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Linear maps and covariance sets

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Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is C^*-Jordan homomorphism if it is a Jordan homomorphism which preserves the adjoint operation. In this note we show that C^*-Jordan homomorphisms -under mild assumptions- preserving covariance set and covariance coset in C^*-algebras.

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1. Introduction and preliminaries

Suppose that $\mathcal{A}$ is a C^*-algebra with identity 1. An element $a \in \mathcal{A}$ is called *regular* if it has a generalized inverse in $\mathcal{A}$, i.e. there exists $b \in \mathcal{A}$ such that

$$aba = a.$$

We say that an element $a \in \mathcal{A}$ *is Moore-Penrose invertible if there exists* $b \in \mathcal{A}$ *such that*

$$aba = a, \quad bab = b, \quad (ab)^* = ab \quad \text{and} \quad (ba)^* = ba.$$

It is well known that the Moore-Penrose inverse (briefly, MP–inverse) is unique if it exists. We reserve the notation $a^\dagger$ for the MP–inverse of a. In what follows, we will denote by $\mathcal{A}^{-1}$ the subset of invertible elements of $\mathcal{A}$ and by $\mathcal{A}^\dagger$, the set of all MP–invertible elements of $\mathcal{A}$. The *commutator* of a pair of elements x and y in $\mathcal{A}$ is given by

$$[x, y] = xy - yx.$$

Note that $[x, y] = 0$ if and only if x and y commute.

In the next section we need the following definition of covariance set which was studied in [2]

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Definition 1.1. *[2] For a given element $a \in \mathcal{A}^\dagger$ with MP–inverse $a^\dagger$ we will denote the covariance set by $\mathfrak{C}(a)$ and define;*

$$\mathfrak{C}(a) = \{b \in \mathcal{A}^{-1} : (bab^{-1})^\dagger = ba^\dagger b^{-1}\}. \tag{1.1}$$

Also the notion of covariance coset was introduced and studied in [1]. In fact, this set is defined by reversing the roles of a and b in $\mathfrak{C}(a)$ and is denoted by $\mathfrak{B}(b)$. i.e.,

$$\mathfrak{B}(b) = \left\{a \in \mathfrak{A}^\dagger : (bab^{-1})^\dagger = ba^\dagger b^{-1}\right\}. \tag{1.2}$$

The porpose of this work is to show that under weak assumptions, C^*-Jordan homomorphisms preserving covariance set and covariance coset in C^*-algebras.

2. Main results

We recall the following definitions and theorems which will be needed to prove some of our results.

Definition 2.1. *[3] We say that a C^*-algebra $\mathcal{A}$ is of real rank zero if the set formed by all the real linear combinations of (orthogonal) projections is dense in the set of self-adjoint elements of $\mathcal{A}$.*

Remark 2.2. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. It is well known that (see [3]) the property of the above definition is satisfied by every von Neumann algebra, and in particular by the C^*-algebra $B(H)$ of all bounded linear operators on a Hilbert space H, and by the Calkin algebra $C(H) = \frac{B(H)}{K(H)}$.*

Definition 2.3. *We say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is C^*-Jordan homomorphism if it is a, Jordan homomorphism which preserves the adjoint operation, i.e.*

$$\phi(x^*) = (\phi(x))^* \qquad \forall x \in \mathcal{A}.$$

The C^*-homomorphism and C^*-anti-homomorphism are analogously defined.

In 2012, Boudi and Mbekhta [3] proved the following theorem.

Theorem 2.4. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero and $\mathcal{B}$ a prime C^* -algebra. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective, unital linear map. Then the following conditions are equivalent:*

1) $\phi(x^\dagger) = (\phi(x))^\dagger$ for all $x \in A^\dagger$;

2) ϕ is either a C^*-homomorphism or a C^*-anti-homomorphism.

Proof . See [3, Theorem 3.3]. □

The next proposition describes a relation between the covariance set $\mathfrak{C}(a)$, and commutators. It was proved in [2].

Proposition 2.5. *Let $a \in \mathcal{A}^\dagger$ with MP–inverse $a^\dagger$. Then the following statements are equivalent:*

(i) $b \in \mathfrak{C}(a)$;

(ii) $[b^*b, aa^\dagger] = 0$ and $[b^*b, a^\dagger a] = 0$.

A similar result also is true for covariance coset:

Proposition 2.6. *[1] Assume $b \in \mathfrak{A}^{-1}$. Then the following statements are equivalent:*

(i) $a \in \mathfrak{B}(b)$;

(ii) $\left[a^\dagger a, b^*b\right] = 0$ and $\left[aa^\dagger, b^*b\right] = 0$.

