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A graph associated to proper non-small subsemimodules of a semimodule

Ahmed H. Alwan[a,*]

[a]*Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq*

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Abstract

Let M be a unitary left R-semimodule where R is a commutative semiring with identity. The small intersection graph $G(M)$ of a semimodule M is an undirected simple graph with all non-small proper subsemimodules of M as vertices and two distinct vertices N and L are adjacent if and only if $N \cap L$ is not small in M. In this paper, we investigate the fundamental properties of these graphs to relate the combinatorial properties of $G(M)$ to the algebraic properties of the R-semimodule M. We determine the diameter and the girth of $G(M)$. Moreover, we study cut vertex, clique number, domination number and independence number of the graph $G(M)$. It is shown that the independence number of small graph is equal to the number of its maximal subsemimodules.

Keywords: small subsemimodule, small intersection graph, clique number, domination number, independence number.
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1. Introduction

In 1988, Beck [6] introduced the concept of the zero-divisor graph, but this work was mostly concerned with colorings of rings. Recently, the study of such graphs of rings are extended to include semirings and modules as in [4, 5, 10].

In 1964, Bosak [8] defined the intersection graph of semigroups. In 2009, the intersection graph of ideals of a ring was considered by Chakrabarty et al. [9].

The intersection graph of ideals of rings and submodules of modules has been investigated by several authors ([1, 13, 19]). Atani et al. [11] studied small intersection graph of ideals.

*Corresponding author
Email address: ahmedha_math@utq.edu.iq (Ahmed H. Alwan)

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In this paper, we introduce small intersection graph of subsemimodules of a semimodule M, denoted by $G(M)$, as a natural extension of the small intersection graph of ideals of a commutative ring. In particular, we define $G(R)$ the small intersection graph of ideals of a semiring R in a analogous manner.

In Section 2, we show that the small intersection graph of a semimodule M is connected if and only if $|\max(M)| \neq 2$. Also if $G(M)$ is a connected graph, then $diam(G(M)) \leq 2$ and $gr(G(M)) = 3$ provided $G(M)$ contains a cycle. For a semimodule M, it is proved that $G(M)$ cannot be a complete r-partite graph and $G(M)$ has no cut vertex. Also, if M is a semimodule with finitely many maximal subsemimodules, then $G(M)$ cannot be complete.

In Section 3, it is proved that if $\omega(G(M))$ is finite, then the number of maximal subsemimodules of R-semimodule M is finite, R and so ${}_RR$ is semiperfect and R has finitely many maximal ideals. This enables us to show that, if the set of proper non-small ideals is non-empty and finite, then the set of ideals of R is finite. Other results, it is shown that the domination number of a small graph is at most 2 and the independence number of a small graph of semimodule is equal to the number of its maximal subsemimodules.

Throughout this paper R is a commutative semiring with identity and M is a unitary left R-semimodule. A *commutative semiring* R is defined as an algebraic system $(R, +, \cdot)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected by $a(b+c) = ab + ac$ for all $a, b, c \in R$, and there exists $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$. A nonempty subset I of R is defined to be an *ideal* of R if $a, b \in I$ and $r \in R$ implies that $a + b, ra \in I$.

Let $(M, +)$ be an additive abelian monoid with additive identity 0_M, then M is a *left semimodule* over a semiring R (*left R-semimodule*) and denoted by ${}_RM$ if there exists a scalar multiplication $R \times M \to M$ denoted by $(r, m) \mapsto rm$, such that $(rr')m = r(r'm)$; $r(m + m') = rm + rm'$; $(r + r')m = rm + r'm$; and $r0_M = 0_M = 0m$ for all $r, r' \in R$ and all $m, m' \in M$. If the condition $1m = m$ for all $m \in M$ hold then the semimodule M is said to be *unitary*. A subset N of R-semimodule M is called a *subsemimodule* of M if for $n, n' \in N$ and $r \in R$, $n + n' \in N$ and $rn \in N$. Thus every semiring R is a left semimodule over itself, and each ideal I of R is a subsemimodule of ${}_RR$. A *subtractive* subsemimodule (or k-subsemimodule) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$. In similar manner we defined the k-ideals of R [3]. We say an R-semimodule is *subtractive* if each of its R-subsemimodules is subtractive ([3, 15]). In particular, if ${}_RR$ is a subtractive semimodule, we say that the semiring R is a subtractive semiring.

A subsemimodule N of M ($N \leq M$) is *small* (or superfluous) (denoted by $N \ll M$) if $N + L = M$, for some subsemimodule L of M, implies $L = M$ [15]. A semimodule M is said to be *hollow* semimodule if every proper subsemimodule of M is a small subsemimodule. A nonzero semimodule ${}_RM$ is called *simple* if it has no proper subsemimodules, and ${}_RM$ is said to be *semisimple* if it is a direct sum of its simple R-subsemimodules; in particular, R is *semisimple* if ${}_RR$ is, See [15].

An R-semimodule M is called *finitely generated* if there exists a non-empty finite subset S of M satisfying $RS = M$. If $S = \{s\}$ and $Rs = M$ then M is called *cyclic* [2]. An R-subsemimodule P of a semimodule M is *maximal* if and only if it is not properly contained in any other subsemimodule of M. In our investigation of $G(M)$, maximal subsemimodules play an important role to find some connections between the graph theoretic properties of this graph and some algebraic properties of semimodules. An R-semimodule M is said to be *local* if it has a unique maximal subsemimodule P and we denote it by (M, P). The set of maximal subsemimodules of M is denoted by $\max(M)$, and the intersection of all maximal subsemimodules of M is called the *Jacobson radical* of M and is denoted by $J(M)$. Similarly the Jacobson radical of R will be denoted by $J(R)$. A semiring R is *Artinian* if and only if every non-empty set of ideals of R has a minimal element, see [15, Proposition 2.1 (iv)].

