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Investigating 1-perfect code using Dominating set

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Abstract

In this article, we investigate the relation between dominating sets and 1-perfect codes. We also study perfect colorings of some Johnson graphs in two colors using linear programming problem.

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1. Introduction

All over this paper, G is a finite connected simple graph with vertex set V and edge set E. The distance $d(x, y)$ between any two vertices x and y of G is the length of a shortest path from x to y in G. The diameter of G is the maximal distance occurring in G [7]. We will need the concept of neighborhood of x, written $N(x)$, that is the set of vertices adjacent to x. Also $N[x] = N(x) \cup \{x\}$.

A perfect coloring of a graph G with m colors (a perfect m-coloring) with matrix $P = [p_{ij}]$; $i, j = 1, 2, \dots, m$ is a coloring of V with the colors $\{1, 2, \dots, m\}$ such that every vertex of color i has p_{ij} neighbors of color j. The matrix P is called the parameter matrix of a perfect m-coloring. In this paper, we study the perfect 2-colorings and call the first color white, and the second color black [2, 5].

Consider E^n as the set of binary vectors of length n. The weight of a vector x in E^n is the number of nonzero coordinates of x. The Johnson graph $J(n, \omega)$ is a graph whose vertex set consists of all vectors in E^n of weight ω. The set of edges of this graph consists of the pairs of vectors differing in exactly two coordinates. It is easy to check that $J(n, \omega)$ is a regular graph of degree $\omega(n - \omega)$ and diameter ω. Note that $J(n, n - \omega)$ and $J(n, \omega)$ are isomorphic; therefore, without loss of generality, we may consider the Johnson graphs $J(n, \omega)$ with $2\omega \leqslant n$ [1].

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Given a graph G with the vertex set V, a sphere of radius r centered at v is defined as $B_r(v) = \{u \in V : d(u,v) \leqslant r\}$. A set $C \subseteq V$ is called an e-perfect code in G if spheres of radius e centered at the vertices of C form a partition of V. If C either coincides with V ($e = 0$) or consists of at most two vertices, such a perfect code is called trivial [4, 5, 6].

A set $S \subset V$ is a dominating set if every vertex not in S has a neighbor in S. The minimum size of a dominating set in G, is called the domination number γ [3, 7]. There are several ways to find γ. One way to compute it, is using linear programming that the following theorem describes it.

Theorem 1.1. *Let G be a graph with vertex set V and adjacency matrix $A = [a_{ij}]_{n\times n}$. The domination number γ of G is obtained by solving the following linear programming problem:*

$$\gamma = \min z = \sum_{i=1}^{n} v_i \tag{1.1}$$

$$s.t.(A + I_n)N \geq 1_n$$

$$v_i \in \{0,1\}$$

where I_n is the identity matrix of order n, $N = (v_1, v_2, \dots, v_n)^T$ and $1_n = (1,1,\dots,1)^T$.

Proof . Consider variables $\{v_1, v_2, \dots, v_n\}$ corresponding to vertices of G; if $v_i = 1$ then $v_i \in S$; otherwise $v_i \notin S$. According to this, each of the constraints corresponds to a vertex of G and indicates that for any $v \in V$, at least one vertex of $N[v]$ belongs to S. This is consistent with the definition of dominating set. Besides, with regard to the objective function, solving the linear programming problem 1.1 gives the domination number of G. $\square$

Assume $B_1(v)$ is a sphere of radius 1 centered at the vertex v belonging to the dominating set S; then $V(G) \subset \cup_{v\in S} B_1(v)$. Notice that the considered spheres can also have intersection. We will show that if this spheres do not have intersection, they form an 1-perfect code for G.

2. How to get the 1-perfect codes

According to the previous section, a 1-perfect code is a dominating set, but the converse of this is not true. Since the neighborhoods of a 1-perfect code do not have intersection; but the spheres of radius 1 centered at the vertices of the dominating set can also have intersection.

Theorem 2.1. *Let G be a graph with the vertex set V such that $|V| = n$. If there exist 1-perfect code in G, it will be achieved from the following linear system:*

$$(A + I_n)N = 1_n; \tag{2.1}$$

$$v_i \in \{0,1\}$$

where $A = [a_{ij}]_{n\times n}$ is the adjacency matrix of G, I_n is the identity matrix of order n, $N = (v_1, v_2, \dots, v_n)^T$ and $1_n = (1,1,\dots,1)^T$ So that $C = \{v_i | v_i = 1\}$ is 1-perfect code in G.

