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The outbreak disease due to the contamination environment and effect on dynamical behavior of prey-predator model

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Abstract

In this study, a predator-prey model (PPMD) was formulated and studied along with infectious on populations of prey and predator, since each one is splited into two sub-populations i.e., healthy and infected. It is presumed that only healthy predator of ability to predate the healthy prey and consume both healthy and prey being infected. Mathematically, the model solutions uniqueness, existence, and bounded-ness are conversed. All probable equilibrium model points are defined. The stability analyses as local and the regions of worldwide stability of each point of equilibriums are inspected. Lastly, few simulations as numerical were offered for validation the geted results theoretically.

Keywords: Prey-Predator model, Stability, ecology, Disease.
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1. Introduction

The study of mathematical models that combine the prey-predator systems and the spread of infectious diseases are greatly important to many of the animal populations as well as fishing operations. Currently such studies are creating a new study field recognized as eco-epidemiology. With trade and economy globalization, pollution of environment progressively turns out to be the mostly severe issue facing by all world countries. For example, the acid rain, ozone depSupposeion, and greenhouse effect are the environmental air pollution effects. Another issue in which the world as w-hole is concerning is what is the way for protecting species in danger of extinction. Global concern is

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increasing over natural and anthropogenic environmental toxins effects on health of ecosystem, such as numerous countries are suffering and still from the parasitic diseases spreading in both animals and humans [1].

Truly, many factors are there which impacts the prey-predator dynamics associations i.e., harvesting, disease, delay, prey refuge, environmental pollution, predation, and lack of food, fishing, etc. besides the infectious diseases spreading between the preys alone, predators, or both. Thus, the high interest support researchers to invistigate the diseases spread affect at such type for of study called as eco-epidemiological. Mathematical modeling often is process as evolving. Mathematical systematic analysis can frequently lead to better bio-economic models understanding. Differential formulas system has, to a definite degree, described successfully the associations among species. Huge literature exist documenting mathematical and ecological results offered by the model.

Despite the theory of predator-prey has been greatly developed, several long standing ecological and mathematical issues stay open. In addition, infectious diseases became as vital regulating aspect for animal and human populations sizes. Especially, for ecosystems of prey-predator, infectious diseases combined with prey-predator association to yield a complex joint effect as prey and predator sizes regulators. There is several ecological prey-predator systems studies alongwith disease. This factor (Disease) therefore, was invited to the attention of veterinary medicine and the provision of vaccines for these diseases. In following years, several workers invistigated the models of environmente along with prey being infected and the reports that relate focusing on subject [2].

Muhseen and Aaid [3] formulated (PPMD) with Epidemic Disease as SIS in involvment of predator Holling kind 2 of functional response. In 1994, Venturino [4] deliberated diseases influence on systems of Lotka-Volterra. In the following time, numerous workers are suggested and invistigated diverse PPMDs with disease spreading in population of prey. Das [5] was invistigated a PPMD with disease in predator. Moreover, several investigations regarding PPMD with disease were done in population of predator. Kuang and Beretta [6] regarded the ratiodependent prey-predator system solutions global behaviors. Chattopadhyay and Orino [7] analyzed and suggested a 3 dimensional PPMD with disease just in population of prey. Chen and You [8] were invistigated the extinction, permanence, and periodic periodic predatorprey system solution along with BeddingtonDeAngelis functional responsing and structure of stage for prey. They got a necessary and sufficient conditions set that guarantee the system permanent. Numbers of workers in the previous decade were discussed systems of simple multi-species comprising 3 trophic levels of food chain [9]. Kesh et al. [10] analyzed and suggested a model as mathematical of one predator and 2 contesting prey species in which prey species are following dynamics of Lotka-Volterra and predator functions uptake are ratio dependent. Hsu et al. [11] were invistigated the 3 trophic levels food chain with ratio-dependent as functional response MechaelisMenten type and its applications to be controlled biologically. Gakkhar and Naji [12] studied a 3 species ratio-dependent food chain. PPMD is a vital tool in ecology being mathematical and precisely for our biological phenomena understanding. Actually, several factors are there that impacting the prey-predator dynamics associations i.e., harvesting, disease, delay, prey refuge, etc.

In the current work, a mathematical model was proposed and analyzed describing PPMD with liable predator (prey) and prey being infected (predator) and the transmitted disease among them. The modified model as world and local stability analysis are analytically and numerically investigated.

2. Formulation of model

Regard model as ecological containing two levels, in the 1 st level the preys is splitted in to 2 classes namely, susceptible and infected, that are signified to their sizes of population at time t by

$X_1(t)$ and $X_2(t)$, correspondingly. Furthermore, in the 2 nd level, the predator is splitted in to two classes namely, infected and susceptible, here $Y_1(t)$ and $Y_2(t)$ signify the density of population at time t for the predator as susceptible and infected, correspondingly. At this point, for the purpose of formulating mathematically the foregoing model, assumptions as follow are taking into account:

1. In the predation absence, the 2 preys that contesting at the 1st level logistically grow with an intrinsic rates growth $r > 0, M > 0$ and loading capacities $K > 0, e > 0$ for $X_1(t)$ and $X_2(t)$ correspondingly. Nevertheless, they are contesting each other with competition rates intensity $\beta > 0$ and $\beta_1 > 0$ correspondingly.

2. In predator existence case in the 2^{nd} level, it is presumed that the predator is splitted into 2 compartments namely helthy predator $Y_1(t)$ and disease predator $Y_2(t)$. The Healthy predator consuming the 1^{st} and 2^{nd} preys based on Lotka-Volterra functional response type along with maximum rates of attack $> 0, \beta_1 > 0$ and $\gamma_1 > 0$ then the food is taken up via the predator with take up rates $0 < e_1 < 1$ and $0 < e_2 < 1$ correspondingly. Furthermore, the disease predator can not attack any Healthy preys, so that it feeds on the disease preys.

Lastly, predators (both healthy and disease predator) are exponentially decay along with normal death rates and correspondingly in their food absence.

Based on such expectations the foregoing defined food web system dynamics can be mathematically formulated along with differential formulas set as follow:

$$
\begin{aligned}
\frac{dX_1}{dT} &= rX_1\left(1-\frac{X_1}{k}\right) - M(1-e)X_1E - \beta X_1Y_1 \\
\frac{dX_2}{dT} &= M(1-e)X_1E - \beta_1(1-\psi)X_2Y_1 - (\mu_1+\gamma_1)X_2 \\
\frac{dY_1}{dT} &= e_1\beta X_1Y_1 + e_2\psi\beta_1X_2Y_1 - \mu_1Y_1 \\
\frac{dY_2}{dT} &= e_2(1-\psi)\beta_1X_2Y_1 - (\mu_1+\gamma_2)Y_2 \\
\frac{dE}{dT} &= \theta(1-\epsilon) - \alpha E
\end{aligned}
\tag{2.1}
$$

Subject to the initial conditions with $X_1(0) \geq 0, X_2(0) \geq 0, Y_1(0) \geq 0, Y_2(0) \geq 0, E(0) \geq 0$

3. Mathematical analysis

3.1. Boundedness of the solution.

As long as whole parameters are non-negative as well as the association functions are continuously differentiable the system right hand side (2.1) is variables smooth function (X_1, X_2, Y_1, Y_2, E) in the +ve octant,

$$\Omega = \{((X_1, X_2, Y_1, Y_2, E)) \mid X_1 \geq 0, X_2 \geq 0, Y_1 \geq 0, Y_2 \geq 0, E \geq 0\}$$

Furthermore, it is easy to prove that Ω is an invariant set. In addition, it is easy to prove that, whole functions of association are globally Lipschitz and then the system (2.1) of solution being unique. At this point, we will verify the system boundedness (2.1).