References for graph theory are [7] and [17]; for commutative semiring theory and semimodules, see [12] and [15].

Let G be a graph. Then $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively. In addition, for two distinct vertices u and v in G, the notation $\{u, v\} \in E(G)$ means that u and v are adjacent. The degree of a vertex v of any graph G is denoted by $\deg(v)$ and defined as the number of edges incident on v. A vertex of degree 0 is called *isolated.* The complete graph of order n, denoted by K_n, is a graph with n vertices in which every two distinct vertices are adjacent.

For a positive integer n, an n-partite graph is one whose vertex set $V(G)$ can be partitioned into n subsets $V_1, V_2, \dots, V_n$ (called partite sets) such that every element of $E(G)$ joins a vertex of V_i to a vertex of $V_j, i \neq j$. The *complete bipartite* graph (2-partite graph) with exactly two partitions of size m and n is denoted by $K_{m,n}$. A graph G is said to be *star* if $G = K_{1,n}$. Two vertices u and v of a graph G are said to be connected in G if there exists a path between them. A graph G is called *connected* if there exists a path between any two distinct vertices. Otherwise, G is called disconnected. Let G be a connected graph.

The distance between two distinct vertices u and v of G, denoted by $d(u, v)$, is the length of the shortest path connecting u and v, if such a path exists; otherwise, we set $d(u, v) = \infty$. The *diameter* of a connected graph G is defined by $\text{diam}(G) = Max\{d(u, v) : u, v \in V(G)\}$. A vertex v of a connected graph G is a cut-vertex if the components of $G\backslash\{v\}$ are more than the components of G.

The *girth* of a graph G, denoted by $\text{gr}(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise; $\text{gr}(G) = \infty$. A *complete* subgraph K_n of a graph G is called a *clique*, and $\omega(G)$ is the clique number of G, which is the greatest integer $r \geq 1$ such that $K_r \subseteq G$. Note that a graph whose vertices-set is empty is a *null* graph and a graph whose edge-set is empty is an *empty* graph.

2. Fundamental properties of $G(M)$

Let M be an R-semimodule. In this section, we introduce some basic definitions and properties of the small intersection graph $G(M)$.

The next result shows the existence of a maximal subsemimodule in a semimodule which is similar to the case of semirings with identity [12, Proposition 6.59].

Proposition 2.1. *If M is a non-zero finitely generated R-semimodule, then M possesses a maximal subsemimodule.*

Proof . By Proposition 2.1 in [15]. □

We remark that any R-semimodule M in this paper possesses a maximal R-submodule, and every proper subsemimodule of M is contained in a maximal subsemimodule of M.

In the following remark we recall the definition of factor semimodule see [12, Example 15.3].

Remark 2.2. *If N is a subsemimodule of a left R-semimodule M, then N induces an R-congruence relation $\equiv_N$ on M, called the Bourne relation, defined by setting $m \equiv_N m'$ if and only if there exist elements n and n' of N such that $m + n = m' + n'$. If $m \in M$ then we write $m/N = m + N$ instead of $m/\equiv_N$. The factor semimodule $M/\equiv_N$ is denoted by M/N.*

Note that if N is a k-subsemimodule of an R-semimodule M, then M/N is an R-semimodule.

Remark 2.3. **(i)** *Let M be an R-semimodule and N, L be two subsemimodules of M. If P is a maximal subsemimodule of M, then $N \cap L \subseteq P$ implies $N \subseteq P$ or $L \subseteq P$.*

(ii) *Let M be an R-semimodule with $max(M) = \{M_i\}_{i\in I}$ and ν be a proper finite subset of I. Then $\cap M_i$ is a non-small subsemimodule of M. Otherwise, if $\cap_\nu M_i \ll M$, then $\cap_\nu M_i \subseteq M_j$ for each $j \in I\backslash\nu$. So $M_i \subseteq M_j$ for some $i \in \nu$, which is a contradiction.*

Now, we give the definition of small intersection graph of subsemimodules of a semimodule.

Definition 2.4. *Let M be an R-semimodule. The small intersection graph $G(M)$ is the graph with all non-small proper subsemimodules of M as vertices and two distinct vertices N and L are adjacent if and only if $N \cap L$ is not small in M.*

Proposition 2.5. *Let M be an R-semimodule. Then $G(M)$ is a null graph if and only if M is a local semimodule.*

Proof . Clear. □

Example 2.6. *Here, we will give two semimodules with its null graphs.*

(1) *Let $\mathbb{N}$ be the semiring of nonnegative integers and consider $M = \mathbb{N}$ be an $\mathbb{N}$-semimodule. It is clear that M is a local semimodule with maximal subsemimodule $\mathbb{N}\backslash\{1\}$. Thus $G(M)$ is a null graph.*

(2) *Let $R = \{0, x, 1\}$, define operations of addition and multiplication on R as follows.*

(a) $0_R = 0$, $1_R = 1$;

(b) $1 + 1 = 1 + x = 1$, $x + 0 = x + x = x$;

(c) $0 \times 0 = 0 \times 1 = 0 \times x = 0$, $1 \times 1 = 1$, $1 \times x = x \times x = x$.