Proof . We prove that 1-perfect code is equivalent to the solution of the system 2.1. Consider the vertex set of G by $V = \{v_1, v_2, \dots, v_n\}$. Let v_k; $k = 1, 2, \dots, n$ has degree d_k. Suppose first that C be 1-perfect code for G. Consider variables $\{v_1, v_2, \dots, v_n\}$ corresponding to each of the vertices G. If $v_k \in C$, then $v_k = 1$; otherwise $v_k = 0$. We show that this is a solution for the system 2.1. Let v_i and v_j be arbitrary elements belonging to C. Therefore $v_i, v_j = 1$. v_i and v_j appear in $d_i + 1$ and $d_j + 1$ equations, respectively. According to the structure of the system, there is an equation corresponding to each vertex so that the variables in that equation are related vertex and its neighbors. On the other hand, since none of the vertices belonging to C are adjacent, equations in which there is v_i, are satisfied. Also, since $B_1(v_i) \cap B_1(v_j) = \emptyset$, there is no equation that includes both v_i and v_j. As well as $\cup_{v \in C} B_1(v) = V(G)$, therefore all of the equations are satisfied. So, we find a solution to the system. Vice versa, let's assume the system has a solution. We consider $C = \{v_k | v_k = 1\}$ and prove C is a 1-perfect code for G. First, we prove for every two vertices v_i and v_j belonging to C, we have $B_1(v_i) \cap B_1(v_j) = \emptyset$. Proof by contradiction: we suppose $v_k \in B_1(v_i) \cap B_1(v_j)$. Since $v_i, v_j \in C$ in the k^{th} equation of the system $v_i, v_j = 1$ which completes the contradiction. Also, if there is $v_k \in V(G)$ such that is not covered by $\cup_{v \in C} B_1(v)$, so the k^{th} equation is not satisfied; that also yields the contradiction. Therefore, the spheres of radius 1 centered at C form a partition of V. So C is a 1-perfect code. $\square$

Theorem 2.2. *$J(n, \omega), 3|n$, does not have a nontrivial 1-perfect code.*

Proof . Since $diam(J(n, \omega)) = \omega$, $diam(J(n, 3)) = 3$. Therefore, if there is a 1-perfect code, the distance between every two vertices belonging to this set is 3. We construct the 1-perfect code. Let $n = 3k$. Consider the set of vertices $\{v_1, \dots, v_k\}$ as follows: v_1 is a vertex that in the first three places gives $1, \dots, v_k$ is a vertex that in the k^{th} three places gives 1. In this case, for each $i, j = 1, 2, \dots, k; i \neq j$, $d(v_i, v_j) = 3$. Therefore, we have: $B_1(v_i) \cap B_1(v_j) = \emptyset$ for each $i, j = 1, 2, \dots, k; i \neq j$. The neighbors $\{v_1, \dots, v_k\}$ involve all cases that have exactly two non-zero elements. The vertex set of $\cap_{i=1}^{k} B_1(v_i)$ does not cover vertex that there is maximum one non-zero element in any of its classes. However, there exist a neighbor for any vertex that has two nonzero elements in one of the classes. Therefore, we cannot extend $\{v_1, \dots, v_k\}$. $\square$

Theorem 2.3. *Consider the graph $J(n, \omega)$. If $\binom{n}{\omega} = 2(\omega(n - \omega) + 1)$ then there exist a trivial 1-perfect code with two vertices in $J(n, \omega)$. So, it is perfect 2-colorable with the parameter matrix*

$$\begin{bmatrix} 0 & \omega(n-\omega) \\ 1 & \omega(n-\omega)-1 \end{bmatrix}$$

Proof . Since $\binom{n}{\omega} = 2(\omega(n - \omega) + 1)$, $\omega \geq 3$. that $d(v_i, v_j) \geq 3$. We can say $d(v_i, v_j) = 3$ because there is a vertex v_k such that v_k is not neighbor of v_i and v_j. Therefore $\{v_i, v_j\}$ is a trivial 1-perfect code in $J(n, \omega)$. Now suppose that $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ is the parameter matrix of this graph, the set C is a 1-perfect code and the other vertices. Since any two vertices are not adjacent in C (since if x and y are adjacent in C then $x, y \in B_1(x) \cap B_1(y)$ and obtaining a contradiction), so $S_{11} = 0$. On the other hand, since $J(n, \omega)$ is $\omega(n - \omega)$-regular and each vertex in C. No vertex has neighbor in C. Therefore, all neighbors of each vertex in C lie on C'; So $S_{12} = \omega(n - \omega)$. Now, suppose vertex z in C' is adjacent to x and y. So $z \in B_1(x) \cap B_1(y)$ that leads to a contradiction (C is a 1-perfect code); therefore $S_{12} < 2$. If there is $w \in C'$ that does not have any neighbor in C then w belongs to no sphere of radius 1 centered at C which leads to a contradiction; so $S_{12} = 1$. Also according to the regularity of graph $S_{22} = \omega(n - \omega) - 1$. $\square$

As a result, in general the graph $J(6, 3)$ just has one trivial 1-perfect code. since $\omega \geq 3$, we have
$$\frac{n(n-1)(n-2)}{3!} = 2(3(n-3)+1) \longrightarrow n^3 - 3n^2 - 34n + 96 = 0 \longrightarrow (n-6)(n^2+3n-16) = 0$$
Its only integer root is 6.

