Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 511-526
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2019.18153.1991

Strong convergence theorems for minimization, variational inequality and fixed point problems for quasi-nonexpansive mappings using modified proximal point algorithms in real Hilbert spaces

Thierno Mohamadane Mansour Sow[a]

[a] *Amadou Mahtar Mbow University, Senegal*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, we investigate the problem of finding a common element of the solution set of convex minimization problem, the solution set of variational inequality problem and the solution set of fixed point problem with an infinite family of quasi-nonexpansive mappings in real Hilbert spaces. Based on the well-known proximal point algorithm and viscosity approximation method, we propose and analyze a new iterative algorithm for computing a common element. Under very mild assumptions, we obtain a strong convergence theorem for the sequence generated by the proposed method. Application to convex minimization and variational inequality problems coupled with inclusion problem is provided to support our main results. Our proposed method is quite general and includes the iterative methods considered in the earlier and recent literature as special cases.

Keywords: Convex minimization problem, Proximal point algorithm, Common fixed points, Quasi-nonexpansive mappings, Variational inequality problem.
2010 MSC: 47J05, 47H09.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|.\|$ respectively. Let K be a nonempty closed convex subset of H. An operator $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall\, x, y \in K,$$

*Thierno Mohamadane Mansour Sow
Email address: thierno.sow@uam.edu.sn (Thierno Mohamadane Mansour Sow)

Received: June 2019 *Accepted:* October 2019

A is said *α-inverse strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall\ x, y \in K.$$

It is immediate that if A is α- inverse strongly monotone, then A is monotone and Lipschitz continuous.

The problem find $u \in K$ such that

$$\langle Au, v - u\rangle \geq 0, \ \ \forall v \in K, \tag{1.1}$$

is called a variational inequality problem. We denote the set of solutions of variational inequality problem (1.1) by $VI(A, K)$. Please note that on the one hand, this problem takes into account some special cases, in signal processing, networking, resource allocation, image recovery, and so on, the constraints can be expressed as variational inequality problems. Consequently, the problem of finding solutions of variational inequality problems has become a flourishing area of contemporary research for numerous mathematicians working in nonlinear operator theory (see, for example, [6, 21] and the references contained in them). In most of the early results on iterative methods for approximating solutions of variational inequality problem, the map A was often assumed to be inverse strongly monotone.
A well known method for solving the variational inequality problem is the projection algorithm which starts with $x_1 \in K$ and generates a sequence $\{x_n\}$ using the following recursion formula,

$$x_{n+1} = P_K(x_n - \lambda_n Ax_n), \ n \geq 1, \tag{1.2}$$

where $\{\lambda_n\}$ a sequence of positive numbers satisfying appropriate conditions. In the case that A is α-inverse strongly monotone, Iiduka et al. [12] proved that the sequence $\{x_n\}$ generated by (1.2) converges weakly to an element of $VI(A, K)$. Furthermore, it is worth pointing out that related iterative methods for solving variational inequality can be found in [1, 7, 20, 22].
Let E be a real normed space, K be a nonempty subset of E. A map $T : K \to E$ is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \forall x, y \in K, \tag{1.3}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive.
We denote by $Fix(T)$ the set of fixed points of the mapping T, that is $Fix(T) := \{x \in D(T) : x = Tx\}$. We assume that $Fix(T)$ is nonempty. If T is nonexpansive mapping, it is well known $Fix(T)$ is closed and convex. A map T is called quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ holds for all x in K and $p \in Fix(T)$.
The mapping $T : K \to K$ is said to be firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \forall x, y \in K.$$

We note that the following inclusions hold for the classes of the mappings:

firmly nonexpansive $\subset$ nonexpansive $\subset$ quasi-nonexpansive.

We illustrate these by the following example.