Theorem 3.1. *All system (1) solutions that initiate in $\Re^5_+$ are bounded uniformly.*

Proof . Suppose $(X_1(t), X_2(t), Y_1(t), Y_2(t), E(t))$ is any system (2.1) solution along with non-negative primary condition $(X_1(0), X_2(0), Y_1(0), Y_2(0), E(0))$, , From the 1st formula, we get as $t \to \infty$

$$\sup\left[rX_1\left(1-\frac{x_1}{k}\right)\right] \leq \frac{rk}{4} \tag{3.1}$$

Suppose $N(t) = X_1(t) + X_2(t) + Y_1(t) + Y_2(t)$, then from the model we get

$$\frac{dN}{dt} = rX_1\left(1-\frac{X_1}{k}\right) - (\mu_1+\gamma_1)\,X_2 - \mu_1 Y_1 - (\mu_1+\gamma_2)\,Y_2$$

Assuming a +ve constant $q > 0$ and $q = \min\{1, \mu_1+\gamma_1, \mu_1, \mu_1+\gamma_2\}, e_2+\psi < 1$ we get

$$\frac{dN}{dt} + qN \leq H\left(=\frac{rk}{4} + X_1\right) \tag{3.2}$$

At this point via utilizing Gronweall Lemma it gets that

$$N(t) \leq \frac{H}{q} + \left(N_0 - \frac{H}{q}\right)e^{-qt} \tag{3.3}$$

Therefore, $N(t) \leq \frac{\mathrm{H}}{q}$, as $t \to \infty$. At this point from the last system (2.1) formula we get

$$\frac{dE}{dt} = \theta(1-\varepsilon) - \alpha E$$

Then $\frac{dE}{dt} + \alpha E \leq \widetilde{H}(=\theta(1-\epsilon))$
By similar way as above we get:

$$E(t) \leq \frac{\theta}{\alpha}\cdot\frac{H}{q},\ \text{ as } t \to \infty \tag{3.4}$$

Thus, all system (2.1) solution which initiate in $\Re_+^5$ are regionaly confined

$$\Omega = \left\{(X_1, X_2, Y_1, Y_2, E) \in \Re_+^5 : N \leq \frac{H}{q}, 0 \leq E \leq \frac{\theta}{\alpha}\cdot\frac{H}{q}\right\} \tag{3.5}$$

Therefore, such solutions are bounded uniformly and the evidence is complete. □

3.2. . point of equilibriums existence.

It is easy to verify that the system (2.1) has at most five points of biological feasible equilibrium. The existence conditions of each of them along with their local stability analyses are discussed as following:

1. The vanishing equilibrium point $E_0 = (0,0,0,0)$ always exists, where $\theta = 0$.

2. The axial point of equilibrium $E_1 = \left(\check{X}_1, 0, 0, 0\right)$, since $\check{X}_1 = k, \theta = 0$.

3. The 1st 2 species point of equilibrium $\mathrm{E}_2 = (\dddot{\mathrm{X}}_1, \dddot{\mathrm{X}}_2, 0, \ddot{\mathrm{E}})$,
since $\dddot{\mathrm{E}} = \frac{\theta(1-\epsilon)}{\alpha}, \dddot{X}_1 = k\left[1 - \frac{M\theta(1-e)(1-\epsilon)}{r\alpha}\right] = u_1, \dddot{X}_2 = \frac{ru_1(k-u_1)}{k(\mu_1+\gamma_1)}$ exists, provided

$$u_1 < k \tag{3.6}$$

4. The 2nd two species point of equilibrium $E_3 = \left(\hat{X}_1, 0, \hat{Y}_1, 0\right)$, since $\hat{X}_1 = \frac{\mu_1}{e_1\beta}$, $\hat{Y}_1 = \frac{r\left(ke_{1\beta}-\mu_1\right)}{e_1\beta^2 k}$ exists provided

$$\mu_1 < ke_{1\beta} \tag{3.7}$$

5. Finally the+ve point of equilibrium $E_4 = \left(\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{E}\right)$, since $\bar{E} = \frac{\theta(1-\epsilon)}{\alpha}, \bar{X}_2 = \frac{\mu_1 - e_1\beta\bar{X}_1}{e_2\psi\beta_1}$, $\bar{Y}_1 = \frac{r\alpha\left(k-\bar{X}_1\right)-kM\theta(1-e)(1-\epsilon)}{\alpha k\beta}$ provided that the following conditions hold

$$M\theta(1-e)(1-\epsilon) < r\alpha \tag{3.8}$$

$$\bar{X}_1 < k, e_1\beta\bar{X}_1 < \mu_1 \tag{3.9}$$

While $\bar{X}_1$, signifies +ve root of the quadratic following formula

$$A_1\bar{X}_1^2 + A_2\bar{X}_1 + A_3 = 0 \tag{3.10}$$

Here

$$A_1 = -\frac{re_1(1-\psi)}{e_2\psi k}$$

$$A_2 = \frac{r\mu_1(1-\psi)}{e_2\psi k\beta} + \frac{e_1\beta\left(\mu_1+\gamma_1\right)}{e_2\psi\beta_1} + \frac{M\theta(1-e)(1-\psi)}{\alpha} + \frac{e_1\beta(1-\psi)}{e_2\psi}\left[\frac{r}{\beta} - \frac{M\theta(1-e)(1-\epsilon)}{\alpha\beta}\right]$$

$$A_3 = \frac{\mu_1(1-\psi)}{e_2\psi}\left[\frac{M\theta(1-e)(1-\epsilon)}{\alpha\beta} - \frac{r}{\beta}\right] - \frac{\mu_1\left(\mu_1+\gamma_1\right)}{e_2\psi\beta_1}$$

Obviously, E_4 uniquely exists in interior of $XY-$ plane when $A_3 > 0$.

4. Local stability analysis:

At this part, the local equilibrium system (2.1) points stability is established utilizing the method of linearization. It is simple to prove that the variational system (2.1) matrix, at the general point (X_1, X_2, Y_1, E), can be stated as $J = (a_{ij})_{4\times4}\,; i, j = 1, 2, 3, 4$, since

$$J = \begin{bmatrix} r\left(1-\frac{2X_1}{k}\right) - M(1-e)E - \beta Y_1 & 0 & -\beta X_1 & -M(1-e)X_1 \\ M(1-e)E & -\beta(1-\psi)Y_1 - \mu_1 - \gamma_1 & -\beta_1(1-\psi)X_2 & M(1-e)X_1 \\ e_1\beta Y_1 & e_2\psi\beta_1 Y_1 & e_1\beta X_1 + e_2\psi\beta_1 X_2 - \mu_1 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix} \tag{4.1}$$

Therefore, the variational system (2.1) matrix at $E_0 = (0, 0, 0, 0)$ is set through;

$$J\left(E_0\right) = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & -\left(\mu_1+\gamma_1\right) & 0 & 0 \\ 0 & 0 & -\mu_1 & 0 \\ 0 & 0 & \theta & -\alpha \end{bmatrix} \tag{4.2}$$

Then the eigenvalues of $J\left(E_0\right)$ are set through;

$$\left.\begin{array}{l} \lambda_1 = r \\ \lambda_2 = -\left(\mu_1+\gamma_1\right) \\ \lambda_3 = -\mu_1 \\ \lambda_4 = -\alpha \end{array}\right\} \tag{4.3}$$

So, $E_0 = (0,0,0,0)$ is saddle point.
The variational matrix of the system (2.1) at $E_1 = \left(\check{X}_1, 0, 0, 0\right)$ is set through;

$$J\left(E_1\right) = \begin{bmatrix} -r & 0 & -\beta \check{X}_1 & -M(1-e)\check{X}_1 \\ 0 & -\left(\mu_1+\gamma_1\right) & 0 & M(1-e)\check{X}_1 \\ 0 & 0 & e_1\beta\check{X}_1-\mu_1 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix} \tag{4.4}$$

