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Fractional dynamical systems: A fresh view on the local qualitative theorems

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Abstract

The aim of this work is to describe the qualitative behavior of the solution set of a given system of fractional differential equations and limiting behavior of the dynamical system or flow defined by the system of fractional differential equations. In order to achieve this goal, it is first necessary to develop the local theory for fractional nonlinear systems. This is done by the extension of the local center manifold theorem, the stable manifold theorem and the Hartman-Grobman theorem to the scope of fractional differential systems. These latter two theorems establish that the qualitative behavior of the solution set of a nonlinear system of fractional differential equations near an equilibrium point is typically the same as the qualitative behavior of the solution set of the corresponding linearized system near the equilibrium point. Furthermore, we discuss the stability conditions for the equilibrium points of these systems. We point out that, the fractional derivative in these systems is in the Caputo sense.

Keywords: Fractional differential systems; Stable manifold theorem; Hartman-Grobman theorem; Local center manifold theorem; Local qualitative theorey.
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1. Introduction

In recent years, fractional differential equations have attracted increasing interest due to the fact that many mathematical problems in science and engineering can be modeled by fractional differential equations, see e.g. [16, 17, 19, 21, 24, 26, 29]. Although, several results on asymptotic behavior of fractional differential equations are already published (e.g. on stability theory [20], linear theory

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[16, 21, 24, 29], Lyapunov exponents [20], etc.), the development of a qualitative theory for fractional differential equations is still in its infancy.

This paper mainly deals with generalization three important results in the local qualitative theory of ordinary differential equations to differential equations of fractional order; i.e., the stable manifold theorem, the Hartman-Grobman theorem and the local center manifold theorem.

We know that any linear system

$$x^{(\alpha)} = Ax, \quad 0 < \alpha \leq 1, \tag{1.1}$$

has a unique solution in each point x_0 in $\mathbb{R}^n$, the solution is given by $x(t) = E_\alpha(At^\alpha)x_0$. In this paper, we would study nonlinear systems of fractional differential equations

$$x^{(\alpha)} = f(x), \quad 0 < \alpha \leq 1, \tag{1.2}$$

near a hyperbolic equilibrium point x_0, where $f : E \to \mathbb{R}^n$ and E is an open subset of $\mathbb{R}^n$. We represent the stable manifold theorem and the Hartman-Grobman theorem [28, 11] for nonlinear systems of fractional order which show that topologically the local behavior of the nonlinear system (1.2) near an equilibrium point x_0 where $f(x_0) = 0$ is typically determined by the behavior of the linear system (1.1) near the origin when the matrix $A = Df(x_0)$. The stability of any hyperbolic equilibrium point x_0 of (1.2) is determined. Finally, we represent the local center manifold theorem for fractional system (1.2), which shows that the qualitative behavior in a neighborhood of a nonhyperbolic critical point x_0 of the nonlinear system (1.2) with $x \in \mathbb{R}^n$ is determined by its behavior on the center manifold near x_0.

First we recall some preliminaries and notations regarding fractional calculus. For more details, see [4, 5, 7, 8, 14, 18, 25, 27, 31].

Definition 1.1. A function $f(x)$, $x > 0$, is said to be in the space C_α, $\alpha \in \mathbb{R}$, if there exists a real number $p(> \alpha)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it said to be in the space C_α^m, $m \in \mathbb{N} \cup \{0\}$, if and only if $f^{(m)}(x) \in C_\alpha$ [15].

Definition 1.2. The Riemann-Liouville integral operator of order α of $f(x) \in C_\alpha$, $\alpha \geq -1$ is defined as [15]

$$I_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi, & \alpha > 0, \ x > 0, \\ \\ f(x), & \alpha = 0. \end{cases}$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 1.3. The left sided Caputo fractional derivative of order α of $f(x) \in C_{-1}^m$, $m \in \mathbb{N} \cup \{0\}$ is defined as [15]

$$D_x^\alpha f(x) = f^{(\alpha)}(x) = \begin{cases} \left[I_x^{m-\alpha} f^{(m)}(x) \right], & m-1 < \alpha < m, \quad m \in \mathbb{N}, \\ \\ \dfrac{d^m}{dx^m} f(x), & \alpha = m. \end{cases}$$

Definition 1.4. The Mittag-Leffler function $E_{\alpha,\beta}(z)$ with $\alpha > 0$, $\beta > 0$ is defined by the following series representation, valid in the whole complex plane [24]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad z \in \mathbb{C}.$$

For $\beta = 1$, we obtain the Mittag-Leffler function in one parameter:

$$E_{\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} \equiv E_\alpha(z).$$

2. Fractional stable manifold theorem

The stable manifold theorem [6, 22] is one of the most important results in the local qualitative theory of ordinary differential equations. In fractional differential equations this theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system (1.2) has stable and unstable manifolds S and U tangent at x_0 to the stable and unstable subspaces E^s and E^u of the linearized system (1.1) where $A = Df(x_0)$ and $x^{(\alpha)}$ is fractional derivative in the sense of Caputo. Furthermore, S and U are of the same dimensions as E^s and E^u, and if ϕ_t is the flow of the nonlinear system (1.2), then S and U are positively and negatively invariant under ϕ_t respectively and satisfy

$$\lim_{t\to\infty} \phi_t(c) = x_0,$$

for all $c \in S$ and

$$\lim_{t\to-\infty} \phi_t(c) = x_0,$$

for all $c \in U$.

Definition 2.1. Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the fractional differential equation (1.2) with the initial condition

$$x(0) = x_0,$$

defined on its maximal interval of existence $I(x_0)$. Then, for $t \in I(x_0)$, the set of mappings ϕ_t defined by

$$\phi_t(x_0) = \phi(t, x_0),$$

is called the flow of the differential equation (1.2) or the flow defined by the differential equation (1.2), ϕ_t is also referred to as the flow of the vector field $f(x)$.