Example 1.1. *Let* $X = l_\infty$ *and* $C := \{x \in l_\infty : \|x\|_\infty \leq 1\}$ *. Define* $T : C \to C$ *by* $Tx = (0, x^2{}_1, x^2{}_2, x^3{}_3, ...)$ *for* $x = (x_1, x_2, x_3, ...)$ *in* C*. Then, it is clear that* T *is continuous and map* C *into* C*. Moreover,* $Tp = p$ *if and only if* $p = 0$*. Futhermore,*

$$\begin{aligned} \|Tx - p\|_\infty &= \|Tx\|_\infty = \|(0, x^2{}_1, x^2{}_2, x^2{}_3, ...)\|_\infty \\ &\leq \|(x_1, x_2, x_3, ...)\|_\infty = \|x\|_\infty \\ &= \|x - p\|_\infty. \end{aligned}$$

Therefore, T *is quasi-nonexpansive. However,* T *is not nonexpansive.*

One of the most investigated methods for approximating fixed points of nonexpansive mappings is known as viscosity approximation method, in light of Moudafi [24]. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$ and $f : C \to C$ be a contraction. The viscosity approximation method is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Under certain conditions, then, the sequence $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T.

Zeng and Yao [34] introduced a new extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. They obtained the following strong convergence theorem.

Theorem 1.2 (see Zeng and Yao [34]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $A : C \to H$ *be a monotone* k*-Lipschitz continuous mapping, and let* $T : C \to C$ *be a nonexpansive mapping such that* $Fix(T) \cap VI(A, C) \neq \emptyset$*. Let the sequences* $\{x_n\}$, $\{y_n\}$ *be generated by*

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)TP_C(x_n - \lambda_n Ay_n), \end{cases} \tag{1.5}$$

where $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *satisfy the following conditions:*
(a) $\{\lambda_n k\} \subset (0, 1 - \delta)$ *for some* $\delta \in (0, 1)$*,*
(b) $\{\alpha_n\} \subset (0, 1),\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0.$
Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $P_{Fix(T)\cap VI(A,K)}(x_0)$ *provided that*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{1.6}$$

Remark 1.3. *The iterative scheme* (1.5) *in Theorem 1.2 has strong convergence but imposed the assumption* (1.6) *on the sequence* $\{x_n\}$*.*

The minimization problem (MP) is one of the most important problems in nonlinear analysis and optimization theory. The MP is defined as follows: find $x \in H$, such that

$$g(x) = \min_{y\in H} g(y),$$

where $g : H \to (-\infty,\ +\infty]$ is a proper convex and lower semi-continuous. The set of all minimizers of g on H is denoted by $\operatorname{argmin}_{y\in H} g(y)$. A successful and powerful tool for solving this problem is the well-known Proximal Point Algorithm (shortly, the PPA) which was initiated by Martinet [23] in 1970 and later studied by Rockafellar [5] in 1976. The PPA is defined as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \operatorname{argmin}_{y\in H}\Big[g(y) + \dfrac{1}{2\lambda_n}\|x_n - y\|^2\Big], \end{cases} \tag{1.7}$$

where $\lambda_n > 0$ for all $n \geq 1$. In [5] Rockafellar proved that the sequence $\{x_n\}$ given by (1.7) converges weakly to a minimizer of g. He then posed the following question:
Q1: does the sequence $\{x_n\}$ converges strongly? This question was resolved in the negative by Güler [13] (1991). He produced a proper lower semi continuous and convex function g in l_2 for which the PPA *converges weakly* but not *strongly*. This leads naturally to the following question:
Q2: Can the PPA be modified to guarantee *strong convergence*? In response to Q2, several works have been done (see, *e.g.,* Güler [13], Kamimura and Takahashi [19], Chidume and Djitte [10] and the references therein). In the recent years, the problem of finding a common element of the set of solutions of convex minimization, variational inequality and the set of fixed point problems in real Hilbert spaces, Banach spaces and complete CAT(0) (Hadamard) spaces have been intensively studied by many authors; see, for example, [32, 31, 31, 15, 14, 4, 16] and the references therein.

Motivated and inspired by the above results, we introduce and study an iterative algorithm and prove that the sequence generated by our iterative process converges strongly to a common element of the set of solution of variational inequality problem, the set of minimizers of proper lower semicontinuous convex function and the set of common fixed points of an infinite family of quasi-nonexpansive mappings in real Hilbert spaces. No compactness assumption is made. The algorithm and results presented in this paper improve and extend some recents results. Application is also included. Finally, our method of proof is of independent interest.