Then eigenvalues are set through;

$$\left.\begin{array}{l} \check{\lambda}_1 = -r \\ \check{\lambda}_2 = -\left(\mu_1+\gamma_1\right) \\ \check{\lambda}_3 = e_1\beta\check{X}_1-\mu_1 \\ \check{\lambda}_4 = -\alpha \end{array}\right\}$$

So, $E_1 = \left(\check{X}_1, 0, 0, 0\right)$ is asymptotically local stable equilibrium when

$$k < \frac{\mu_1}{e_1\beta}. \tag{4.5}$$

The variational system (2.1) matrix at $E_2 = \left(\dddot{X}_1, \dddot{X}_2, 0, \dddot{E}\right)$, is set through;

$$J\left(E_2\right) = \begin{bmatrix} -\frac{r}{k}\dddot{X}_1 & 0 & \beta\dddot{X}_1 & -M(1-e)\dddot{X}_1 \\ M(1-e)\dddot{E} & -\left(\mu_1+\gamma_1\right) & -\beta_1(1-\psi)\dddot{X}_2 & M(1-e)\dddot{X}_1 \\ 0 & 0 & e_1\beta\dddot{X}_1+e_2\psi\beta_1\dddot{X}_2-\mu_1 & 0 \\ 0 & 0 & \theta & -\alpha \end{bmatrix} \tag{4.6}$$

Then eigenvalues are set through;

$$\left.\begin{array}{l} \dddot{\lambda}_1 = \frac{M\theta(1-e)(1-\epsilon)}{\alpha} - r \\ \dddot{\lambda}_2 = -\left(\mu_1+\gamma_1\right) \\ \dddot{\lambda}_3 = e_1\beta\dddot{X}_1+e_2\psi\beta_1\dddot{X}_2-\mu_1 \\ \dddot{\lambda}_4 = -\alpha \end{array}\right\} \tag{4.7}$$

So, $E_2 = \left(\dddot{X}_1, \dddot{X}_2, 0, \dddot{E}\right)$ is a asymptotically locally stable equilibrium if

$$e_1\beta k + \frac{e_2\psi\beta_1 r}{\mu_1+\gamma_1} < \mu_1 + \frac{e_1\beta k M\theta(1-e)(1-\epsilon)}{r\alpha}. \tag{4.8}$$

The variational system (2.1) matrix at $E_3 = \left(\hat{X}_1, 0, \hat{Y}_1, 0\right)$ is set through;

$$J\left(E_3\right) = \begin{bmatrix} -\frac{r\mu_1}{e_1\beta k} & 0 & -\beta\hat{X}_1 & -M(1-e)\hat{X}_1 \\ 0 & -\beta_1(1-\psi)\hat{Y}_1-\mu_1-\gamma_1 & 0 & M(1-e)\hat{X}_1 \\ e_1\beta\hat{Y}_1 & e_2\psi\beta_1\hat{Y}_1 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix} \tag{4.9}$$

The characteristic formula is set through;

$$(-\alpha-\lambda)\left[\lambda^3+\hat{A}_1\lambda^2+\hat{A}_2\lambda+\hat{A}_3\right] = 0 \tag{4.10}$$

Here

$$\begin{aligned}\bar{A}_1 &= -\left(\hat{a}_{11}+\hat{a}_{22}\right)\\ \bar{A}_2 &= \hat{a}_{11}\hat{a}_{22}+\hat{a}_{13}\hat{a}_{31}\\ \bar{A}_3 &= \hat{a}_{13}\hat{a}_{22}\hat{a}_{31}\end{aligned}$$

So either $(-\alpha-\lambda)=0$, which gives the eigenvalue in the $X-$ direction by $\hat{\lambda}_X=-\alpha$ or $\lambda^3+\hat{A}_1\lambda^2+\hat{A}_2\lambda+\hat{A}_3=0$.
At this point, based on Criterion of Routh-Hawirtiz all $J(E_3)$ eigenvalues of roots with real negative parts if and only if $\hat{\lambda}_i(i=1,3)>0$ and $\Delta=\hat{A}_1\hat{A}_2-\hat{A}_3>0$. So, $E_3=\left(\hat{X}_1,0,\hat{Y}_1,0\right)$ is a asymptotically locally stable equilibrium if

$$f_1<f_2 \tag{4.11}$$

since

$$\begin{aligned}f_1 &= \frac{r\mu_1}{e_1\beta k}\left(\beta_1(1-\psi)\hat{Y}_1+\mu_1+\gamma_1\right)\left[\left(\frac{r\mu_1}{e_1\beta k}\right)-\left(\beta_1(1-\psi)\hat{Y}_1+\mu_1+\gamma_1\right)\right]\\ f_2 &= \left(-e_1\beta^2\hat{X}_1\hat{Y}_1\right)\left[2\left(-\beta_1(1-\psi)\hat{Y}_1-\mu_1-\gamma_1\right)+\frac{r\mu_1}{e_1\beta k}\right]\end{aligned} \tag{4.12}$$

From ecological stand point, such equilibrium is so vital. The cause is quite clear that in such case all 4 populations will simultaneously exist. The variational the system (2.1) matrix at $E_4=\left(\bar{X}_1,\bar{X}_2,\bar{Y}_1,\bar{E}\right)$ can be written as;

$$J(E_3)=\begin{bmatrix} -\frac{r\bar{X}_1}{k} & 0 & -\beta\bar{X}_1 & -M(1-e)\bar{X}_1\\ M(1-e)\bar{E} & -\beta_1(1-\psi)\bar{Y}_1-\mu_1-\gamma_1 & -\beta_1(1-\psi)\bar{X}_2 & M(1-e)\bar{X}_1\\ e_1\beta\bar{Y}_1 & e_2\psi\beta_1\bar{Y}_1 & 0 & 0\\ 0 & 0 & 0 & -\alpha\end{bmatrix} \tag{4.13}$$

The characteristic formula is set through;

$$(-\alpha-\lambda)\left[\lambda^3+\bar{A}_1\lambda^2+\bar{A}_2\lambda+\bar{A}_3\right]=0 \tag{4.14}$$

Here

$$\begin{aligned}\bar{A}_1 &= -\left(\bar{a}_{11}+\bar{a}_{22}\right)\\ \bar{A}_2 &= \bar{a}_{11}\bar{a}_{22}+\bar{a}_{13}\bar{a}_{31}-\bar{a}_{23}\bar{a}_{32}\\ \bar{A}_3 &= \bar{a}_{11}\bar{a}_{23}\bar{a}_{32}+\bar{a}_{13}\left(\bar{a}_{22}\bar{a}_{31}-\bar{a}_{21}\bar{a}_{32}\right)\end{aligned}$$

So either $(-\alpha-\lambda)=0$, which gives the eigenvalue in the $X-$ direction by $\bar{\lambda}_X=-\alpha$ or $\lambda^3+\overline{A_1}\lambda^2+\bar{A}_2\lambda+\bar{A}_3=0$.
At this point, based on the Criterion of Routh-Hawirtiz all the $J(E_4)$ eigenvalues of roots with real negative parts if and only if $\bar{\lambda}_i(i=1,3)>0$ and $\Delta=\bar{A}_1\bar{A}_2-\bar{A}_3>0$.
So, $E_4=\left(\bar{X}_1,\bar{X}_2,\bar{Y}_1,\bar{E}\right)$ is a asymptotically locally stable equilibrium if

$$D_2<D_1 \tag{4.15}$$

since

$$\begin{aligned}D_1 &= \beta_1(1-\psi)\bar{Y}_1+\mu_1+\gamma_1\left[\frac{r\bar{x}_1}{k}\left(\frac{r\bar{X}_1}{k}+\beta_1(1-\psi)\bar{Y}_1+\mu_1+\gamma_1\right)+\left(\beta_1(1-\psi)\bar{X}_2e_2\psi\beta_1\bar{Y}_1\right)\right]\\ D_2 &= -\beta\bar{X}_1\left[e_1\beta\bar{Y}_1\left(-\frac{r\bar{X}_1}{k}+2\left(-\beta_1(1-\psi)\bar{Y}_1-\mu_1-\gamma_1\right)\right)-M(1-e)e_2\psi\beta_1\bar{Y}_1\bar{E}\right]\end{aligned} \tag{4.16}$$

5. Analysis of global stability

At this part, the global stability region (attraction basin) of every equilibrium system (2.1) points is displayed as illustrated in formulas as follow.