Definition 2.2. A point $x_0 \in \mathbb{R}^n$ is called an equilibrium point or critical point of (1.2) if $f(x_0) = 0$. An equilibrium point x_0 is called a hyperbolic equilibrium point of (1.2) if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part. The linear system (1.1) with the matrix $A = Df(x_0)$ is called the linerization of (1.2) at x_0.

Definition 2.3. An equilibrium point x_0 of (1.2) is called a sink if all of the eigenvalues of the matrix $Df(x_0)$ have negative real part; it is called a source if all of the eigenvalues of $Df(x_0)$ have positive real part; and it is called a saddle if it is a hyperbolic equilibrium point and $Df(x_0)$ has at least one eigenvalue with a positive real part and at least one with a negative real part.

Definition 2.4. Let X be a metric space and let A and B be subsets of X. A homeomorphism of A onto B is a continuous one-to-one map of A onto B, $h : A \to B$, such that $h^{-1} : B \to A$ is continuous. The sets A and B are called homeomorphic or topologically equivalent if there is a homeomorphism of A onto B. If we wish to emphasize that h maps A onto B, we write $h : A \to B$.

Definition 2.5. An n-dimensional differentiable manifold, M (or a manifold of class C^k_α), is a connected metric space with an open covering $\{U_\beta\}$, i.e., $M = \cup_\beta U_\beta$, such that
(i) for all β, U_β is homeomorphic to the open unit ball in $\mathbb{R}^n$, $B = \{x\in\mathbb{R}^n \mid \ |x| < 1\}$, i.e., for all θ there exists a homeomorphism of U_β onto B, $h_\beta : U_\beta \to B$, and
(ii) if $U_\beta \cap U_\theta \neq 0$ and $h_\beta : U_\beta \to B$, $h_\theta : U_\theta \to B$ are homeomorphisms, then $h_\beta(U_\beta\cap U_\theta)$ and $h_\theta(U_\beta\cap U_\theta)$ are subsets of $\mathbb{R}^n$ and the map

$$h = h_\beta \ o \ h_\theta : \ h_\theta(U_\beta\cap U_\theta) \to h_\beta(U_\beta\cap U_\theta),$$

is differentiable (or of class C^k_α) and for all $x\in h_\theta(U_\beta\cap U_\theta)$, the Jacobian determinant $\det Dh(x) \neq 0$. The manifold M is said to be analytic if the maps $h = h_\beta \ o \ h_\theta^{-1}$ are analytic.

Theorem 2.6. *(The stable manifold theorem for fractional differential systems). Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f\in C^1(E)$, and let ϕ_t be the flow of the nonlinear system* (1.2). *Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then, there exists a k-dimensional manifold S of class C^1_α tangent to the stable subspace E^s of the linear system* (1.1) *at* 0 *such that for all $t\geq 0$, $\phi_t(S)\subset S$ and for all $x_0\in S$*

$$\lim_{t\to\infty} \phi_t(x_0) = 0,$$

and there exists an $n-k$-dimensional manifold U of class C^1_α tangent to the unstable subspace E^u of (1.1) *at* 0 *such that for all $t\leq 0$, $\phi_t(U)\in U$ and for all $x_0\in U$,*

$$\lim_{t\to-\infty} \phi_t(x_0) = 0.$$

Proof . Before proving this theorem, we remark that if $f\in C^1(E)$ and $f(0) = 0$, then, the system (1.2) can be written as

$$x^{(\alpha)} = Ax + F(x), \ 0 < \alpha \leq 1, \tag{2.1}$$

where $A = Df(0)$, $F(x) = f(x) - Ax$, $F\in C^1(E)$, $F(0) = 0$ and $DF(0) = 0$. Then, given any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y\in N_\delta(0)$ we have

$$||F(x) - F(y)||\leq\epsilon||x - y||. \tag{2.2}$$

Furthermore, by the Jordan canonical form theorem, there is an $n\times n$ invertible matrix C such that

$$C^{-1}AC = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} = B,$$

where the eigenvalues $\lambda_1, \cdots, \lambda_k$ of the $k\times k$ matrix P have negative real part and the eigenvalues $\lambda_{k+1}, \cdots, \lambda_n$ of the $(n-k)\times(n-k)$ matrix Q have positive real part. We can choose $\beta > 0$ sufficiently small that for $j = 1, \cdots, k$

$$Re(\lambda_j) < -\beta < 0. \tag{2.3}$$

Letting $y = C^{-1}x$, the system (2.1) has the form

$$y^{(\alpha)} = By + G(y),\ 0 < \alpha \leq 1, \tag{2.4}$$

where $G(y) = C^{-1}F(Cy) \in C^1(\widehat{E})$, $\widehat{E} = C^{-1}(E)$ and G satisfies the Lipschitz-type condition above.

Consider the system (2.4). Let us denote

$$U(t) = \begin{pmatrix} E_\alpha(Pt^\alpha) & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{U}(t) = \begin{pmatrix} E_{\alpha,\alpha}(Pt^\alpha) & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$V(t) = \begin{pmatrix} 0 & 0 \\ 0 & E_\alpha(Qt^\alpha) \end{pmatrix}, \quad \tilde{V}(t) = \begin{pmatrix} 0 & 0 \\ 0 & E_{\alpha,\alpha}(Qt^\alpha) \end{pmatrix},$$

so that $E_\alpha(Bt^\alpha) = U(t) + V(t)$.

