge(edge) selection, then:

- *the pre α -complement of U is the α -closed subset $X \setminus \bigcup_{U \in \mathcal{U}} U$.*
- *the induced α -hyperedge(edge) decomposition is the partition whose parts are the pre $\alpha-$ hyperedges(edge) and the pre α -complement.*
- *the α -clumps are the components of the pre α -complement.*
- *the induced α -clump-hyperedge(edge) decomposition is the partition whose parts are the pre α hyperedges(edge) and the α -clumps.*
- *the induced α -hyperedge(edge) quotient and α -clump-hyperedge(edge) quotient are the quotient spaces induced by the respective decompositions α -hyperedge(edge) and α -clumphyperedge(edge).*

Definition 4.4. *Let X be an α -topological space, a subset of X is an H_α -set if it is the union of the pre α -hyperedges of some pre α -hyperedge selection, and a (proper) E_α -set when all these α hyperedges(edges) are proper. Moreover, a subset is a V_α -set if it is the complement of some H_α-set.*

Proposition 4.5. *Let X be an α- topological space, $\mathcal{U}$ be pre α -hyperedge selection in X such that all the pre α -hyperedges have finite boundaries. Let $q : X \to Q$ be the induced quotient map onto the α -clump-hyperedge quotient Q, then a vertex $v \in V_Q$ is incident with an α -hyperedge $e \in Q$ if and only if the pre α -hyperedge whose image is e has a boundary point in the α -clump whose image is v.*

Proof . Let v and e be an arbitrary vertex and edge in Q, then $q(C)=\{v\}$ for some α -clump C and $q(H)=\{e\}$ for some pre α -hyperedge H in X. Assume that $x\in C\cap\partial(H)$ in X, and U is an $\alpha-$ neighbourhood of v in Q. Then $q^{-1}(U)$ is an α -neighbourhood of C such that $x\in C\subseteq q^{-1}(U)$ also $p\in q^{-1}(U)$ for some point p. So $q(p)=e\in q(q^{-1}(U))=U$, hence any α -neighbourhood of v contains e, means $v\in\partial(e)$.
Conversely, suppose that $C\cap\partial(H)=\emptyset$, then $\partial(H)$ is α -closed since it is finite, also all α -clumps being components of the pre α -complement are α -closed in X. So, the union of $H(\alpha-Cl(H))$ with all α -clumps containing a boundary point of H which denoted as B is α -closed, and $A=X\backslash B$ is α -open. Since B and C are disjoint, then $C\subseteq A$. Furthermore, A and B are saturated with respect to the α -clump-hyperedge decomposition of X. Therefore $q(A),q(B)$ are disjoint and $A=q^{-1}(q(A))$, so $v\in q(A)$ is α -open and disjoint from $e\in q(B)$. That implies $v\notin\partial(e)$, but when $v\in\partial(e)$ implies that $C\cap\partial(H)\neq\emptyset$. □

Definition 4.6. *Let X be an α -topological space, a pre α -hyperedge selection $\mathcal{U}=\{U\}_{i\in I}$ satisfies the strong pre α -hyperedge convergence property if:*
$\forall J\subseteq I,\{v_j\}_{j\in J}\subseteq X$ and $x\in X$ such that $x\in\alpha-Cl\left(\cup_{j\in J}U_j\right)\backslash\cup_{j\in J}\alpha-Cl\left(U_j\right)$, and $v_j\in\partial\left(U_j\right)\forall j\in J$, then $x\in\alpha-Cl\left(\{v_j\mid j\in J\}\right)$.
We have that x contained in the pre α -complement, since pre α -edges are disjoint and α -open. Also, when U_j 's are the singletons, the definition equal to definition 3.5 ofα -quasiregular.

Proposition 4.7. *Let $\mathcal{U}$ be a pre α -hyperedge selection in α -topological space X, then the induced α -hyperedge quotient is:*

1) *α -quasiregular if and only if u satisfies the strong α -hyperedge convergence property.*
2) *α -feebly regular if and only if the pre α -complement is α -feebly regular and $\mathcal{U}$ satisfies the strong pre α -hyperedge convergence property.*

Proof . This comes from definition 3.5 and corollary 3.11. □

Definition 4.8. *Let X be an α -topological space, $\mathcal{U}=\{U\}_{i\in I}$ a pre α -hyperedge selection and $\mathcal{Z}$ be the collection of components of the pre α -complement Z, then $\mathcal{U}=\{U\}_{i\in I}$ satisfies the weak pre α -hyperedge convergence property if:*
$\forall J\subseteq I,\{K_j\}_{j\in J}\subseteq\mathcal{Z}$ and $K\in\mathcal{Z}$ such that $K\cap\alpha-Cl\left(\bigcup_{j\in J}U_j\right)\neq\emptyset$, and $\alpha-Cl\left(U_j\right)\cap K_j\neq\emptyset,\forall j\in J$, but $K\cap\alpha-Cl\left(U_j\right)=\emptyset$, then $\alpha-Cl\left(\bigcup_{j\in J}K_j\right)\cap K\neq\emptyset$.