2. Preliminaries

In this section, we give some preliminaries, definitions and results which will be needed in the sequel. Let K be a nonempty, closed convex subset of H. For any $y \in H$, there exists a unique point in K, denoted by $P_K(u)$, such that

$$\|y - P_K(u)\| \leq \|y - x\|,\ \ \forall\, x \in K.$$

It is well known that the projection operator can be characterized by the following properties

(i) $\langle x - P_K x, y - P_K x\rangle \leq 0\ \forall\, x \in K$;
(ii) $\langle P_K x - P_K y, x - y\rangle \leq \|P_K x - P_K y\|^2\ \forall\, y, x \in K$;
(iii) $\|P_K y - x\|^2 \leq \|x - y\|^2 - \|P_K y - y\|^2,\ \forall\, x, y \in K$.

Remark 2.1. *In the context of variational inequality problem* (1.1)*, we have*

$$u \in VI(A, K) \Longleftrightarrow u \in Fix(P_K(I - \theta A)),\ \ \theta > 0. \tag{2.1}$$

The demiclosedness of a nonlinear operator T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 2.2. *Let K be a nonempty, closed convex subset of a real Hilbert space H and let $T : K \to K$ be a single-valued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in Fix(T)$.*

Lemma 2.3 (Demiclosedness principle [5]). *Let K be a nonempty, closed convex subset of a real Hilbert space H and let $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demiclosed.*

Lemma 2.4 ([9]). *Let H be a real Hilbert space. Then for any $x, y \in H$, the following inequalities hold:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2, \;\; \lambda \in (0, 1).$$

Lemma 2.5 (Xu, [33]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$, *(b)* $\displaystyle\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\displaystyle\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. *Then* $\displaystyle\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (Aoyama et. al [3], Nilsrakoo et al. [26]). *Let K be a nonempty closed subset of a Banach space and let $\{T_n\}_{n\geq 0}$ be a sequence of mappings of K into itself. Suppose that $\displaystyle\sum_{n=0}^{\infty} \sup\Big\{\|T_{n+1}x - T_n x\| : \; x \in B\Big\} < \infty$ for any bounded subset B of K. Then, for any $x \in K$ $\{T_n x\}$ converges strongly to some point of K. Moreover, let T be a mapping of K into itself defined by $Tx = \displaystyle\lim_{n\to\infty} T_n x$ for all $x \in K$. Then,*

$$\lim_{n\to\infty} \sup_{x\in K} \|T_n x - Tx\| = 0.$$

Lemma 2.7. *(Rockafellar, [28]) Let K be a nonempty closed and convex subset of a real Hilbert space H and A is a monotone, hemicontinuous map of C into H. Let $B \subset H \times H$ be an operator defined as follows:*

$$Bz = \begin{cases} Az + N_K(z) & \text{if} \quad z \in K, \\ \emptyset & \text{if} \quad z \notin K, \end{cases} \tag{2.2}$$

where $N_K(z)$ is the normal K at z and is defined as follows:

$$N_K(z) = \{w \in H \;:\; \langle w, z - v\rangle \geq 0 \;\; \forall\, v \in K\}.$$

Then, B is maximal monotone and $B^{-1}(0) = VI(A, K)$.

Lemma 2.8. *Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $A : K \to H$ be an α-inverse strongly monotone mapping. Then, $I - \theta A$ is nonexpansive mapping for all $x, y \in K$ and $\theta \in [0, 2\alpha]$.*

Proof . For all $x, y \in K$, we have

$$\begin{aligned} \|(I - \theta A)x - (I - \theta A)y\|^2 &= \|(x - y) - \theta(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\theta\langle Ax - Ay, x - y\rangle + \theta^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \theta(\theta - 2\alpha)\|Ax - Ay\|^2. \end{aligned}$$

This shows that $I-\theta A$ is nonexpansive. $\square$ Let $g: H \to (-\infty,\ +\infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of g in a real Hilbert space H as follows:

$$J_\lambda^g x = \text{argmin}_{u\in H}\Big[g(u) + \frac{1}{2\lambda}\|x-u\|^2\Big],$$

for all $x \in H$. It was shown in [13] that the set of fixed points of the resolvent associated with g coincides with the set of minimizers of g. Also, the resolvent J_λ^g of g is nonexpansive for all $\lambda > 0$ (see [18]).