We can choose $k > 0$ sufficiently large and $\sigma > 0$ sufficiently small that

$$\|U(t)\| \leq kE_\alpha(-(\beta + \sigma)t^\alpha), \quad \|\tilde{U}(t)\| \leq kE_{\alpha,\alpha}(-(\beta + \sigma)t^\alpha), \quad \forall t \geq 0, \tag{2.5}$$

where $\beta > 0$ chosen as in (2.3), and

$$\|V(t)\| \leq kE_\alpha(\sigma t^\alpha), \quad \|\tilde{V}(t)\| \leq kE_{\alpha,\alpha}(\sigma t^\alpha), \quad \forall t \leq 0.$$

Now consider the integral equation

$$u(t, a) = U(t)a + \int_0^t (t-s)^{\alpha-1}\tilde{U}(t-s)G(u(s,a))ds$$
$$- \int_t^\infty (t-s)^{\alpha-1}\tilde{V}(t-s)G(u(s,a))ds. \tag{2.6}$$

If $u(t, a)$ is a continuous solution of this integral equation, then, it is a solution of the differential equation (2.4). Here is some intuition on why the particular integral equation in chosen. We basically want to remove the parts that blow up as $t \to \infty$. In general, the solution of this system satisfies

$$u(t, a) = \begin{pmatrix} E_\alpha(Pt^\alpha) & 0 \\ 0 & E_{\alpha,}(Qt^\alpha) \end{pmatrix} a$$
$$+ \int_0^t (t-s)^{\alpha-1} \begin{pmatrix} E_{\alpha,\alpha}(P(t-s)^\alpha) & 0 \\ 0 & E_{\alpha,\alpha}(Q(t-s)^\alpha) \end{pmatrix} G(u(s,a))ds.$$

Separate the convergent and non-convergent parts

$$u(t, a) = U(t)a + V(t)a + \int_0^t (t-s)^{\alpha-1}\tilde{U}(t-s)G(u(s,a))ds$$
$$+ \int_0^t (t-s)^{\alpha-1}\tilde{V}(t-s)G(u(s,a))ds$$
$$= U(t)a + V(t)a + \int_0^t (t-s)^{\alpha-1}\tilde{U}(t-s)G(u(s,a))ds$$
$$+ \int_0^\infty (t-s)^{\alpha-1}\tilde{V}(t-s)G(u(s,a))ds - \int_t^\infty (t-s)^{\alpha-1}\tilde{V}(t-s)G(u(s,a))ds.$$

Remove contributions that will cause it not to converge to the origin

$$u(t,a) = U(t)a + \int_0^t (t-s)^{\alpha-1}\tilde{U}(t-s)G(u(s,a))ds$$

$$-\int_t^\infty (t-s)^{\alpha-1}\tilde{V}(t-s)G(u(s,a))ds.$$

We now solve this integral equation by the method of successive approximation. Let

$$u^0(t,a) = 0,$$

$$u^{(j+1)}(t,a) = U(t)a + \int_0^t (t-s)^{\alpha-1}\tilde{U}(t-s)G(u^j(s,a))ds$$

$$-\int_t^\infty (t-s)^{\alpha-1}\tilde{V}(t-s)G(u^j(s,a))ds.$$

Since by assumption $G(0) = 0$, we have

$$||u^1(t,a) - u^0(t,a)|| = ||U(t)a|| \leq k||a||E_\alpha(-(\beta+\sigma)t^\alpha) \leq k||a||E_\alpha(-\beta t^\alpha),$$

and

$$||u^2(t,a) - u^1(t,a)|| \leq \int_0^t (t-s)^{\alpha-1}||\tilde{U}(t-s)||\ ||G(u^1(s,a)) - G(u^0(s,a))||ds$$

$$+\int_t^\infty (t-s)^{\alpha-1}||\tilde{V}(t-s)||\ ||G(u^1(s,a)) - G(u^0(s,a))||ds$$

$$\leq \epsilon k^2||a|| \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(\beta+\sigma)(t-s)^\alpha)E_\alpha(-\beta s^\alpha)ds$$

$$+\epsilon k^2||a|| \int_t^\infty (t-s)^{\alpha-1}E_{\alpha,\alpha}(\sigma(t-s)^\alpha)E_\alpha(-\beta s^\alpha)ds$$

$$\leq \frac{2\epsilon k^2||a||}{\sigma}E_\alpha(-\beta t^\alpha).$$

Let us assume that $\epsilon k/\sigma < 1/4$, then

$$||u^2(t,a) - u^1(t,a)|| \leq \frac{k||a||}{2}E_\alpha(-\beta t^\alpha).$$

Assume that the induction hypothesis

$$||u^{(j)}(t,a) - u^{(j-1)}(t,a)|| \leq \frac{k||a||E_\alpha(-\beta t^\alpha)}{2^{j-1}}, \tag{2.7}$$

holds for $j = 1, 2, \cdots, m$ and $t \in [0, \infty)$. Then, it follows that

$$||u^{(m+1)}(t,a) - u^{(m)}(t,a)|| \leq \int_0^t (t-s)^{\alpha-1}||\tilde{U}(t-s)||\epsilon||u^{(m)}(s,a) - u^{(m-1)}(s,a)||ds$$

$$+\int_t^\infty (t-s)^{\alpha-1}||\tilde{V}(t-s)||\epsilon||u^{(m)}(s,a) - u^{(m-1)}(s,a)||ds$$

$$\leq \epsilon \int_0^t (t-s)^{\alpha-1} k E_{\alpha,\alpha}(-(\beta+\sigma)(t-s)^{\alpha}) \frac{k||a||E_{\alpha}(-\beta s^{\alpha})}{2^{m-1}} ds$$
$$+ \epsilon \int_t^{\infty} (t-s)^{\alpha-1} k E_{\alpha,\alpha}(\sigma(t-s)^{\alpha}) \frac{k||a||E_{\alpha}(-\beta s^{\alpha})}{2^{m-1}} ds$$
$$\leq \frac{\epsilon k^2 ||a|| E_{\alpha}(-\beta t^{\alpha})}{\sigma 2^{m-1}} + \frac{\epsilon k^2 ||a|| E_{\alpha}(-\beta t^{\alpha})}{\sigma 2^{m-1}}$$
$$< (\frac{1}{4} + \frac{1}{4}) \frac{k||a||E_{\alpha}(-\beta t^{\alpha})}{2^{m-1}} = \frac{k||a||E_{\alpha}(-\beta t^{\alpha})}{2^{m}}. \tag{2.8}$$