Proposition 4.9. *Let $\mathcal{U}$ be a pre α -hyperedge selection in α -connected topological space X such that the boundary of each pre α -hyperedge is finite, then the induced α -clump-hyperedge quotient is α -quasiregular if and only if $\mathcal{U}$ satisfies the weak pre α -hyperedge convergence property.*

Proof . This comes from proposition 4.5 and definition 3.5. □

Definition 4.10. *If X is α -topological space, then X is α -feebly normal if for any two closed sets $C\neq D$ in X, there exist two α -open sets U_C,U_D such that $C\subseteq U_C,D\subseteq U_D$ and $(U_C\cap U_D)\subseteq C^{\alpha\diamond}$.*

The normality property is not hereditary in general, the next corollary presents it with necessary condition.

Proposition 4.11. *Let X be an α -topological space, S be any subspace of X. If X is α -feebly normal and S is α -closed, then S is α -feebly normal.*

Proof . the same way as the proof of α -feebly regular proposition 3.3. □

Theorem 4.12. *If X is α -quasiregular hyperedge space, V is α -feebly normal, then X is α -feebly normal.*

Proof . this is the same proof of regularity case of theorem 3.10. □

The next proposition shows that α -feebly normal property is a topological property under α-closed quotient map.

Proposition 4.13. *If $p : X \to Y$ is α -closed quotient mapping, and X is α -feebly normal, then Y is α -feebly normal.*

Proof . Assume that C_1, C_2 is any disjoint α -closed subsets of Y, and $K_1 = p^{-1}(C_1), K_2 = p^{-1}(C_2)$ in X. Since X is α -feebly normal, then there exist α -open sets U_1, U_2 in X such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$ respectively with $U_1 \cap U_2 \subseteq K_1^{\alpha\diamond} \cap K_2^{\alpha\diamond}$. Also, p is α -closed, gives that the decomposition of X into parts with respect to p is α -closed. So, there exists an α -open set W_y which is the union of equivalence classes and such that $p^{-1}(y) \subseteq W_y \subseteq U_i$, for any $y \in C_i (i = 1, 2)$. Putting $W_i = \bigcup_{y \in C_i} W_y$ gives an α -open set which is the union of parts and such that $K_i \subseteq W_i \subseteq U_i$. So the set $Y \setminus (p(X \setminus W_i)) = p(W_i)$ is α -open in Y, since p is α -closed, but W_i is the union of parts, hence $C_i \subseteq p(W_i)$. Now Assume that $y \in p(W_1) \cap p(W_2)$, so $p^{-1}(y) \subseteq W_1 \cap W_2 \subseteq U_1 \cap U_2$. Hence if A is any α -open set in Y containing C_1, gives that $p^{-1}(y) \in K_1 \subseteq p^{-1}(A)$, and $y \in p(p^{-1}(A)) = A$. So $p(W_1) \cap p(W_2) \subseteq C_1^{\alpha_\diamond}$, since A and y are arbitrary (α -open set containing C_1 and point in $p(W_1) \cap p(W_2)$. □

Proposition 4.14. *Let X be an α -feebly normal topological space, W is a V_α -set in X. Assume that the components of W by decomposition is upper semi α -continuous, and that the induced pre α -hyperedges have finite boundaries. Then the induced α -clump-hyperedge quotient is α -feebly normal if and only if the induced pre α -hyperedge selection satisfies the weak pre α hyperedge convergence property.*

Proof . Let Q be the induced clump α -hyperedge quotient. Since W is an α -closed subset of an α- feebly normal space, then by proposition 4.11, W is α -feebly normal. Since the components of W by D is upper semi α -continuous, then the quotient map restricted to W is α -closed by lemma 4.2, so $W/D \cong V_Q$ is α -feebly normal by proposition 4.13. Suppose that the α -pre-hyperedge selection satisfies the weak pre α -hyperedge convergence property, then is α -quasiregular by proposition 4.9, and it is α -feebly normal by theorem 3.10. Conversely, if the clump of α- hyperedge quotient is not α -feebly normal, then it is not α -quasiregular by theorem 3.10, and the induced pre α -hyperedge selection is not satisfying the weak pre α-hyperedge convergence property by proposition 4.9. □