Theorem 5.1. *Adopt that the point of equilibrium E_1 is asymptotically being local. Then it is a asymptotically globally stable in the subregion of $\Re_+^4$ provided that*

$$\check{X}_1 < \min\left\{\frac{\mu_1}{\beta}, \frac{\alpha}{M(1-e)}\right\} \tag{5.1}$$

Proof . Regard the following +ve as function being definite

$$V_1\left(X_1, X_2, Y_1, E\right) = \left(X_1 - \check{X}_1 - X_1 \ln \frac{X_1}{\check{X}_1}\right) + X_2 + Y_1 + E$$

Obviously, $V_1 : \Re_+^4 \rightarrow \Re$ is a differentiable continuous function such that $V_1\left(\check{X}_1, 0, 0, 0\right) = 0$ and $V_1\left(X_1, X_2, Y_1, E\right) > 0, \forall\left(X_1, X_2, Y_1, E\right) \neq \left(\check{X}_1, 0, 0, 0\right)$. Further,

$$\begin{aligned}\frac{dV_1}{dt} = &\left(\frac{x_1 - \check{X}_1}{X_1}\right)\left[rX_1 - \frac{r}{k_1}X_1^2 - M(1-e)X_1E - \beta X_1Y_1\right] \\ &+ M(1-e)X_1E - \beta_1(1-\psi)X_2Y_1 - \left(\mu_1 + \gamma_1\right)X_2 \\ &+ e_1\beta X_1Y_1 + e_2\psi\beta_1X_2Y_1 - \mu_1Y_1 - \alpha E\end{aligned}$$

At this point, through making some algebraic manipulations and utilizing the condition (5.1), we acquire

$$\begin{aligned}\frac{dV_1}{dt} \leq &-\frac{r}{k}\left(X_1 - \check{X}_1\right)^2 - \left(\alpha - M(1-e)\check{X}_1\right)E - \left(\mu_1 - \beta\check{X}_1\right)Y_1 \\ &-\left(\mu_1 + \gamma_1\right)X_2 - \beta_1(1-\psi(1-e))X_2Y_1\end{aligned} \tag{5.2}$$

Consequently, due to the condition above $\frac{dV_1}{dt} < 0$ is ve definite and thus V_1 is Lyapunov function with respect to E_1 in the region that satisfies the given condition. Thus E_1 is asymptotically globally stable and the evidence is complete. □

Theorem 5.2. *Adopt that the point of equilibrium E_2 is asymptotically locally stable. Then it is asymptotically locally stable in the sub-region of $\Re_+^4$ that satisfied the following conditions*

$$\begin{aligned}&\beta\dddot{X}_1 + e_2\psi\beta_1X_2 + \mu_1 < \beta_1(1-\psi)X_2\left(X_2 - \dddot{X}_2\right) \\ &\dddot{X}_2 < X_2 \\ &\theta(1-\epsilon)(E - \dddot{E}) < Y_1\left[\beta_1(1-\psi)X_2\left(X_2 - \dddot{X}_2\right) - \beta\ddot{X}_1 - e_2\psi\beta_1X_2 - \mu_1\right] \\ &q_{12}^2 < 4q_{11}q_{22} \\ &q_{14}^2 < 4q_{11}q_{44} \\ &q_{24}^2 < 4q_{22}q_{44}\end{aligned} \tag{5.3}$$

Proof . Regard the following +ve definite function

$$V_2\left(X_1, X_2, Y_1, E\right) = \left(X_1 - \dddot{X}_1 - \dddot{X}_1 \ln \frac{X}{\dddot{X}_1}\right) + \frac{1}{2}\left(X_2 - \dddot{X}_2\right)^2 + Y_1 + \frac{1}{2}(E - \dddot{E})^2$$

Obviously, $V_2 : \Re_+^4 \to \Re$ is a differentiable continuouse function such that $V_2\left(\dddot{X}_1, \dddot{X}_2, 0, \dddot{E}\right) = 0$ and $V_2\left(X_1, X_2, Y_1, E\right) > 0, \forall\left(X_1, X_2, Y_1, E\right) \in \Re_+^4$ and $\left(X_1, X_2, Y_1, E\right) \neq \left(\dddot{X}_1, \dddot{X}_2, 0, \dddot{E}\right)$
Considering the derivative to the time and shortening the resulting terms, we obtain that

$$\begin{aligned}
\frac{dV_2}{dt} &= \left(X_1 - \dddot{X}_1\right)\left[r - \frac{r}{k_1}X_1 - M(1-e)E - \beta Y_1\right] \\
&+ \left(X_2 - \dddot{X}_2\right)\left[M(1-e)X_1E - \beta_1(1-\psi)X_2Y_1 - \left(\mu_1 + \gamma_1\right)X_2\right] \\
&+ \left[e_1\beta X_1Y_1 + e_2\psi\beta_1X_2Y_1 - \mu_1Y_1\right] + (E - \dddot{E})[\theta(1-\epsilon) - \alpha E]
\end{aligned}$$

$$\begin{aligned}
\frac{dV_2}{dt} = &- \left[\frac{q_{11}}{2}\left(X_1 - \dddot{X}_1\right)^2 - q_{12}\left(X_1 - \dddot{X}_1\right)\left(X_2 - \dddot{X}_2\right) + \frac{q_{22}}{2}\left(X_2 - \dddot{X}_2\right)^2\right] \\
&- \left[\frac{q_{11}}{2}\left(X_1 - \dddot{X}_1\right)^2 + q_{14}\left(X_1 - \dddot{X}_1\right)(E - \dddot{E}) + \frac{q_{44}}{2}(E - \dddot{E})^2\right] \\
&- \left[\frac{q_{22}}{2}\left(X_2 - \dddot{X}_2\right)^2 - q_{24}\left(X_2 - \dddot{X}_2\right)(E - \dddot{E}) + \frac{q_{44}}{2}(E - \dddot{E})^2\right] \\
&- Y_1\left[\beta(1-\psi)X_2\left(X_2 - \dddot{X}_2\right) - \beta\dddot{X}_1 - e_2\psi\beta_1X_2 - \mu_1\right] + \theta(1-\epsilon)(E - \dddot{E})
\end{aligned}$$

Consequently by using (5.3) conditions we get that

$$\begin{aligned}
\frac{dV_2}{dt} \leq &- \left[\sqrt{\frac{q_{11}}{2}}\left(X_1 - \ddot{X}_1\right) - \sqrt{\frac{q_{22}}{2}}\left(X_2 - \ddot{X}_2\right)\right]^2 - \left[\sqrt{\frac{q_{11}}{2}}\left(X_1 - \dddot{X}_1\right) - \sqrt{\frac{a_{44}}{2}}(E - \dddot{E})\right]^2 \\
&- \left[\sqrt{\frac{a_{22}}{2}}\left(X_2 - \ddot{X}_2\right) - \sqrt{\frac{a_{44}}{2}}(E - \dddot{E})\right]^2 + \theta(1-\epsilon)(E - \dddot{E}) \\
&- Y_1\left[\beta(1-\psi)X_2\left(X_2 - \dddot{X}_2\right) - \beta\ddot{X}_1 - e_2\psi\beta_1X_2 - \mu_1\right]
\end{aligned} \tag{5.4}$$

since

$$\begin{aligned}
&q_{11} = \frac{r}{k}, q_{12} = M(1-e)\dddot{E}, q_{22} = \mu_1 + \gamma_1 \\
&q_{14} = M(1-e), q_{44} = \alpha, q_{24} = M(1-e)X_1
\end{aligned} \tag{5.5}$$