If we choose $k||a|| < \delta/2$, then, the Lipschitz-type condition (2.2) holds for the function G and therefore, (2.7) holds for all $j = 1, 2, \cdots$ and $t \in [0, \infty)$. Thus, for $n > m \geq N$

$$||u^{(n)}(t,a) - u^{(m)}(t,a)|| \leq \sum_{j=m}^{n-1} ||u^{(j+1)}(t,a) - u^{(j)}(t,a)||$$

$$\leq \sum_{j=N}^{\infty} ||u^{(j+1)}(t,a) - u^{(j)}(t,a)|| \leq k||a|| \sum_{j=N}^{\infty} \frac{1}{2^j} = \frac{k||a||}{2^{N-1}}.$$

Hence, $u^{(j)}(t,a)$ is a Cauchy sequence of continuous functions for $t \in [0, \infty)$ and therefore, is uniformly convergent to $u(t,a)$. Then, it follows that $u(t,a)$ satisfies the integral equation (2.6) and hence, the differential equation (2.4), furthermore, by setting $m = 0$ in (2.8) we have

$$||u(t,a)|| \leq k||a|| E_{\alpha}(-\beta t^{\alpha}), \tag{2.9}$$

for $t \geq 0$ and hence, $u(t,a)$ converges to the origin as $t \to \infty$.

It is clear from (2.6) that the last $n-k$ components of the vector a do not enter the computation and hence, they may be taken as zero. Thus, the components $u_j(t,a)$ of the solution $u(t,a)$ satisfy the initial conditions

$$u_j(0,a) = a_j \ , j = 1, \cdots, k,$$

and

$$u_j(0,a) = -\int_0^{\infty} V(-s) G(u(s, a_1, \cdots, a_k, 0)) \ , j = k+1, \cdots, n.$$

For $j = k+1, \cdots, n$ we define the functions

$$\psi_j(a_1, \cdots, a_k) = u_j(0, a_1, \cdots, a_k, 0, \cdots, 0), \tag{2.10}$$

then, the initial values $y_j = u_j(0, a_1, \cdots, a_k, 0, \cdots, 0)$ satisfy

$$y_j = \psi_j(y_1, \cdots, y_k) \ , j = k+1, \cdots, n,$$

according to the definition (2.10). These equation then define a manifold $\widehat{S}$ of class C_{α}^1 for y sufficiently near the origin. The manifold S of class C_{α}^1 in x-space is then obtained from $\widehat{S}$ under the linear transformation on coordinates $x = Cy$.

The existence of the unstable manifold $\widehat{U}$ of (2.4) is established in exactly the same way by considering the system (2.4) with $t \to -t$, i.e.,

$$y^{(\alpha)} = -By - G(y), \quad 0 < \alpha \leq 1.$$

The stable manifold for this system will then be the unstable manifold $\widehat{U}$ for (2.4). This completes the proof of the stable manifold Theorem. $\square$

The stable and unstable manifolds S and U are only defined in a small neighborhood of the origin in the proof of the stable manifold theorem. S and U are therefore referred to as the local stable and unstable manifolds of (1.2) at the origin or simply as the local stable and unstable manifolds of he origin. We define the global stable and unstable manifolds of (1.2) at 0 by letting points in S flow backward in time and those in U flow forward n time.

Definition 2.7. Let ϕ_t be the flow of the nonlinear system (1.2). The global stable and unstable manifolds of (1.2) at 0 are defined by

$$W^s(0) = \cup_{t\leq 0}\phi_t(S),$$

and

$$W^u(0) = \cup_{t\geq 0}\phi_t(U),$$

respectively; $W^s(0)$ and $W^u(0)$ are also referred to as the global stable and unstable manifolds of the origin respectively. It can be shown that the global stable and unstable manifolds $W^s(0)$ and $W^u(0)$ are unique and that they are invariant with respect to the flow ϕ_t; furthermore, for all $x\in W^s(0)$, $\lim_{t\to\infty}\phi_t(x) = 0$ and for all $x\in W^u(0)$, $\lim_{t\to-\infty}\phi_t(x) = 0$.

As in the proof of the stable manifold theorem, it can be shown that in a small neighborhood, N, of a hyperbolic critical point at the origin, the local stable and unstable manifolds, S and U, of (1.2) at the origin are given by

$$S = \{x\in N \mid \phi_t(x)\to 0 \text{ as } t\to\infty \text{ and } \phi_t(x)\in N \text{ for } t\geq 0\},$$

and

$$U = \{x\in N \mid \phi_t(x)\to 0 \text{ as } t\to-\infty \text{ and } \phi_t(x)\in N \text{ for } t\leq 0\},$$

respectively [10].

Corollary 2.8. *Under the hypotheses of the stable manifold theorem, if S and U are the stable and unstable manifolds of* (1.2) *at the origin and if $Re(\lambda_j) < -\beta < 0 < \theta < Re(\lambda_m)$ for $j = 1,\cdots,k$ and $m = k+1,\cdots,n$, then given $\epsilon > 0$ there exists a $\delta > 0$ such that if $x_0\in N_\delta(0)\cap S$ then*

$$||\phi_t(x_0)|| \leq \epsilon E_\alpha(-\beta t^\alpha),$$

for all $t\geq 0$ and if $x_0\in N_\delta(0)\cup U$ then

$$||\phi_t(x_0)|| \leq \epsilon E_\alpha(\theta t^\alpha),$$

for all $t\leq 0$.

Proof . It follows from equation (2.9) in the proof of the stable manifold theorem that if $x(t)$ is a solution of the differential equation (2.4) with $x(0)\in S$, i.e., if $x(t) = Cy(t)$ with $y(0) = u(0,a)\in\tilde{S}$, then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x(0)| < \delta$ then

$$||x(t)|| \leq \epsilon E_\alpha(-\beta t^\alpha),$$

for all $t\geq 0$. Just as in the proof of the stable manifold theorem, β is any positive number that satisfies $Re(\lambda_j) < -\beta$ for $j = 1,\cdots,k$ where λ_j, $j = 1,\cdots,k$ are the eigenvalues of $Df(0)$ with negative real part. This result shows that solutions starting in S, sufficiently near the origin, approach the origin in terms of Mittag-Leffler function as $t\to-\infty$. $\square$

3. Fractional Hartman-Grobman theorem

The Hartman-Grobman theorem is another very important result in the local qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system (1.2) has the same qualitative structure as the linear system (1.1) with $A = Df(x_0)$. Throughout this section we shall assume that the equilibrium point x_0 has been translated to the origin.