Now we are going to prove the main result.

Theorem 2.7. *Let $\mathcal{A}$ be a C^*-algebra of real rank zero and $\mathcal{B}$ a prime C^* -algebra. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective, unital linear map. If $\phi(x^\dagger) = (\phi(x))^\dagger$ for all $x \in A^\dagger$, then $\phi(\mathfrak{C}(a)) = \mathfrak{C}(\phi(a))$ and $\phi(\mathfrak{B}(a)) = \mathfrak{B}(\phi(a))$.*

Proof . By Theorem 2.4, ϕ is either a C^*-homomorphism or a C^*-anti-homomorphism. First we assume that ϕ is a C^*-homomorphism. Let $b \in \mathfrak{C}(a)$. By Proposition 2.5

$$b^*baa^\dagger = aa^\dagger b^*b, \qquad b^*ba^\dagger a = a^\dagger ab^*b \tag{2.1}$$

Since ϕ is a C^*-homomorphism, from (2.1) we get

$$\phi(b)^* \phi(b) \phi(a) \phi(a)^\dagger = \phi(a) \phi(a)^\dagger \phi(b)^* \phi(b),$$
$$\phi(b)^* \phi(b) \phi(a)^\dagger \phi(a) = \phi(a)^\dagger \phi(a) \phi(b)^* \phi(b)$$

which means that $\phi(b) \in \phi(\mathfrak{C}(a))$ i.e. $\phi(\mathfrak{C}(a)) \subset \mathfrak{C}(\phi(a))$. Since ϕ is surjective we get $\phi(\mathfrak{C}(a)) = \mathfrak{C}(\phi(a))$.

Now we suppose that ϕ is a C^*-anti-homomorphism. Let $b \in \mathfrak{C}(a)$. Again by Proposition 2.5 we have (2.1). Applying ϕ on (2.1) we get

$$\phi\left(aa^\dagger\right) \phi(b^*b) = \phi(b^*b) \phi\left(aa^\dagger\right), \quad \phi\left(a^\dagger a\right) \phi(b^*b) = \phi(b^*b) \phi\left(a^\dagger a\right). \tag{2.2}$$

Since ϕ is a C^*-anti-homomorphism and $\phi(x^\dagger) = (\phi(x))^\dagger$ from (2.2) we obtain

$$\phi\left(a\right)^\dagger \phi(a) \phi(b)\phi(b^*) = \phi(b)\phi(b^*) \phi\left(a\right)^\dagger \phi(a),$$
$$\phi(a) \phi(a)^\dagger \phi(b)\phi(b^*) = \phi(b)\phi(b^*) \phi(a) \phi(a)^\dagger$$

Now by using Proposition 2.5 we conclude that $\phi(b) \in \phi(\mathfrak{C}(a))$ i.e. $\phi(\mathfrak{C}(a)) = \mathfrak{C}(\phi(a))$.

Applying Proposition 2.6, a similar argument shows that $\phi(\mathfrak{B}(a)) = \mathfrak{B}(\phi(a))$. $\square$

By Theorem 2.7 and Remark 2.2, we deduce the following results.

Corollary 2.8. *Assume that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and also von Neumann algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective, unital linear map. If $\phi(x^\dagger) = (\phi(x))^\dagger$ for all $x \in A^\dagger$, then $\phi(\mathfrak{C}(a)) = \mathfrak{C}(\phi(a))$ and $\phi(\mathfrak{B}(a)) = \mathfrak{B}(\phi(a))$.*

Corollary 2.9. *Suppose that H and K are Hilbert spaces. Let $\phi : B(H) \to B(K)$ be a surjective linear map. If $\phi(T^\dagger) = (\phi(T))^\dagger$ for all $T \in B(H)^\dagger$, then $\phi(\mathfrak{C}(T)) = \mathfrak{C}(\phi(T))$ and $\phi(\mathfrak{B}(T)) = \mathfrak{B}(\phi(T))$.*

Let $n \in \mathbb{N}$. We say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is n-C^*-Jordan homomorphism if it is a, n-Jordan homomorphism (for more detail see [4]) which preserves the adjoint operation.

Question: For wich $n \in \mathbb{N}$, the above results are true for n-C^*-Jordan homomorphism?

In connection with Theorem 2.7, we conclude the paper by the following conjecture:

Conjecture 2.10. *Assume that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective, unital linear map. If $\phi(x^\dagger) = (\phi(x))^\dagger$ for all $x \in A^\dagger$, then $\phi(\mathfrak{C}(a)) = \mathfrak{C}(\phi(a))$ and $\phi(\mathfrak{B}(a)) = \mathfrak{B}(\phi(a))$.*

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