Then $(R, +, \times)$ is a commutative semiring. Let $M =_R R$. It is not difficult to see that M is a local semimodule with maximal subsemimodule $\{0, x\}$. Thus $G(M)$ is a null graph.

Since all definitions of graph theory are for non-null graph, so we remark that all graphs in this paper are considered non-null ([7]).

Proposition 2.7. *Let P be a proper subsemimodule of R-semimodule M. Then P is maximal if and only if for each $a \in M\backslash P$, $Ra + P = M$.*

Proof . the proof follows directly from the definition of a maximal subsemimodule. □

Now, we have a further important Statement for cyclic subsemimodules which are not small. The proof of the following lemma as in modules see [14, Lemma 5.1.4].

Lemma 2.8. *For $a \in {}_RM$ we have: Ra is not small in M if and only if there is a maximal subsemimodule C of M with $a \notin C$.*

Proof . ($\Rightarrow$) If C is a maximal subsemimodule of M with $a \notin C$ then it follows that $Ra + C = M$, thus Ra is not small in M.

($\Leftarrow$) Proof by the use of Zorn's Lemma. Let

$$\Omega = \{N | N \subsetneqq M \wedge Ra + N = M\}.$$

Since Ra is not small, there is a $N \in \Omega$, i.e. $\Omega \neq \emptyset$.

Let $\Lambda \neq \emptyset$ be a totally ordered (with respect to inclusion) subset of Ω. Then

$$N_0 = \cup_{N\in\Lambda} N$$

is an upper bound of Λ. Assume $a \in N_0$, then a must already be contained in N; from which it would follow that $Ra \leq N$, hence $N = Ra + N = M$, a contradiction.

As $a \notin N_0$ it follows that $N_0 \lneqq M$. Since $N \leq N_0$ for any $N \in \Lambda$, then $Ra + N_0 = M$, thus we have $N_0 \in \Omega$, i.e. Λ has an upper bound in Ω. Zorn's Lemma implies then that Ω contains a maximal element C.

We claim that C is in fact a maximal subsemimodule of M. Let $C \lneqq B \leq M$, then it follows that $B \notin \Omega$, since C is maximal in Ω. From $M = Ra + C \leq Ra + B \leq M$ it follows that $Ra + B = M$ and as $B \notin \Omega$ we must have $B = M$. This completes the proof. □

Theorem 2.9. *Let M be an R-semimodule in which every maximal subsemimodule is subtractive. Then $G(M)$ is an empty graph if and only if $max(M) = \{M_1, M_2\}$,where M_1 and M_2 ($M_1 \neq M_2$) are finitely generated hollow R-semimodules.*

Proof . Let $G(M)$ be an empty graph. If $|\max(M)| = 1$, then $G(M)$ is a null graph by Proposition 2.5, a contradiction. Assume, $|\max(M)| \geq 3$ and M_1, M_2 and $M_3 \in \max(M)$. By Remark 2.3, M_1 and M_2 are adjacent, a contradiction. So $|\max(M)| = 2$. Let $\max(M) = \{M_1, M_2\}$ with $M_1 \neq M_2$. We show that M_1 and M_2 are hollow R-semimodules. Since $\frac{M}{M_2} = \frac{M_1+M_2}{M_2} \cong \frac{M_1}{M_1 \cap M_2}$, $M_1 \cap M_2$ is a maximal subsemimodule of M_1. We show that this is the only maximal subsemimodule of M_1. Let N be a maximal subsemimodule of M_1. If N is not small in M, then $N \cap M_1 = N$ implies N and M_1 are adjacent in $G(M)$, a contradiction. So $N \ll M$. Hence $N \subseteq J(M) = M_1 \cap M_2$, which implies that $N = M_1 \cap M_2$ by maximality of N. So M_1 is a local R-semimodule with maximal subsemimodule $M_1 \cap M_2$. Now, we show that M_1 is a finitely generated R-semimodule. Let $a \in M_1 \backslash M_2$, so Ra is not small of T because $Ra \nsubseteq M_1 \cap M_2 = J(M)$. If $Ra \neq M_1$, then $Ra \cap M_1 = Ra$ which implies Ra and M_1 are adjacent in $G(M)$, a contradiction. So $Ra = M_1$. Thus M_1 is a finitely generated local R-semimodule. Therefore as in modules [18], then M_1 is a finitely generated hollow R-semimodule. By the similar manner M_2 is a finitely generated hollow R-semimodule.

Conversely, let $\max(M) = \{M_1, M_2\}$, where M_1 and M_2 are finitely generated hollow R-semimodules. By a similar argument as above, $M_1 \cap M_2$ is a maximal subsemimodule of M_1 and M_2. Since M_1 and M_2 are local, $M_1 \cap M_2$ is the only maximal subsemimodule of M_1 and M_2. Let $N \neq M_1, M_2$ be a non-small subsemimodule of M. Then $N \subseteq M_1$ or $N \subseteq M_2$. Suppose, without loss of generality, $N \subseteq M_1$. Since M_1 is a finitely generated local R-semimodule, $N \subseteq M_1 \cap M_2 = J(M)$. So $N \ll M$, a contradiction. So the only non-small subsemimodules of M are M_1 and M_2 which are not adjacent. So $G(M)$ is an empty graph. □

In the following we give an example of a semimodule M with empty $G(M)$.