Obviously, $\frac{dV_2}{dt}$ is -ve definite and thus V_2 is Layapunov function with regard to E_2. So E_2 is asymptotically locally stable in the sub-region that satisfies the given condition. □

Theorem 5.3. *Adopt that the point of equilibrium E_3 is asymptotically being local. Then it is asymptotically locally stable in the subregion of $\Re_+^4$ provided that*

$$\begin{aligned}
&\hat{X}_1 < \frac{\alpha}{M(1-e)} \\
&\hat{X}_1 < X_1 \\
&\hat{Y}_1 < Y_1 \\
&\psi < \frac{1}{2}
\end{aligned} \tag{5.6}$$

Proof . Regard the following +ve definite function

$$V_3\left(X_1, X_2, Y_1, E\right) = \left(X_1 - \hat{X}_1 - X_1\ln\frac{X_1}{\hat{X}_1}\right) + X_2 + \left(Y_1 - \hat{Y}_1 - Y_1\ln\frac{Y_1}{\hat{Y}_1}\right) + E$$

Obviously, $V_3 : \Re^4_+ \to \Re$ is a differentiable continuouse function such that $V_3\left(\hat{X}_1, 0, \hat{Y}_1, 0\right) = 0$ and $V_3\left(X_1, X_2, Y_1, E\right) > 0, \forall\left(X_1, X_2, Y_1, E\right) \neq \left(\hat{X}_1, 0, \hat{Y}_1, 0\right)$. Further,

$$\begin{aligned}\frac{dV_3}{dt} =& \left(\frac{X_1-\hat{X}_1}{X_1}\right)\left[rX_1 - \frac{r}{k_1}X_1^2 - M(1-e)X_1E - \beta X_1Y_1\right]\\ &+ M(1-e)X_1E - \beta_1(1-\psi)X_2Y_1 - \left(\mu_1+\gamma_1\right)X_2\\ &+ \left(\frac{Y_1-\hat{Y}_1}{Y_1}\right)\left[e_1\beta X_1Y_1 + e_2\psi\beta_1X_2Y_1 - \mu_1Y_1\right] - \alpha E\end{aligned}$$

At this point, by doing some algebraic manipulations and using the condition (5.6), we get

$$\begin{aligned}\frac{dV_3}{dt} \leq& -\frac{r}{k}\left(X_1-\hat{X}_1\right)^2 - \beta\left(1-e_1\right)\left(X_1-\hat{X}_1\right)\left(Y_1-\hat{Y}_1\right) - \left[\alpha - M(1-e)\hat{X}_1\right]E\\ &- \left[\beta_1Y_1(1-2\psi) + e_2\psi\beta_1\hat{Y}_1 + \left(\mu_1+\gamma_1\right)\right]X_2\end{aligned} \tag{5.7}$$

Consequently, due to the condition above $\frac{dV_3}{dt} < 0$ is ve definite and thus V_3 is Lyapunov function with regard to E_3 in the region that satisfies the given condition. Thus E_3 is asymptotically locally stable and the Evidence is complete. □

Theorem 5.4. *Adopt that the point of equilibrium E_4 is asymptotically locally stable. Then it is asymptotically locally stable in the sub-region of $\mathfrak{R}^4_+$ that satisfied the following conditions*

$$\begin{aligned}&e_2\psi\beta < \beta_1(1-\psi)X_2\\ &\bar{X}_2 < X_2\\ &\bar{Y}_1 < Y_1\\ &\bar{E} < E\\ &\mu_1+\gamma_1 < \beta_1(1-\psi)\bar{Y}_1\\ &q_{12}^2 < q_{11}q_{22}\\ &q_{14}^2 < q_{11}q_{44}\\ &q_{24}^2 < q_{22}q_{44}\end{aligned} \tag{5.8}$$

Proof . Regard the following +ve definite function

$$V_4\left(X_1, X_2, Y_1, E\right) = \left(X_1 - \bar{X}_1 - X_1\ln\frac{X_1}{\bar{X}_1}\right) + \frac{\left(X_2-\bar{X}_2\right)^2}{2} + \left(Y_1 - \bar{Y}_1 - Y_1\ln\frac{Y_1}{\bar{Y}_1}\right) + \frac{(E-\bar{E})^2}{2}$$

Obviously, $V_4 : \Re^4_+ \to \Re$ is a differentiable continuouse function such that $V_4\left(\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{E}\right) = 0$ and $V_4\left(X_1, X_2, Y_1, E\right) > 0, \forall\left(X_1, X_2, Y_1, E\right) \in \mathfrak{R}^4_+$ and $\left(X_1, X_2, Y_1, E\right) \neq \left(\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{E}\right)$
Considering the derivative with regard to the time and shortening the resulting terms, we obtain that

$$\begin{aligned}\frac{dV_4}{dt} =& \left(\frac{x_1-\bar{X}_1}{X_1}\right)\left[rX_1 - \frac{r}{k_1}X_1^2 - M(1-e)X_1E - \beta X_1Y_1\right]\\ &+ \left(X_2-\bar{X}_2\right)\left[M(1-e)X_1E - \beta_1(1-\psi)X_2Y_1 - \left(\mu_1+\gamma_1\right)X_2\right]\\ &+ \left(\frac{Y_1-\bar{Y}_1}{Y_1}\right)\left[e_1\beta X_1Y_1 + e_2\psi\beta_1X_2Y_1 - \mu_1Y_1\right] + (E-\bar{E})[\theta(1-\epsilon) - \alpha E]\end{aligned}$$

$$\begin{aligned}\frac{dV_4}{dt} = &- \left[\frac{q_{11}}{2}\left(X_1-\bar{X}_1\right)^2 + q_{12}\left(X_1-\bar{X}_1\right)\left(X_2-\bar{X}_2\right) + \frac{q_{22}}{2}\left(X_2-\bar{X}_2\right)^2\right] \\ &- \left[\frac{q_{11}}{2}\left(X_1-\bar{X}_1\right)^2 + q_{14}\left(X_1-\bar{X}_1\right)(E-\bar{E}) + \frac{q_{44}}{2}(E-\bar{E})^2\right] \\ &- \left[\frac{q_{22}}{2}\left(X_2-\bar{X}_2\right)^2 - q_{24}\left(X_2-\bar{X}_2\right)(E-\bar{E}) + \frac{q_{44}}{2}(E-\bar{E})^2\right] \\ &- [\beta_1(1-\psi)X_2 - e_2\psi\beta]\left(X_2-\bar{X}_2\right)\left(Y_1-\bar{Y}_1\right) + \theta(1-\epsilon)(E-\bar{E})\end{aligned}$$

Consequently by using (5.8) conditions we get that

$$\begin{aligned}\frac{dV_4}{dt} \leq &- \left[\sqrt{\frac{q_{11}}{2}}\left(X_1-\bar{X}_1\right) + \sqrt{\frac{q_{22}}{2}}\left(X_2-\bar{X}_2\right)\right]^2 \\ &- \left[\sqrt{\frac{q_{11}}{2}}\left(X_1-\bar{X}_1\right) + \sqrt{\frac{a_{44}}{2}}(E-\bar{E})\right]^2 - \left[\sqrt{\frac{q_{22}}{2}}\left(X_2-\bar{X}_2\right) - \sqrt{\frac{a_{44}}{2}}(E-\bar{E})\right]^2 \\ &- [\beta_1(1-\psi)X_2 - e_2\psi\beta]\left(X_2-\bar{X}_2\right)\left(Y_1-\bar{Y}_1\right) + \theta(1-\epsilon)(E-\bar{E})\end{aligned} \tag{5.9}$$

since

$$\begin{aligned}&q_{11} = \frac{r}{k}, q_{12} = M(1-e)\bar{E}, q_{22} = \beta_1(1-\psi)\bar{Y}_1 - (\mu_1+\gamma_1) \\ &q_{14} = M(1-e), q_{24} = M(1-e)X_1, q_{44} = \alpha\end{aligned}$$