Definition 3.1. Two autonomous systems of differential equations such as (1.2) and (1.1) are said to be topologically equivalent in a neighborhood of the origin or to have the same qualitative structure near the origin if there is a homeomorphism H mapping an open set U containing the origin onto an open set V containing the origin which maps trajectories of (1.2) in U onto trajectories of (1.1) in V and preserves their orientation by time in the sense that if a trajectory is directed from x_1 to x_2 in U, then its image is directed from $H(x_1)$ to $H(x_2)$ in V. If the homeomorphism H preserves the parameterization by time, then the systems (1.2) and (1.1) are said to be topologically conjugate in a neighborhood of the origin.

Theorem 3.2. *(The Hartman-Grobman theorem for fractional differential systems). Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (1.2). Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part. Then, there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in U$ and $t \in I_0$*

$$H \; o \; \phi_t(x_0) = E_\alpha(At^\alpha)H(x_0), \quad 0 < \alpha \leq 1.$$

Procedure of construction the homeomorphism for fractional Hartman-Grobman theorem:

Consider the nonlinear system (1.2) with $f \in C^1(E)$, $f(0) = 0$ and $A = Df(0)$.

1. Suppose that the matrix A is written in the form

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where the eigenvalues of P have negative real part and the eigenvalues of Q have positive real part.

2. Let ϕ_t be the flow of the nonlinear system (1.2) and write the solution

$$x(t, x_0) = \phi_t(x_0) = \begin{pmatrix} y(t, y_0, z_0) \\ z(t, y_0, z_0) \end{pmatrix},$$

where

$$x_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in \mathbb{R}^n,$$

$y_0 \in E^s$, the stable subspace of A and $z_0 \in E^u$, the unstable subspace of A.

3. Define the functions

$$Y(y_0, z_0) = y(1, y_0, z_0) - E_\alpha(P)y_0,$$

and

$$Z(y_0, z_0) = z(1, y_0, z_0) - E_\alpha(Q)z_0.$$

Let $B = E_\alpha(P) \equiv E_\alpha(P.1^\alpha)$ and $C = E_\alpha(Q) \equiv E_\alpha(Q.1^\alpha)$.
4. We construct the homeomorphism using the method of successive approximations. For $x \in \mathbb{R}^n$, let

$$H_0(x) = \begin{pmatrix} \Phi(y,z) \\ \Psi(y,z) \end{pmatrix}.$$

Then, $H_0 \; o \; \phi_{t=1}(x_0) = E_\alpha(A) H_0(x_0)$ is equivalent to the pair of equations

$$B\Phi(y,z) = \Phi(By + Y(y,z), Cz + Z(y,z)), \tag{3.1}$$

$$C\Psi(y,z) = \Psi(By + Y(y,z), Cz + Z(y,z)).$$

Define the successive approximations for the second equation by

$$\Psi_0(y,z) = z,$$

$$\Psi_{k+1}(y,z) = C^{-1}\Psi_k(By + Y(y,z), Cz + Z(y,z)).$$

Furthermore, the equation (3.1) can be written as

$$B^{-1}\Phi(y,z) = \Phi(B^{-1}y + \widehat{Y}(y,z), C^{-1}z + \widehat{Z}(y,z)). \tag{3.2}$$

Then, equation (3.2) can be solved for $\Phi(y,z)$ by the method of successive approximations with $\Phi_0(y,z) = y$. We therefore obtain

$$H_0(y,z) = \begin{pmatrix} \Phi(y,z) \\ \Psi(y,z) \end{pmatrix}.$$

In the case $\alpha = 1$, Hartman on [[12], pp. 248-249] showed that H_0 is a homeomorphism on $\mathbb{R}^n$. In the same manner, one can verify that for $0 < \alpha < 1$, H_0 is a homemorphism on $\mathbb{R}^n$.
5. Define

$$H = \int_0^1 E_\alpha(-As^\alpha) \; H_0 \; \phi_s ds,$$

then, H satisfies

$$H \; o \; \phi_t(x_0) = E_\alpha(At^\alpha) H(x_0),$$

and it can be shown that H is a homeomorphism on $\mathbb{R}^n$; cf. [[12], pp. 250-251]. This completes the procedure of construction the homeomorphism for fractional Hartman-Grobman theorem.

The above procedure for the Hartman-Grobman theorem is a generalization proof of P. Hartman; cf. [[12], pp. 244-251]. It was proved independently by P. Hartman [12] and the Russian mathematician D.M. Grobman [9] in 1959. This theorem with f, H and H^{-1} analytic was proved by H. Poincare in 1879, cf. [23], under the assumptions that the elementary divisors of A (cf. [**3**, p. 219]) are simple and that the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A lie in a half plane in C and satisfy

$$\lambda_j \neq m_1\lambda_1 + \cdots + m_n\lambda_n, \tag{3.3}$$

for all sets of non-negative integers $(m_1, \cdots, m_n)$ satisfying $m_1 + \cdots + m_n > 1$. An analogous result for smooth f, H and H^{-1} can be established and proved that there exists a map H of class C^1_α with an inverse H^{-1} of class C^1_α (i.e., a C^1_α-difeomorphism) satisfying the conclusions of the above theorem even without the Diophantine conditions (3.3) on λ_j:

Theorem 3.3. (The Hartman theorem for fractional differential systems). Let E be an open subset of $\mathbb{R}^n$ containing the point x_0, let $f \in C^2(E)$, and let ϕ_t be the flow of the nonlinear system (1.2). Suppose that $f(x_0) = 0$ and that all of the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the matrix $A = Df(x_0)$ have negative (or positive) real part. Then there exists a C^1_α-difeomorphism H of a neighborhood U of xo onto an open set V containing the origin such that for each $x \in U$ there is an open interval $I(x) \subset \mathbb{R}$ containing zero such that for all $x \in U$ and $t \in I(x)$

$$H\ o\ \phi_t(x) = E_\alpha(At^\alpha)H(x), \quad 0 < \alpha \leq 1.$$

4. Stability conditions

In this section we discuss the stability of the equilibrium points of the nonlinear system (1.2). The stability of any hyperbolic equilibrium point x_0 of (1.2) is determined by the signs of the real parts of the eigenvalues λ_j of the matrix $Df(x_0)$. A hyperbolic equilibrium point x_0 is asymptotically stable if $Re(\lambda_j) < 0$ for $j = 1, \cdots, n$; i.e., if x_0 is a sink. And a hyperbolic equilibrium point x_0 is unstable if it is either a source or a saddle.