Example 2.10. *Consider $M = \mathbb{Z}_6$ as a $\mathbb{Z}$-semimodule. It is clear that $\max(M) = \{(2), (3)\}$, and $J(M) = (0)$. It is easy to see that $G(M)$ is an empty graph with two vertices and (2), (3) are hollow.*

The next result shows the relationship between the number of maximal subsemimodules of M and the connectivity of $G(M)$.

Theorem 2.11. *Let M be a non-zero R-semimodule. The following statements are equivalent:*

(1) *$G(M)$ is not connected;*
(2) *$|max(M)| = 2$;*
(3) *$G(M) = G_1 \cup G_2$, where G_1, G_2 are two disjoint complete subgraphs of $G(M)$.*

Proof . (1) $\Rightarrow$ (2) Assume that $G(M)$ is not connected. Let G_1 and G_2 be two components of $G(M)$ and N, L be two subsemimodules of M such that $N \in G_1$ and $L \in G_2$. Let M_1, M_2 be maximal subsemimodules of M such that $N \subseteq M_1$ and $L \subseteq M_2$. If $M_1 = M_2$, then $N - M_1 - L$ is a path in $G(M)$ which is a contradiction. So $M_1 \neq M_2$. If $M_1 \cap M_2$ is not small in M, then $N - M_1 - M_2 - L$ is a path between G_1 and G_2, which is a contradiction. Therefore $M_1 \cap M_2 \ll M$, which gives $|\max(M)| = 2$.

(2) $\Rightarrow$ (3) Let $|\max(M)| = 2$ and $J(M) = M_1 \cap M_2$, where M_1, M_2 are two maximal subsemimodules of M. Let $G_i = \{N_k : N_k \subseteq M_i$ and N_k is a non-small subsemimodule of $M\}$ for $i = 1, 2$. Let N, L be elements of G_1. If N and L are not adjacent then $N \cap L \ll T$, which implies $N \cap L \subseteq M_1 \cap M_2$. Hence $N \cap L \subseteq M_2$, which gives $N \subseteq M_2$ or $L \subseteq M_2$ by Remark 2.3. So $N \ll M$ or $L \ll M$, a contradiction. So G_1 is a complete subgraph of $G(M)$. By the similar manner G_2 is a complete subgraph of $G(M)$. Now, we show that there is no path between G_1 and G_2. Suppose, on the contrary, N and L are adjacent for some subsemimodules $N \in G_1$ and $L \in G_2$ (note that each vertex in $G(M)$ is contained in G_1 or G_2). Since $N \cap L \subseteq M_1 \cap M_2 = J(M)$, so $N \cap L \ll M$, a contradiction with adjacency of N and L. So none of elements of G_1 and G_2 are adjacent. Hence $G(M) = G_1 \cup G_2$, where $Gi's$ are disjoint complete subgraph of $G(M)$.

(3) $\Rightarrow$ (1) Clear. □

In the following we provide an example of a semimodule M with two maximal subsemimodules such that $G(M)$ is not connected.

Example 2.12. *Let $M = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ be a $\mathbb{Z}$-semimodule. It is clear that* $\max(M) = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 2\mathbb{Z}_4\}$ *and $G(M)$ is disconnected. See that $V(G(M)) = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 2\mathbb{Z}_4, 0 \oplus \mathbb{Z}_4, \mathbb{Z}_4 \oplus 0\}$, and $G(M) = G_1 \cup G_2$, where $G_1 = \{2\mathbb{Z}_4 \oplus \mathbb{Z}_4, 0 \oplus \mathbb{Z}_4\}$ and $G_2 = \{\mathbb{Z}_4 \oplus 2\mathbb{Z}_4, \mathbb{Z}_4 \oplus 0\}$.*

Theorem 2.13. *Let M be an R-semimodule and $G(M)$ be a connected graph, then diam$(G(M)) \leq 2$.*

Proof . Let N and L be two non-adjacent vertices of $G(M)$. Hence $N \cap L \ll M$. Assume that $N \subseteq M_1$ and $L \subseteq M_2$ for some maximal subsemimodules M_1, M_2 of M. If $N \cap M_2$ is not small in M, then $N - M_2 - L$ is a path in $G(M)$, thus $d(N, L) = 2$. By the similar way if $L \cap M_1$ is a non-small subsemimodule of M, then $d(N, L) = 2$. Suppose $N \cap M_2 \ll M$ and $L \cap M_1 \ll M$. Since $G(M)$ is connected by Theorem 2.11, $|\max(M)| \geq 3$. Let $M_3 \in \max(M)$. Since $N \cap L \ll M$, so $N \cap L \subseteq J(M) \subseteq M_3$ which implies $N \subseteq M_3$ or $L \subseteq M_3$. Assume, without loss of generality, $N \subseteq M_3$. Now, we show that $L \cap M_3$ is a non-small subsemimodule of M. If $L \cap M_3 \ll M$, then $L \cap M_3 \subseteq J(M) \subseteq M_1$, which implies $L \subseteq M_1$. Thus $L = L \cap M_1 \ll M$, a contradiction. So $L \cap M_3$ is not small in M. Thus $N - M_3 - L$ is a path in $G(M)$. Hence $d(N, L) = 2$. □

Theorem 2.14. *Let M be an R-semimodule. If $G(M)$ contains a cycle, then $gr(G(M)) = 3$.*