Obviously, $\frac{dV_4}{dt}$ is -ve definite and thus V_4 is Layapunov function with regard to E_4. So E_4 is asymptotically locally stable in the subregion that satisfies the given condition. □

6. Numerical Simulation

For visualizing the foregoing analytical results and understands the influence of variable the parameters on the global system (2.1) dynamics, the numerical simulation is pereformed at this part. The study objectives are endorsing our obtained analytical results and detecting the control parameters set that affect the system dynamics. Recalling system (2.1) that containing 2 enter-specific competitions associations, the 1st one between the 2 preys (healthy and infected) at the 1st level, whereas the 2nd one between the healthy and infected predator in the 2nd level. It was found the data set that satisfies the coexistence for 4 populations of them as shown below. Furthermore, as long as we offerings the conditions that render the system to be asymptotically stable +ve equilibrium point analytically; thus, still possibility is there for having such data. Consequently, system (2.1) is solved numerically for different sets of primary conditions and for various set of feasible parameters biologically hypothetical. It is noticed that for the following hypothetical parameters set, the system (2.1) has a globally asymptotically stable +ve point of equilibrium as illustrated in the below figures:

$$\begin{aligned}&r = 1.1, k = 0.7, m = 0.4, e = 0.2, \beta = 0.4 \\ &\beta_1 = 0.6, \psi = 0.4, \mu_1 = 0.1, \gamma_1 = 0.2, \epsilon = 0.6 \\ &e_1 = 0.4, e_2 = 0.3, \gamma_2 = 0.1, \theta = 0.3, \alpha = 0.5\end{aligned} \tag{6.1}$$

We got that the system (2.1) trajectories with 3 different sets of +ve primary conditions asymptotically approach to the +ve point of equilibrium E_4 as illustrated in Fig.1

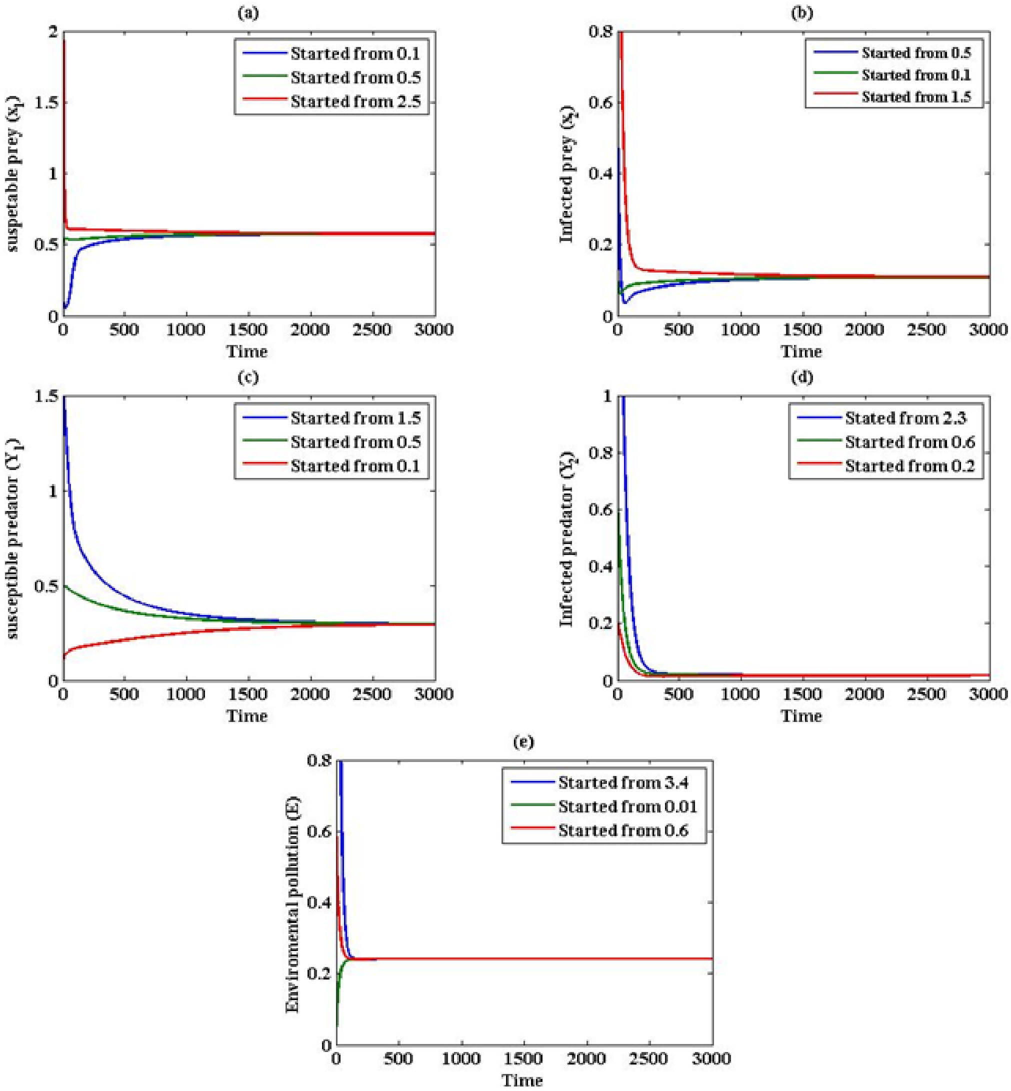

Figure 1: Asymptotically locally stable +ve point of equilibrium E_4 of system (2.1) for: (a) Trajectories of $X_1(t)$ (b) Trajectories of $X_2(t)$ (c) Trajectories of $Y_1(t)$ (d) Trajectories of $Y_2(t)$ (e) Trajectories of $E(t)$.

Obviously, Fig. 1 verifies our gotten analytical results in respect to existence that +ve point of equilibrium is asymptotically locally stable. However, for the data by formula (6.1) with $\theta = 0$ and $\mu_1 = 0.2$, the solution of system (2.1) approaches asymptotically to the vanishing equilibrium point E_1 shown in the following typical, figure 2