Example 2.4. *Consider the graph $J(6, 3)$ with vertex set:*

$$
\begin{aligned}
&v_1+v_2+v_3+v_4+v_5+v_6+v_7+v_{18}+v_{19}+v_{20}=1\\
&v_1+v_2+v_3+v_4+v_6+v_8+v_9+v_{15}+v_{17}+v_{20}=1\\
&v_1+v_2+v_3+v_4+v_7+v_8+v_{10}+v_{15}+v_{16}+v_{19}=1\\
&v_1+v_2+v_3+v_4+v_5+v_9+v_{10}+v_{16}+v_{17}+v_{18}=1\\
&v_1+v_4+v_5+v_6+v_7+v_9+v_{10}+v_{12}+v_{13}+v_{18}=1\\
&v_1+v_2+v_5+v_7+v_8+v_9+v_{13}+v_{14}+v_{20}=1\\
&v_1+v_3+v_5+v_6+v_7+v_8+v_{10}+v_{12}+v_{14}+v_{19}=1\\
&+v_2+v_3+v_6+v_7+v_8+v_9+v_{10}+v_{11}+v_{14}+v_{15}=1\\
&+v_2+v_4+v_5+v_6+v_8+v_9+v_{10}+v_{11}+v_{13}+v_{17}=1\\
&+v_3+v_4+v_5+v_7+v_8+v_9+v_{10}+v_{11}+v_{12}+v_{16}=1\\
&+v_8+v_9+v_{10}+v_{11}+v_{12}+v_{13}+v_{14}+v_{15}+v_{16}+v_{17}=1\\
&+v_5+v_7+v_{10}+v_{11}+v_{12}+v_{13}+v_{14}+v_{17}+v_{18}+v_{19}=1\\
&+v_5+v_6+v_9+v_{11}+v_{12}+v_{13}+v_{14}+v_{17}+v_{18}+v_{20}=1\\
&+v_6+v_7+v_8+v_{11}+v_{12}+v_{13}+v_{14}+v_{15}+v_{19}+v_{20}=1\\
&+v_2+v_3+v_8+v_{11}+v_{14}+v_{15}+v_{16}+v_{17}+v_{19}+v_{20}=1\\
&+v_3+v_4+v_{10}+v_{11}+v_{12}+v_{15}+v_{16}+v_{17}+v_{18}+v_{19}=1\\
&+v_2+v_4+v_9+v_{11}+v_{13}+v_{15}+v_{16}+v_{17}+v_{18}+v_{20}=1\\
&v_1+v_4+v_5+v_{12}+v_{13}+v_{16}+v_{17}+v_{18}+v_{19}+v_{20}=1\\
&v_1+v_3+v_7+v_{12}+v_{14}+v_{15}+v_{16}+v_{18}+v_{19}+v_{20}=1\\
&v_1+v_2+v_6+v_{13}+v_{14}+v_{15}+v_{17}+v_{18}+v_{19}+v_{20}=1
\end{aligned}
$$

$$
\begin{array}{lllll}
V=\{ & v_1=(000111), & v_2=(001110), & v_3=(001101), & v_4=(001011)\\
& v_5=(010011), & v_6=(010110), & v_7=(010101), & v_8=(011100)\\
& v_9=(011010), & v_{10}=(011001), & v_{11}=(111000), & v_{12}=(110001)\\
& v_{13}=(110010), & v_{14}=(110100), & v_{15}=(101100), & v_{16}=(101001)\\
& v_{17}=(101010), & v_{18}=(100011), & v_{19}=(100101), & v_{20}=(100110)\}
\end{array}
$$

The system will be as what is shown above. Therefore, if we put $v_1 = 1$, *and* $v_j = 0, j = 1, 2, \cdots, n$, $j \neq 1, 11$, *then* $C = \{v_1, v_{11}\}$ *is a 1-perfect code. Also* $B_1(v_1) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_{18}, v_{19}, v_{20}\}$ *and* $B_1(v_{11}) = \{v_{11}, v_8, v_9, v_{10}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}\}$. *As it is illustrated,* $B_1(v_1)$ *and* $B_1(v_{11})$ *form a partition of* V. *If we color* $\{v_1, v_{11}\}$ *in white and others in black, we obtain a perfect 2-coloring with the parameter matrix*

$$P = \begin{bmatrix} 0 & 9 \\ 1 & 8 \end{bmatrix}$$

Notice that, in this example, $C = \{v_i, v_{i+10}\}; i = 1, 2, \cdots, 10$ *are also 1-perfect codes.*

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