Proof . If $|\max(M)| = 2$, then $G(M)$ is a union of two disjoint complete subgraph by Theorem 2.11. Thus if $G(M)$ contains a cycle, then $gr(G(M)) = 3$. If $|\max(M)| \geq 3$, then by Remark 2.3, $M_1 - M_2 - M_3 - M_1$ is a cycle in $G(M)$, where $M_i \in \max(M)$. Therefore $gr(G(M)) = 3$. □

Theorem 2.15. *Let M be an R-semimodule with $G(M)$ connected. Then $G(M)$ has no cut vertex.*

Proof . Let B be a cut vertex of $G(M)$, so $G(M) \setminus \{B\}$ is not connected. Therefore there exist vertices N, L such that B lies on every path from L to N. By Theorem 2.13, the shortest path from B to N is of length 2. So $N - B - L$ is a path between N, L. Thus $N \cap L \ll M$, $N \cap B$ is not small in M and $L \cap B$ is not small in M. Firstly, we prove that B is a maximal subsemimodule of M. If not, so there exists a subsemimodule H of M such that $B \subseteq H$ (as B is a non-small subsemimodule of M, H is non-small). Since $N \cap B \subseteq N \cap H$ and $N \cap B$ is not small in M, $N \cap H$ is not small in M. By a analogous way $L \cap H$ is a non-small subsemimodule of M. Hence $N - H - L$ is a path in $G(M)\backslash\{B\}$, a contradiction. So B is a maximal subsemimodule of M. We claim that there exists a maximal subsemimodule $M_i \neq B$ of M such that $N \nsubseteq M_i$. Otherwise, if $N \subseteq M_i$ for each $B \neq M_i \in \max(M)$, then $N \subseteq (\cap_{M_i \neq B} M_i)$, so $N \cap B \subseteq \cap_{M_i \in max(M)} M_i = J(M)$. Hence $N \cap B \ll M$, a contradiction. By the similar way there exists a maximal subsemimodule $M_j \neq B$ of M such that $L \nsubseteq M_j$. Now, we show that for each $M_t \in \max(M)$, $L \subseteq M_t$ or $N \subseteq M_t$. Because $N \cap L \ll M$, hence $N \cap L \subseteq J(M) \subseteq M_t$ for each $M_t \in \max(M)$. So $N \subseteq M_t$ or $L \subseteq M_t$ for each $M_t \in \max(M)$. Since $G(M)$ is connected, $|\max(M)| \geq 3$ by Theorem 2.11. Now, let $B \neq M_i, M_i \in \max(M)$ such that $L \nsubseteq M_i$ and $N \nsubseteq M_j$. Thus $L \subseteq M_j$ and $N \subseteq M_i$. So $N - M_i - M_j - L$ is a path in $G(M)\backslash\{B\}$, a contradiction. Hence $G(M)$ has no cut vertex. □

Theorem 2.16. *Let M be an R-semimodule. Then $G(M)$ cannot be a complete n-partite graph (n is a positive integer).*

Proof . Suppose that $G(M)$ is a complete n-partite graph with n parts $V_1, V_2, \ldots, V_n$. By Remark 2.3, M_i and M_j are adjacent, for each M_i, $M_j \in \max(M)$. So each V_i contains at most one maximal subsemimodule of M. Hence by Pigeon hole principle $|\max(M)| \leq n$. Now, we prove that $|\max(M)| = n$. In contrary way, assume $\max(M) = \{M_1, M_2, \ldots, M_m\}$, where $m < n$. Let $M_i \in V_i$ for $1 \leq i \leq m$. Hence V_{m+1} contains no maximal subsemimodule. Since $|\max(M)|$ is finite, by Remark 2.3, then $\cap_{j\neq i} M_j$ is a non-small subsemimodule of M. Since $\cap_{j\neq i} M_j \cap M_i = J(M) \ll M$, so $\cap_{j\neq i} M_j$ and M_i are not adjacent. Hence $\cap_{j\neq i} M_j \in V_i$, because $M_i \in V_i$. Let N be a vertex in V_{m+1} and $N \subseteq M_k$ for some $M_k \in \max(M)$. So N is adjacent to M_k. Since $G(M)$ is a complete n-partite graph and $M_k \in V_k$, so N is adjacent to all elements of V_k. Thus N is adjacent to $\cap_{j\neq k} M_j$, a contradiction, because $N \cap (\cap_{j\neq k} M_j) \subseteq M_k \cap (\cap_{j\neq k} M_j) = J(M) \ll M$. Thus $|\max(M)| = n$. Now, assume the subsemimodule $L = \cap_{i=3}^{n} M_i$. By Remark 2.3, L is not small in M. Since $L \cap M_1 = \cap_{i\neq 2} M_i$ is not small in M, L is adjacent to M_1. By the analogous way L is adjacent to M_2. So $L \notin V_1, V_2$. Further, $L \cap M_i = L$ is not small in M, for each $3 \leq i \leq n$. So L is adjacent to all maximal subsemimodules M_i of M. So $L \notin V_i$ for each $1 \leq i \leq n$, which is a contradiction. □

Theorem 2.17. *Let M be an R-semimodule with finitely many maximal subsemimodules. Then*

(1) *There is no vertex in $G(M)$ which is adjacent to every other vertex,*
(2) *$G(M)$ cannot be a complete graph.*

Proof . (1) Assume $\max(M) = \{M_1, M_2, \ldots, M_m\}$, where $m \leq n$. In contrary way, assume that $G(M)$ is a complete graph. So, any vertex N in $G(M)$ is adjacent to every other vertex. It is Clear that $N \subseteq M_i$ for some $M_i \in \max(M)$. By Remark 2.3, $H = \cap_{j\neq i} M_j$ is a non-small subsemimodule of M. Since N is adjacent to every vertex, N and K are adjacent. Hence $N \cap H$ is a non-small subsemimodule of M. But $N \cap H \subseteq M_i \cap (\cap_{j\neq i} M_j) = J(M)$. Thus $N \cap H \ll M$, which is a contradiction. Hence there is no vertex in $G(M)$ which is adjacent to every other vertex.