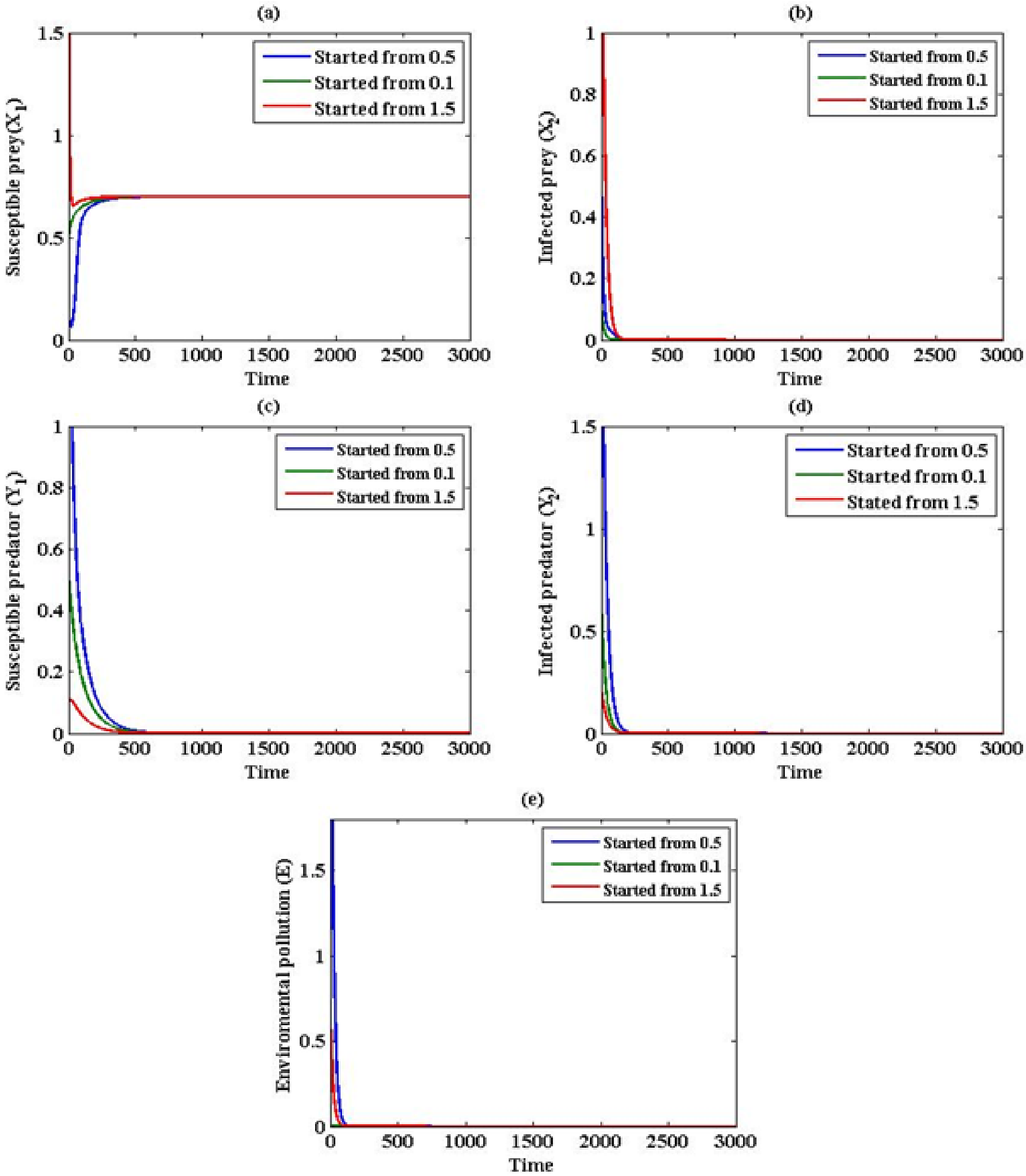

Figure 2: Asymptotically locally stable of vanishing equilibrium point E_1 of system (2.1) for:(a) Trajectories of $X_1(t)$ (b) Trajectories of $\mathrm{X}_2(t)$ (c) Trajectories of $\mathrm{Y}_1(t)$ (d) Trajectories of $Y_2(\mathrm{t})$ (e) Trajectories of $E(t)$.

At this point and for the purpose of investigating the varying parameters effect value at a time on the dynamical system (2.1) behavior, results as follow are noticed. Based on the figure 3, it is obvious that the system (2.1) solution approaches asymptotically to the 1st 2 species point of equilibrium for the parameters values shown in Eq. (6.1) with varying $\mu_1 = 0.2$, to obtain the trajectories of system (2.1) approach asymptotically to the E_2 as shown in Figure. 3

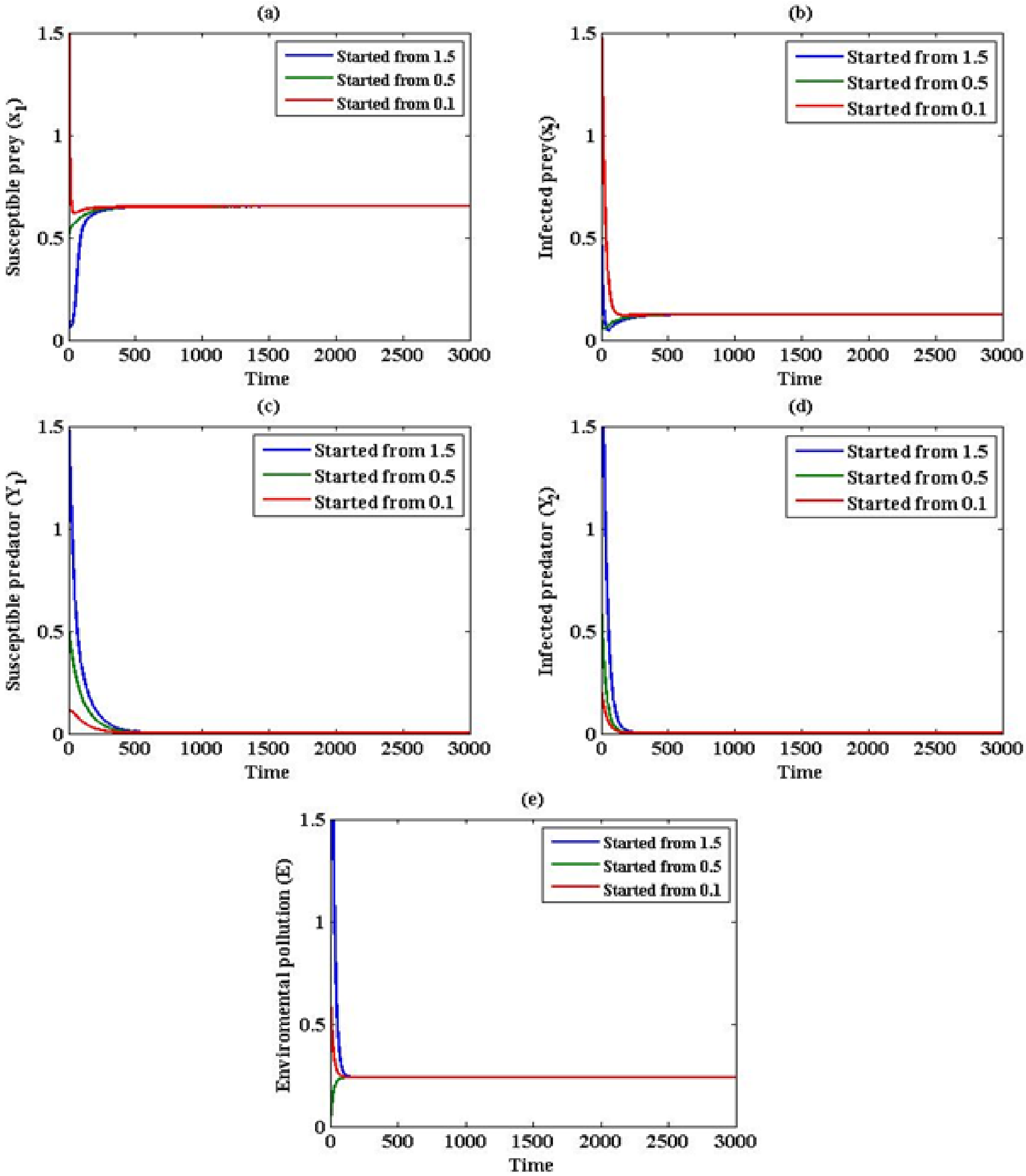

Figure 3: Asymptotically locally stable of the 1st 2 species equilibrium E_2 of system (2.1) for: (a) Trajectories of $X_1(t)$ (b) Trajectories of $X_2(t)$ (c) Trajectories of $Y_1(t)$ (d) Trajectories of Y_2(t) (e) Trajectories of $E(t)$.