Definition 4.1. Let ϕ_t denote the flow of the differential equation (1.2) defined for all $t \in \mathbb{R}$. An equilibrium point x_0 of (1.2) is stable if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ and $t > 0$ we have

$$\phi_t(x) \in N_\epsilon(x_0).$$

The equilibrium point x_0 is unstable if it is not stable. And x_0 is asymptotically stable if it is stable and if there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ we have

$$\lim_{t \to \infty} \phi_t(x) = x_0.$$

It follows from the stable manifold theorem and the Hartman-Grobman theorem that any sink of (1.2) is asymptotically stable and any source or saddle of (1.2) is unstable. Hence, any hyperbolic equilibrium point of (1.2) is either asymptotically stable or unstable. The corollary in Section 2 provides even more information concerning the local behavior of solutions near a sink:

Theorem 4.2. *If x_0 is a sink of the nonlinear system (1.2) and $Re(\lambda_j) < -\beta < 0$ for all of the eigenvalues λ_j of the matrix $Df(x_0)$, then, given $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$, the flow $\phi_t(x)$ of (1.2) satisfies*

$$||\phi_t(x) - x_0|| \leq \epsilon E_\alpha(-\beta t^\alpha), \ 0 < \alpha \leq 1, \tag{4.1}$$

for all $t \geq 0$.

Proof . Inserting $x - x_0 = a$ and $\phi_t(x) - x_0 = u(t, a)$ and by notice to this fact that, all the eigenvalues λ_j have negative real parts, it follows from (2.6) that

$$u(t, a) = U(t)a + \int_0^t (t-s)^{\alpha-1}\tilde{U}(t-s)G(u(s, a))ds,$$

and hence

$$||u(t, a)|| \leq ||U(t)||\ ||a|| + \left|\left| \int_0^t (t-s)^{\alpha-1}\tilde{U}(t-s)\ G(u(s, a))ds \right|\right|$$

$$\leq ||U(t)||\ ||a|| + \epsilon \int_0^t (t-s)^{\alpha-1} ||\tilde{U}(t-s)||\ ||u(s,a)|| ds. \tag{4.2}$$

Now, we set $A(t) = ||u(t,a)||$ and $C = ||U(t)||\ ||a||$. Then, (4.2) can be written as

$$A(t) \leq C + \epsilon \int_0^t (t-s)^{\alpha-1} ||\tilde{U}(t-s)||\ A(s) ds. \tag{4.3}$$

Multiplying (4.3) by $E_\alpha(-\epsilon t^\alpha)$, we find that

$$D_t^\alpha \left[E_\alpha(-\epsilon t^\alpha) \left(\int_0^t (t-s)^{\alpha-1} ||\tilde{U}(t-s)||\ A(s) ds) \right) \right] \leq C E_\alpha(-\epsilon t^\alpha). \tag{4.4}$$

Integrating I_t^α of the inequality (4.4), we deduce that

$$E_\alpha(-\epsilon t^\alpha) \left(\int_0^t (t-s)^{\alpha-1} ||\tilde{U}(t-s)||\ A(s) ds \right) \leq \frac{C}{\epsilon} (1 - E_\alpha(-\epsilon t^\alpha)), \tag{4.5}$$

that is

$$\epsilon \left(\int_0^t (t-s)^{\alpha-1} ||\tilde{U}(t-s)||\ A(s) ds \right) \leq C (E_\alpha(\epsilon t^\alpha) - 1). \tag{4.6}$$

Now, substituting (4.6) into (4.3) gives the following estimate

$$A(t) \leq C E_\alpha(\epsilon t^\alpha). \tag{4.7}$$

Returning to our original notation, we conclude from (2.5) and (4.7) that

$$||\phi_t(x) - x_0|| \leq E_\alpha(-\beta t^\alpha) ||a|| E_\alpha(\epsilon t^\alpha). \tag{4.8}$$

Thus, given $\epsilon > 0$, it is sufficient to choose $\delta = \dfrac{\epsilon}{E_\alpha(\epsilon t^\alpha)}$ to deduce (4.1). $\square$

Since hyperbolic equilibrium points are either asymptotically stable or unstable, the only time that an equilibrium point x_0 of (1.2) can be stable but not asymptotically stable is when $Df(x_0)$ has a zero eigenvalue or a pair of complex-conjugate, pure-imaginary eigenvalues $\lambda = \pm ib$. It follows from the next theorem, proved in [13], that all other eigenvalues λ_j of $Df(x_0)$ must satisfy $Re(\lambda_j) \leq 0$ if x_0 is stable.

Theorem 4.3. If x_0 is a stable equilibrium point of (1.2), no eigenvalue of $Df(x_0)$ has positive real part.

We see that stable equilibrium points which are not asymptotically stable can only occur at nonhyperbolic equilibrium points.