(2) From (1) we have $G(M)$ cannot be a complete graph. □

The condition $|\max(M)|$ is finite of Theorem 2.17 is not superfluous, as the next example shows.

Example 2.18. *Let M be the $\mathbb{Z}$-semimodule $\mathbb{Z}$. It is clear that max(M) is infinite and the only small subsemimodule of M is $\{0\}$. Since for every non-zero subsemimodules N and L of M, $N \cap L \neq \{0\}$, hence N and L are adjacent in $G(M)$. Hence $G(M)$ is a complete graph.*

Theorem 2.19. *Let M be an R-semimodule. Then the following hold:*

(1) $G(M)$ contains an end vertex if and only if $|\max(M)| = 2$ and $G(M) = G_1 \cup G_2$, where G_1, G_2 are two disjoint complete subgraph of $G(M)$ and $|V(G_i)| = 2$ for some $i = 1, 2$;

(2) $G(M)$ cannot be a star graph.

Proof . (1) Suppose that N is an end vertex of $G(M)$. Assume, $|\max(M)| \geq 3$. By Remark 2.3, for any $M_i \in \max(M)$, M_i is adjacent to every other maximal subsemimodules of M, so $deg(M_i) \geq 2$. Thus N is not a maximal subsemimodule of M. Without loss of generality, assume $N \subseteq M_1$, thus N and M_1 are adjacent. Since $deg(N) = 1$, hence M_1 is the only vertex of $G(M)$ which is adjacent to N and there is no maximal subsemimodule $M_i \neq M_1$ of M such that $N \subseteq M_i$. Also $N \cap M_2 \ll M$. Hence $N \cap M_2 \subseteq M_j$ for each $M_j \neq M_1, M_2$. So $N \subseteq M_j$, which is a contradiction. Thus $|\max(M)| = 2$. By Theorem 2.11, $G(M) = G_1 \cup G_2$, where G_1, G_2 are two complete subgraph of $G(M)$. Let $N \in G_i$. Since G_i is a complete subgraph of $G(M)$ and $deg(N) = 1$, $|V(G_i)| = 2$. This completes the proof since the converse is clear.

(2) Suppose that $G(M)$ is a star graph. Hence $G(M)$ contains an end vertex. So $|\max(M)| = 2$ by (1). By Theorem 2.11, $G(M)$ is not connected, which is a contradiction. Thus $G(M)$ cannot be a star graph. □

Proposition 2.20. *Let M be an R-semimodule. If N and L are two vertices of $G(M)$ such that $N \subseteq L$, then $deg(N) \leq deg(L)$.*

Proof . Let N and L be two vertices of $G(M)$ such that $N \subseteq L$. Let H be a vertex adjacent to N. So $N \cap H$ is a non-small subsemimodule of M, which implies $L \cap H$ is a non-small subsemimodule of M. Hence H is adjacent to L. Thus $deg(N) \leq deg(L)$. □

3. Clique number, domination number and independence number

In this section, we obtain some results on the clique number, domination number and independence number of the small graph. In the beginning, we find the clique number of $G(M)$.

Proposition 3.1. *Let M be an R-semimodule. The following statements hold.*

(1) $\omega(G(M)) \geq |max(M)|$.
(2) *If $\omega(G(M)) < \infty$, then the number of maximal subsemimodules of M is finite.*
(3) $\omega(G(M)) = 1$ *if and only if $max(M) = \{M_1, M_2\}$, where M_1 and M_2 are finitely generated subtractive hollow R-semimodules.*
(4) *If the number of maximal subsemimodules of M is finite, then $\omega(G(M)) \geq 2^{|max(M)|-1} - 1$.*

Proof . (1) By Remark 2.3, the subgraph of $G(M)$ with vertex set $\{M_i\}_{M_i \in max(M)}$ is a complete subgraph of $G(M)$. Hence $\omega(G(M)) \geq |max(M)|$.

(2) This is a direct consequence of (1).

(3) This is a direct consequence of Theorem 2.9.

(4) Let $\max(M) = \{M_1, M_2, \ldots, M_r\}$ and for each $1 \leq i \leq r$, consider

$$E_i = \{M_1, M_2, \ldots, M_{i-1}, M_{i+1}, \ldots, M_r\}.$$

Let $P(E_i)$ be the power set of E_i. For each $X \in P(E_i)$, set $S_X = \cap_{S \in X} S$. Then by Remark 2.3, the subgraph of $G(M)$ with vertex set $\{S_X\}_{X \in P(E_i) \setminus \{\emptyset\}}$ is a complete subgraph of $G(M)$. Since $|P(E_i) \setminus \{\emptyset\}| = 2^{|max(M)|-1} - 1$, so $|\{S_X\}_{X \in P(E_i) \setminus \{\emptyset\}}| = 2^{|max(M)|-1} - 1$. Thus $\omega(G(M)) \geq 2^{|max(M)|-1} - 1$. □

Definition 3.2. *An idempotent in a semiring R is an element e with $e^2 = e$. Let I be a k-ideal of a semiring R. Then an idempotent $x + I \in R/I$ can be lifted mod I, if there is an idempotent $e \in R$ such that $e + I = x + I$.*

Lam [16, p. 356] calls a ring R semiperfect if R/I is semisimple and idempotents in R/I can be lifted *mod* I. Analogously, we give the next definition.