We select the environment pollution coefficient values $\theta = 0$,leaving other parameters constant as shown in formula (6.1), we obtain the system (2.1) trajectories still approaches to the 2^{nd} 2 species point of equilibrium. Furthermore the effect of environment is shown in figure 4

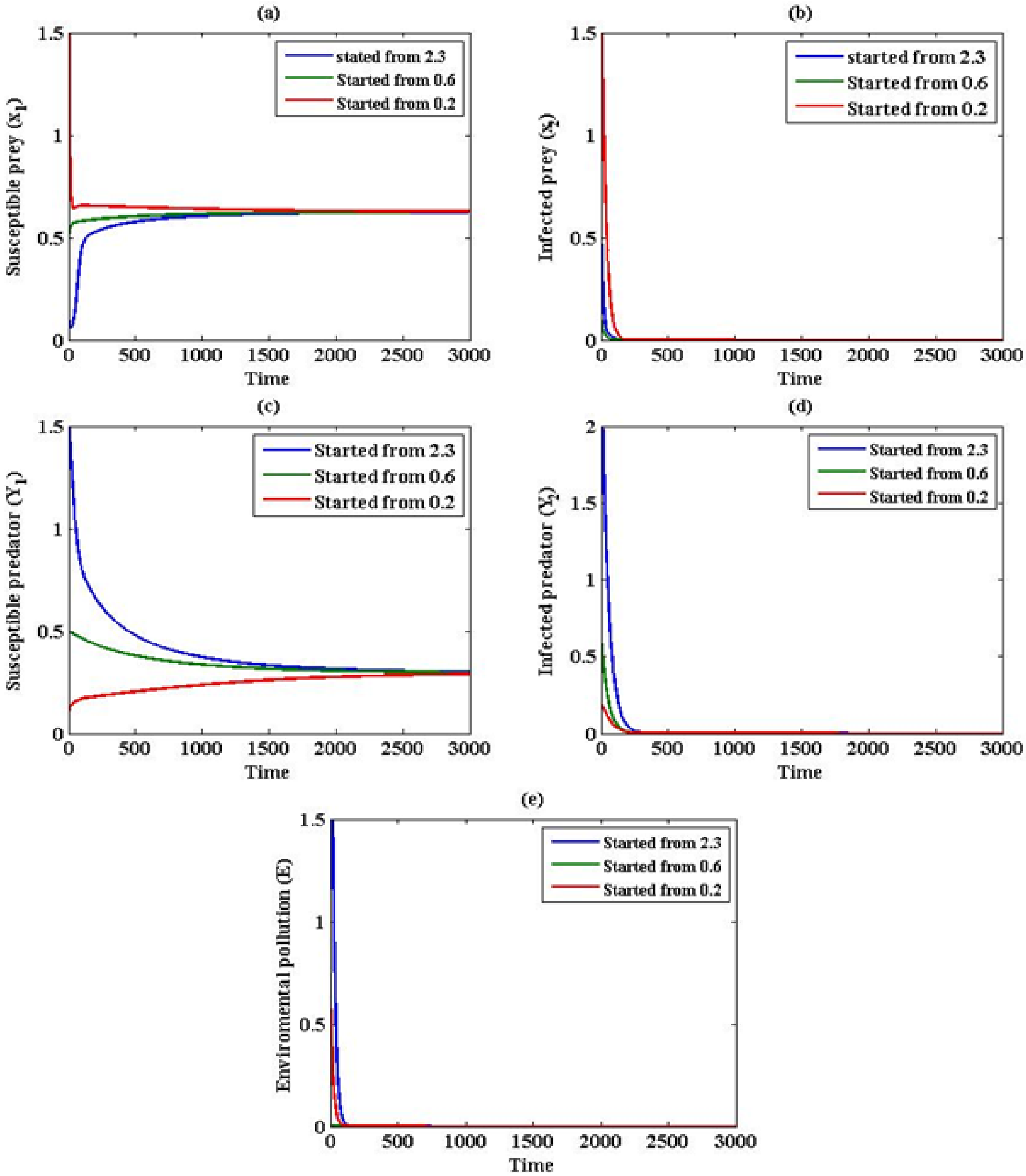

Figure 4: asymptotically locally stable of point of equilibrium E_3 of system (2.1) (a) Trajectories of $X_1(t)$ (b) Trajectories of $X_2(t)$ (c) Trajectories of $Y_1(t)$ (d) Trajectories of Y_2(t) (e) Trajectories of $E(t)$.

7. CONCLUSIONS AND DISCUSSION

At the current study, we have invistigated the diseased susceptible model stability, prey being infected and predators around steady interior state. The model including five non-linear differential autonomous formulas which describing the 4 different populations dynamics, namely Susceptible prey (X_1), , prey being infected (X_2), healthy predator (Y_1) and infected predator (Y_2). The system (2.1) boundedness was discussed. The conditions existences of all possible point of equilibriums are detected. The global and local stability analyses of such points are achieved. Also, we invistigated the stochastic model (2.1) perturbation, that produces an important change in the populations intensity because of low, medium and high oscillations variances. Lastly, for completing our vision to the global dynamical system (2.1) behavior, numerical simulation is applied utilizing parameters values set hypothetically via Eq. (6.1). The numerical simulation results are summarized as follow.

1. (E_4) is mostly significant point of equilibrium since it offers all the 4 species coexistence simultaneously. For ecological eco-system coexistence balance of all the species in respective proportions is very important. The stability of (E_4) specifies all species existence for a long time.

2. The system (2.1) trajectory approaches asymptotically to +ve point of equilibrium starting from diverse primary points utilizing the data Eq. (6.1), that specifies existence of asymptotically locally stable +ve point of equilibrium.

3. Increasing the inhibition disease rate or disease death rate higher than specific value cause extinction in predator species because of food lacking. Additional increasing as a minimum one of such parameters leads to extinction in the prey being infected specie and the system (2.1) trajectory approaches asymptotically to free point of equilibrium. Or else, the system still continues at a +ve point of equilibrium.

4. We observed that environment pollution coefficient (θ)has a leading function in the systems (2.1) equilibria existence and stability.

Keeping the foregoing in mind, all such outcomes relied on the parameters values set hypothetical set by Eq. (6.1), diverse results might be gotten for data different sets.

References

[1] X. Zhou and A. A. Muhseen, *The Global Stable with Bifurcation of Parasitic Diseases Model* World Journal of Modelling and Simulation, Vol. 13 No. 4, (2017), 293-313.

[2] R. K. Naji and H. F. Ridha *The Dynamics of a Prey-Predator Model Incorporating Svis-Type of Disease in Prey* IOSR Journal of Mathematics,vol.12 , No 2, (2016), 90-101.

[3] A. A. Muhseen and I. A. Aaid *Stability of a Prey-Predator Model with SIS Epidemic Disease in Predator Involving Holling Type II Functional Response* IOSR Journal of Mathematics,vol.11, No3, (2015), 38-55.

[4] E. Venturino , *The influence of diseases on Lotka-Volterra systems*, Rocky Mountain J. Math. Vol. 24 , No. 1, (1994), 381-402.

[5] K.P. Das , *A Mathematical study of a predator prey dynamics with disease in predator*. ISRN Applied Mathematics, (2011), 1-16.

[6] Y. Kuang and E. Beretta , *Global qualitative analysis of a ratio-dependent predator-prey system*, J. Math. Biol.Vol. 36 , No. 4,(1998), 389-406.

[7] J. Chattopadhyay and O. Arino , *A predator-prey model with disease in the prey, Nonlinear Anal.* 36 , No. 6, Ser. B: Real World Appl., (1999), 747-766, .

[8] F. Chen and M. You . Permanence, *extinction and periodic solution of the predatorC prey system with Beddington-CDeAngelis functional respon-se and stage structure for prey.* Nonlinear Analysis: Real World Applications ,9:(2008) 207-221 .

[9] R. K. Naji and H. F. Ridha *The Dynamics Of Four Species Food Web Model With Stage Structure*, International Journal of technology Enhancements and Emerging Engineering Research, , 4(2), (2016).

[10] D. Kesh, A.K. Sarkar and A.B. Roy , *Persistent of Two Prey One Predator System with Ratio Dependent Predator Influence*, Math. Appl.Sci., 23, (2000), 347-356.

[11] SB. Hsu , TW. Hwang , Y. Kuang *A ratio-dependent food chain model and its applications to biological control.* Math Biosci, (2002), preprint.

[12] S. Gakkhar , R.K. Naji *Chaos in three species ratio-dependent food chain. Chaos, Solitons & Fractals*;14:(2002) 771-8, .