5. Fractional Center Manifold theorem

In the Section 3 we presented the Hartman-Grobman theorem, which showed that, in a neighborhood of a hyperbolic critical point $x_0 \in E$, the nonlinear system (1.2) is topologically conjugate to the linear system (1.1) with $A = Df(x_0)$, in a neighborhood of the origin. The Hartman-Grobman theorem therefore completely solves the problem of determining the stability and qualitative behavior in a neighborhood of a hyperbolic critical point of a nonlinear system. In the section 4, we gave some results for determining the stability and qualitative behavior in a neighborhood of a hyperbolic critical point of the nonlinear system (1.2). In this section, we present the local center manifold theorem, which shows that the qualitative behavior in a neighborhood of a nonhyperbolic critical point x_0 of the nonlinear system (1.2) with $x \in \mathbb{R}^n$ is determined by its behavior on the center manifold near x_0. Since the center manifold is generally of smaller dimension than the system (1.2), this simplifies the problem of determining the stability and qualitative behavior of the flow near a nonhyperbolic critical point of (1.2).

Theorem 5.1. *(The fractional local center manifold theorem). Let $f \in C^r(E)$, where E is an open subset of $\mathbb{R}^n$ containing the origin and $r \geq 1$. Suppose that $f(0) = 0$ and that $Df(0)$ has c eigenvalues with zero real parts and s eigenvalues with negative real parts, where $c + s = n$. The system* (1.2) *then can be written in diagonal form*

$$\begin{aligned} x^{(\alpha)} &= Cx + F(x,y), \\ y^{(\alpha)} &= Py + G(x,y), \quad 0 < \alpha \leq 1, \end{aligned} \tag{5.1}$$

where $(x,y) \in \mathbb{R}^c \times \mathbb{R}^s$, C is a square matrix with c eigenvalues having zero real parts, P is a square matrix with s eigenvalues with negative mat parts, and $F(0) = G(0) = 0$, $DF(0) = DG(0) = 0$; furthermore, there exists a $\delta > 0$ and a function $h \in C^r_\alpha(N_\delta(0))$ that defines the local center manifold

$$W^c_{loc}(0) = \{(x,y) \in \mathbb{R}^c \times \mathbb{R}^s \mid y = h(x) \text{ for } |x| < \delta\},$$

and satisfies

$$D^\alpha h(x)[Cx + F(x, h(x))] - Ph(x) - G(x, h(x)) = 0,$$

for $|x| < \delta$; and the flow on the center manifold $W^c(0)$ is defined by the system of differential equations

$$x^{(\alpha)} = Cx + F(x, h(x)),$$

for all $x \in \mathbb{R}^c$ with $|x| < \delta$.

Proof . Let $\psi : \mathbb{R}^n \to [0,1]$ be a C^∞ function with $\psi(x) = 1$ when $|x| \leq 1$ and $\psi(x) = 0$ when $|x| \geq 2$. For $\epsilon > 0$ define $\widehat{F}$ and $\widehat{G}$ by

$$\widehat{F}(x,y) = F(x\psi(\frac{x}{\epsilon}), y), \quad \widehat{G}(x,y) = G(x\psi(\frac{x}{\epsilon}), y).$$

The reason that the cut-off function ψ is only a function of x is that the proof of the existence of a center manifold generalizes in an obvious way to infinite dimensional problems.

We prove that the system

$$\begin{aligned} x^{(\alpha)} &= Cx + \widehat{F}(x,y), \\ y^{(\alpha)} &= Py + \widehat{G}(x,y), \quad 0 < \alpha \leq 1, \end{aligned} \tag{5.2}$$

has a center manifold $y = h(x)$, $x \in \mathbb{R}^n$, for small enough ϵ. Since $\widehat{F}$ and $\widehat{G}$ agree with F and G in a neighborhood of the origin, this proves the existence of a local center manifold for (5.1).

For $p > 0$ and $p_1 > 0$ let X be the set of Lipschitz functions $h : \mathbb{R}^n \to \mathbb{R}^m$ with Lipschitz constant p_1, $|h(x)| \leq p$ for $x \in \mathbb{R}^n$ and $h(0) = 0$. With the supremum norm $|| \cdot ||$, X is a complete space.

For $h \in X$ and $x_0 \in \mathbb{R}^n$, let $x(t, x_0, h)$ be the solution of

$$x^{(\alpha)} = Cx + \widehat{F}(x, h(x)), \quad x(0, x_0, h) = x_0. \tag{5.3}$$

The bounds on $\widehat{F}$ and h ensure that the solution of (5.3) exists for all t. We now define a new function Th by

$$(Th)(x_0) = \int_{-\infty}^{0} E_\alpha(-P\tau^\alpha) G(x(\tau, x_0, h), h(x(\tau, x_0, h)))d\tau. \tag{5.4}$$

If h is a fixed point of (5.4) then h is a center manifold for (5.2). We prove that for p, p_1 and ϵ small enough, T is a contraction on X. We recall that a contraction T on X is a function T from X to itself, with the property that there is some non-negative real number $0 \leq \kappa < 1$ such that for all x and y in X, $||T(x) - T(y)|| \leq \kappa \; ||x - y||$. The smallest such value of κ is called the Lipschitz constant of T. A contraction has at most one fixed point. Moreover, the Banach fixed point theorem states that every contraction on a nonempty complete metric space has a unique fixed point.

Using the definitions of $\widehat{F}$ and $\widehat{G}$, there is a continuous function $k(\epsilon)$ with $k(0) = 0$ such that

$$\begin{aligned} &|\widehat{F}(x, y)| + |\widehat{G}(x, y)| \leq \epsilon k(\epsilon), \\ &|\widehat{F}(x, y) - \widehat{F}(x', y')| \leq k(\epsilon)[|x - x'| + |y - y'|], \\ &|\widehat{G}(x, y) - \widehat{G}(x', y')| \leq k(\epsilon)[|x - x'| + |y - y'|], \end{aligned} \tag{5.5}$$

for all $x, x' \in \mathbb{R}^n$ and all $y, y' \in \mathbb{R}^m$ with $|y|, \; |y'| < \epsilon$.