Definition 3.3. *A semiring R is called semiperfect in case $R/J(R)$ is semisimple and every idempotent of $R/J(R)$ can be lifted mod $J(R)$.*

The semiring R is semiperfect if and only if the regular semimodule ${}_RR$ is semiperfect. As in modules, we can see that each subtractive local semimodule is semiperfect.

An ideal I of a semiring R is called *small* if $I + K = R$, for some ideal K of R, implies $K = R$ [15]. We use $\mathbb{I}(R)$ and $\mathbb{NSI}(R)$ to denote the set of ideals of R and the set of proper non-small ideals of R, respectively.

Theorem 3.4. *Let R be a semiring such that $\omega(G(R)) < \infty$. Then the following statements holds.*

(1) *If $J(R)$ is a subtractive ideal of R, then R is semiperfect.*
(2) *If $R = R_1 \times R_2 \times \cdots \times R_r$ where $r \geq 2, (R_i, P_i)$ is a local semiring, then $G(R)$ is finite.*
(3) *If R has the form as in (2), then R is Artinian.*
(4) *If R has the form as in (2), then $\omega(G(R)) \geq \max\{(\prod_{j=1,j\neq i}^{r} |\mathbb{I}(R_i)|) - 1 : 1 \leq i \leq r\}$.*

Proof . (1) Since $J(R)$ is a subtractive ideal of R. Then $R/J(R)$ is a semiring. Since $\omega(G(R)) < \infty$ then by Proposition 3.1, $\max(R)$ is finite. Therefore, $R/J(R)$ is semisimple. Now, we show that idempotent of $R/J(R)$ can be lifted. Let $x + J(R)$ be a nonzero idempotent of $R/J(R)$. Clearly $x \notin J(R)$, so $x^n \notin J(R)$ for each $n \in \mathbb{N}$. Thus $Rx \supseteq Rx^2 \supseteq Rx^3 \supseteq \cdots$ is a descending chain of non-small proper ideals of R (if $Rx^n = R$, then $x + J(R) = 1 + J(R)$) by Lemma 2.8. Since $\omega(G(R)) < \infty$, so there exists $n \in \mathbb{N}$ such that $Rx^n = Rx^{n+1}$. Thus $x^n = x^{n+1}r$ for some $r \in R$. Let $e = x^n r^n$. Then $e = (x^{n+1}r)r^n = x^{n+1}r^{n+1}$. This implies that $e = e^2$ and $x + J(R) = x^n + J(R) = x^{n+1}r + J(R) = (x^{n+1} + J(R))(r + J(R)) = (x + J(R))(r + J(R)) = xr + J(R)$. So, $x + J(R) = (x + J(R))^2 = (x + J(R))^n = (xr + J(R))^n = e + J(R)$. Thus R is semiperfect.

(2) Let $R = R_1 \times R_2 \times \cdots \times R_r$, where (R_i, P_i) is a local semiring for $1 \leq i \leq r$. As $G(R)$ is non-null, $r \geq 2$, by Proposition 2.5. Now, we will show that $G(R)$ is finite. It suffices to show that $\mathbb{I}(R_i)$ is finite for all $1 \leq i \leq r$. Suppose, on the contrary, $\mathbb{I}(R_i)$ is infinite for some $1 \leq i \leq r$. Put

$$\mathbb{E} = \{R_1 \times R_2 \times \cdots \times R_{i-1} \times F \times R_{i+1} \times \cdots \times R_r \mid F \in \mathbb{I}(R_i)\}.$$

Then $\mathbb{E}$ is an infinite clique in $G(R)$, which is a contradiction. Thus $\mathbb{I}(R_i)$ is finite for all $1 \leq i \leq r$. Hence $\mathbb{I}(R)$ is finite and so $G(R)$ is finite.

(3) From the proof of (2), we have $\mathbb{I}(R)$ is finite. Therefore, R is Artinian.

(4) Consider

$$C_j = \{L < R : L = L_1 \times L_2 \times \cdots \times L_{j-1} \times R_j \times L_{j+1} \times \cdots \times L_r, L_t \in \mathbb{I}(R_t), \text{ for } 1 \leq t \neq j \leq r\},$$

for each $1 \leq j \leq r$. As $0 \times 0 \times \cdots \times R_j \times \cdots \times 0 \subseteq L$ for each $L \in C_j$, C_j is a clique in R. Since $|C_j| = (\prod_{i=1,j\neq i}^{r} |\mathbb{I}(R_i)|) - 1$, therefore $\omega(G(R)) \geq \max\{(\prod_{j=1,j\neq i}^{r} |\mathrm{Id}(R_i)|) - 1 : 1 \leq i \leq r\}$. $\square$

Corollary 3.5. *Let $R = R_1 \times R_2 \times \cdots \times R_r$ where $r \geq 2, (R_i, P_i)$ is a local semiring such that $\mathbb{NSI}(R) \neq \emptyset$. Then $\mathbb{NSI}(R)$ is finite if and only if $\mathbb{I}(R)$ is finite.*

Proof . Let $\mathbb{NSI}(R) \neq \emptyset$. Then $G(R)$ is a non-null graph. If $\mathbb{NSI}(R)$ is finite, then by Theorem 3.4, $\omega(G(R))$ is finite and so $G(R)$ is finite. Thus $|\mathbb{I}(R)| < \infty$.