Proposition 4.15. *Let X be an α -quasiregular hyperedge space, if the components of V_X by a decomposition D is upper semi α -continuous, then the clump α -hyperedge decomposition of X is upper semi α -continuous.*

Proof . Suppose D is a part of the α -edge-clump decomposition and U be an α -open set contains D. We hope to find a saturated α -open set contained in U and contains D. When D is a singleton containing an α -hyperedge, it is trivial. If that is not satisfying, then $U \cap V_X$ is α -open in V_X and containing D. Since D is a part with respect to the decomposition of V_X into clumps, there exists a saturated α -open set W contains D and contained in $U \cap V_X$ which is saturated. Since by corollary 3.8, X is α -quasiregular, then $W^{\square}$ is α -open in X and contained in any α -open set contains W. Particularly, U is the union of clumps and singletons consisting of α -hyperedges, that means U is saturated with respect to the α -edge-clump decomposition of X. □

Definition 4.16. *An α -topological space X is α -weakly normal, if for any two disjoint closed sets C and D, there exist α -neighbourhoods (α -open sets) U_C and U_D such that $C \subseteq U_C$ and $D \subseteq U_D$, and $U_C \cap U_D$ is finite. If the intersection is exactly in one point, X is α -almost normal spaces.*

Corollary 4.17. *Let X be an α -topological space, then:*

1- Every α -normal space is α -almost normal space.

2- Every α -almost normal space is α -weakly normal space.

3- Every α -weakly normal space is α -feebly normal space.

Proof . this proof is immediate from their definitions. □

5. Conclusion

In this paper, alpha feebly regular and alpha feebly normal spaces in alpha topological spaces are defined, also some properties and relationships are studied in order to continuum the future works to study new related concepts like alpha compact feebly regular and alpha compact feebly normal spaces.

References

[1] D. Andrijevic, *Some properties of the topology of α-sets*, Math. Vesnik, 36 (1984) 1–10.

[2] C. Dorsett, *Feebly separation axioms, the feebly induced topology and separation axioms and the feebly induced topology*, J . Karadeniz Univ. Math. J. 8 (1985) 43–54.

[3] M. C. Gemignani, *Elementary Topology*, Buffalo. State University of New York, 1972.

[4] F. Harary, *Graph Theory*, Addison–Wesley, Reading, MA, 1969.

[5] N. Levine, *Generalized closed sets in topology*, Rend. Circ. Math. Palermo, 19 (1970) 89–96.

[6] S. N. Maheswari and U. Tapi, *On feebly ρ-spaces*, An. Univ Timisoara. Ser. Stiint Math. 16 (1978) 173–177.

[7] B. K. Mahmoud and Y. Y. Yousif, *Cutpoints and separations in connected α-topological spaces*, Iraq J. Sci., accepted (2020).

[8] B. K. Mahmoud and Y. Y. Yousif, *Compatibility and edge spaces in α-topological spaces*, to appear.

[9] B. K. Mahmoud and Y. Y. Yousif, *Feeble Hausdorff spaces in α-topological spaces using graph*, accepted in IOP Conference (2021).

[10] M. Mrsevic and I. L. Reilly, *Separation properties of a topological spaces and its associatted topology of subsets*, Belgrade. Kyungpook. Math. J. 33(1) (1993) 75–86.

[11] O. Njastad, *On some classes of nearly open sets*, Pacific J. Math. 15 (1965) 961–970.

[12] S. J. Radhi, M. A. Abdlhusein and A. E. Hashoosh, *The arrow domination in graphs*, Int. J. Nonlenear Anal. Appl. 12(1) (2021) 473–480.

[13] T. C. K. Raman, V. Kumari and M. K. Sharma, *α-generalized and α^*-separation axioms for topological spaces*, IOSR J. Math. 10(3) (2014) 32–36 .

[14] U. Tapi, *Feebly Regular and Feebly Normal Spaces*, J. Indain Acad. Math., 9 (1987) 104–109.

[15] U. Tapi, *On Pairwise Feebly Ro- Spaces*, Belgrade. Kyungpook. Math. J., 30 (1990) 81–85.

[16] A. Vella, *A Fundamentally Topological Perspective on Graph Theory*, Ph. D. Thesis, Waterloo, Ontaio, Canada, 2005.

[17] S. Willard, *General topology*, London-Don Mills. Addison-Wesley Publishing Co. Reading Mass. Ontaio, 1970.