Since the eigenvalues of P all have negative real parts, there exist positive constants β, θ such that for $s \leq 0$ and $y \in \mathbb{R}^m$

$$|E_\alpha(-P\tau^\alpha)y| \leq \theta E_\alpha(\beta\tau^\alpha)|y|. \tag{5.6}$$

Since the eigenvalues of C all have zero real parts, for each $r > 0$ there is a constant $M(r)$ such that for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$

$$|E_\alpha(C\tau^\alpha)x| \leq M(r)E_\alpha(r|\tau|^\alpha)|x|. \tag{5.7}$$

Note that in general, $M(r) \to \infty$ as $r \to 0$.

If $p < \epsilon$, then we can use (5.5) to estimate terms involving $G(x(s, x_0, h), h(x(s, x_0$
$, h)))$ and similar terms. We shall suppose that $p < \epsilon$ from now on.

If $x_0 \in \mathbb{R}^n$, then using (5.6) and the estimates on $\widehat{G}$ and h, we have from (5.4) that

$$|Th(x_0)| \leq \theta\beta^{-1}\epsilon k(\epsilon). \tag{5.8}$$

Now let $x_0, x_1 \in \mathbb{R}^n$. Using (5.7) and the estimates on $\widehat{F}$ and h, we have from (5.3) that for $r > 0$ and $t \leq 0$

$$\begin{aligned} |x(t, x_0, h) - x(t, x_1, h)| &\leq M(r)E_\alpha(-rt^\alpha)|x_0 - x_1| \\ &+ (1 + p_1)M(r)k(\epsilon) \int_t^0 (\tau - t)^{\alpha-1} E_\alpha(r(\tau - t)^\alpha)|x(\tau, x_0, h) - x(\tau, x_1, h)|d\tau. \end{aligned}$$

By fractional Gronwall's inequality [30], for $t \leq 0$

$$|x(t, x_0, h) - x(t, x_1, h)| \leq M(r)|x_1 - x_0|E_\alpha(-\gamma t^\alpha), \tag{5.9}$$

where $\gamma = r + (1 + p_1)M(r)k(\epsilon)$. Using (5.9) and the bounds on $\widehat{G}$ and h, it obtain from (5.4) that

$$|Th(x_0) - Th(x_1)| \leq \theta(M(r) + p_1)k(\epsilon)(\beta - \gamma)^{-1}|x_0 - x_1|, \tag{5.10}$$

if ϵ and r are small enough so that $\beta > \gamma$.

Similarly, if $h_1, h_2 \in X$ and $x_0 \in \mathbb{R}^n$, we obtain

$$|Th_1(x_0) - Th_2(x_0)| \leq \theta k(\epsilon)[\beta^{-1} + (1+p_1)M(r)k(\epsilon)r^{-1}(\beta-\gamma)^{-1}].||h_1 - h_2||. \tag{5.11}$$

By a suitable choice of p, p_1, ϵ and γ, we see from (5.8), (5.10) and (5.11) that T is a contraction on X. This proves the existence of a Lipschitz center manifold for (5.2). To prove that h is C^1_α we show that T is a contraction on a subset of X consisting of Lipschitz differentiable functions. The details are straightforward so we omit them. To prove that h is C^r_α, we imitate the proof of Theorem 4.2 on page 333 of [3]. □

The next theorem, which is a special case of the theorem proved by Carr in [1], is analogous to the Hartman-Grobman theorem except that, in order to determine completely the qualitative behavior of the flow near a nonhyperbolic critical point, one must be able to determine the qualitative behavior of the flow on the center manifold, which is determined by the first system of differential equations in the following theorem.

Theorem 5.2. Let E be an open subset of $\mathbb{R}^n$ containing the origin, and let $f \in C^1(E)$; suppose that $f(0) = 0$ and that the $n \times n$ matrix $Df(0) = \mathrm{diag}[C, P, Q]$, where the square matrix C has c eigenvalues with zero real parts, the square matrix P has s eigenvalues with negative real parts, and the square matrix Q has it eigenvalues with positive real parts. Then there exists C^1_α functions $h_1(x)$ and $h_2(x)$ satisfying

$$\begin{aligned} &D^\alpha h_1(x)[Cx + F(x, h_1(x), h_2(x)] - Ph_1(x) - G(x, h_1(x), h_2(x)) = 0,\\ &D^\alpha h_2(x)[Cx + F(x, h_1(x), h_2(x)] - Qh_1(x) - H(x, h_1(x), h_2(x)) = 0, \end{aligned}$$

in a neighborhood of the origin such that the nonlinear system (1.2), which can be written in the form

$$\begin{aligned} x^{(\alpha)} &= Cx + F(x, y, z),\\ y^{(\alpha)} &= Py + G(x, y, z), \quad 0 < \alpha \leq 1,\\ z^{(\alpha)} &= Qz + H(x, y, z), \end{aligned}$$

is topologically conjugate to the C^1_α system

$$\begin{aligned} x^{(\alpha)} &= Cx + F(x, h_1(x), h_2(x)),\\ y^{(\alpha)} &= Py, \quad 0 < \alpha \leq 1,\\ z^{(\alpha)} &= Qz, \end{aligned}$$

for $(x, y, z) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u$ in a neighborhood of the origin.

We add one final result to this paper which establishes the existence of an invariant center manifold $W^c(0)$ tangent to E^c at 0. The next theorem follows from the local center manifold theorem, Theorem 5.2, and the stable manifold theorem in section 2.

Theorem 5.3. (The fractional center manifold theorem). Let $f \in C^r(E)$ where E is an open subset of $\mathbb{R}^n$ containing the origin and $r \geq 1$. Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part, j eigenvalues with positive real part, and $m = n-k-j$ eigenvalues with zero real part. Then there exists an m-dimensional center manifold $W^c(0)$ of class C^r_α tangent to the center subspace E^c of (1.1) at 0, there exists a k-dimensional stable manifold $W^s(0)$ of class C^r_α tangent to the stable subspace E^s of (1.1) at 0 and there exists a j-dimensional unstable manifold $W^u(0)$ of class C^r_α tangent to the unstable subspace E^u of (1.1) at 0; furthermore, $W^c(0)$, $W^s(0)$ and $W^u(0)$ are invariant under the flow ϕ_t of (1.2).

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