Conversely, let $\mathbb{I}(R)$ is finite. Since $\mathbb{NSI}(R) \subseteq \mathbb{I}(R)$, so $|\mathbb{NSI}(R)| < \infty$. $\square$

Proposition 3.6. *Let M be a semisimple R-semimodule isomorphic to $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ where $M_i, i = 1\ldots, n$ is a simple R-semimodule. Then $G(M)$ is a finite graph.*

Proof . Straightforward. □

Let G be a graph. By a dominating set for G we mean a subset D of the vertex set of G such that every vertex not in D is joined to at least one vertex in D by some edge. A dominating set D is called a minimal dominating set if D' is not a dominating set for any subset D' of D with $D' \neq D$. The domination number of G is the smallest of the cardinalities of the minimal dominating sets for G. For a graph G we denote by $\gamma(G)$ the domination number of G. See, for instance, [17]. In the following theorem, for a semimodule M, the domination number of $G(M)$ is determined.

Theorem 3.7. *Let M be an R-semimodule. Then the following hold:*

(1) $\gamma(G(M)) \leq 2$,
(2) *If $J(M)$ is a k-subsemimodule of M, then $max(M)$ is infinite if and only if $\gamma(G(M)) = 1$,*
(3) *If $J(M)$ is a k-subsemimodule of M, then $max(M)$ is finite if and only if $\gamma(G(M)) = 2$.*

Proof . (1) From $G(M)$ is non-null, we have $|\max(M)| \geq 2$ by Proposition 2.5. Consider $S = \{M_1, M_2\}$ where M_1, $M_2 \in \max(M)$. Let N be a vertex of $G(M)$. If $N \subseteq M_1$ or $N \subseteq M_2$, then $N \cap M_1$ is a non-small subsemimodule of M or $N \cap M_2$ is a non-small subsemimodule of M. Thus N is adjacent to M_1 or M_2. Suppose that $N \nsubseteq M_1$ and $N \nsubseteq M_2$. If N is not adjacent to M_1, then $N \cap M_1 \ll M$. So $N \cap M_1 \leq M_2$. This implies $N \subseteq M_2$, a contradiction. So N is adjacent to M_1. Similarly, N is adjacent to M_2. Thus $\gamma(G(M)) \leq 2$.

(2) Let $J(M)$ be a k-subsemimodule of M, then $M/J(M)$ is an R-semimodule. If $\max(M)$ is infinite, then $M/J(M)$ is not semisimple. Hence there is a subsemimodule N of M such that $N/J(M)$ is an essential subsemimodule of $M/J(M)$. So N is not small and for each subsemimodule B of M such that $J(M) \subset B$ we have $B \cap N$ is a non-small subsemimodule of M. Let F be a proper non-small subsemimodule of M. As $N \cap (F + J(M)) = J(M) + N \cap F$ is not small in M, $N \cap F$ is not small in M. So N is adjacent to every other vertex of $G(M)$, and hence $\gamma(G(M)) = 1$.

Conversely, suppose that $\gamma(G(M)) = 1$. Thus there is a subsemimodule which is adjacent to every other vertex of $G(M)$. So $\max(R)$ is infinite by Theorem 2.17.

(3) This is a direct consequence of Theorem 2.17 and (2). □

A graph $G = (V, E)$ is said to be *totally disconnected* if it has no edges. A set $S \subseteq V$ is an *independent set* if the subgraph induced by S is totally disconnected. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G.

Finally, the following result shown that the independence number of $G(M)$ is equal to $|\max(M)|$, for a semimodule M with a finite number of maximal subsemimodules.

Proposition 3.8. *Let M be an R-semimodule with a finite number of maximal subsemimodules. Then $\alpha(G(M)) = |max(M)|$.*

Proof . Assume that $\max(M)$ is finite and $\max(M) = \{M_1, M_2, \ldots, M_n\}$. As $\{\bigcap_{j=1, i\neq j}^{n} M_j\}_{i=1}^{n}$ is an independent set in $G(M)$, $n \leq \alpha(G(M))$. Let $\alpha(G(M)) = m$ and $S = \{N_1, N_2, \ldots, N_m\}$ be a maximal independent set in $G(M)$. For each $N \in S$, N is a non-small subsemimodule of M. So by Lemma 2.8, $N \nsubseteq P$ for some $P \in \max(M)$. If $m > n$, then by Pigeon hole principle, there exist $1 \leq i, j \leq n$ and $P \in \max(M)$ such that $N_i \nsubseteq P$ and $N_j \nsubseteq P$. Thus $N_i \cap N_j \nsubseteq P$. As S is an independent set in $G(M)$, N_i and N_j are not adjacent and $N_i \cap N_j \ll M$. Hence $N_i \cap N_j \subseteq P$, a contradiction. This proves that $\alpha(G(M)) = |\max(M)|$. If $\alpha(G(M)) = \infty$, then by a similar argument as above (by using Pigeon hole principle), we obtain a contradiction. Therefore $\alpha(G(M)) = |\max(M)|$. □

Remark 3.9. *The condition "max(M) is finite" in Proposition 3.8 is not superfluous. To see this, let* $R = \mathbb{Z}$ *and consider the R-semimodule* $M = \mathbb{Z}$*. Clearly,* $|max(M)| = \infty$*, while* $\alpha(G(M)) = 0$*.*

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