Lemma 2.9. *(Miyadera [25]) For any $r > 0$ and $\mu > 0$, the following holds:*

$$J_r^g x = J_\mu^g x(\frac{\mu}{r}x + (1-\frac{\mu}{r})J_r^g x).$$

Lemma 2.10 (Sub-differential inequality, [2]). *Let $g : H \to (-\infty,\ +\infty]$ be a proper convex and lower semicontinuous function. Then, for all $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{\lambda}\|J_\lambda^g x - y\|^2 - \frac{1}{\lambda}\|x-y\|^2 + \frac{1}{\lambda}\|x - J_\lambda^g x\|^2 + g(J_\lambda^g x) \le g(y). \tag{2.3}$$

3. Strong convergence theorems

The following is our main result.

Theorem 3.1. *Let K be a nonempty, closed and convex subset of a real Hilbert H and $A : K \to H$ be an α-inverse strongly monotone operator. Let $f : K \to K$ be a contraction with coefficient b and $g : K \to (-\infty,\ +\infty]$ be a proper convex and lower semi-continuous function. For each $n = 0, 1, ...,$ let $T_n : K \to K$ be a quasi-nonexpansive mapping such that $\Gamma := \bigcap_{n=0}^{\infty} Fix(T_n) \cap VI(A,K) \cap argmin_{u\in K}\, g(u) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} u_n = argmin_{u\in K}\Big[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big], \\ z_n = P_K(I - \theta_n A)u_n, \\ y_n = \beta_n z_n + (1-\beta_n)T_n z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_n y_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, $\theta_n \in [a,b] \subset \big(0,\ min\{1,\ 2\alpha\}\big)$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \ge 1$ and some λ.
Assume that (a) $\sum_{n=0}^{\infty}\sup\Big\{\|T_{n+1}x - T_n x\| \ :\ \ x \in B\Big\} < \infty$ for any bounded subset B of K and $Fix(T) = \bigcap_{n=0}^{\infty} F(T_n)$ where T be a mapping of K into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in K$.
(b) $I - T$ is demiclosed at origin. Then, the sequence $\{x_n\}$ generated by (3.1) converges strongly to an element of Γ.

Proof . We first prove that the sequences $\{x_n\}$ is bounded. Let $p \in \Gamma$. Then, $g(p) \leq g(u)$ for all $u \in K$ This implies that

$$g(p) + \frac{1}{2\lambda_n}\|p - p\|^2 \leq g(u) + \frac{1}{2\lambda_n}\|u - p\|^2$$

and hence $J^g_{\lambda_n} p = p$ for all $n \geq 1$, where $J^g_{\lambda_n}$ is the Moreau-Yosida resolvent of g in K. By using inequality (2.1) and Lemma 2.8, we have

$$\|z_n - p\| = \|P_K(I - \theta_n A)u_n - p\| \leq \|u_n - p\| = \|J^g_{\lambda_n} x_n - p\| \leq \|x_n - p\|, \ \forall n \geq 0.$$

Using (3.1), T_n is quasi-nonexpansive and Lemma 2.8, we have

$$\begin{aligned} \|y_n - p\| &= \|\beta_n z_n + (1 - \beta_n)T_n z_n - p\| \\ &\leq \beta_n\|z_n - p\| + (1 - \beta_n)\|z_n - p\| \\ &= \|z_n - p\|. \end{aligned}$$

Hence,

$$\|y_n - p\| \leq \|z_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.2}$$

Using (3.1) and inequality (3.2), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1 - \alpha_n)T_n y_n - p\| \\ &\leq \alpha_n\lambda_n\|x_n - p\| + (1 - \alpha_n)\|y_n - p\| + \alpha_n\|f(p) - p\| \\ &\leq \alpha_n b\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\leq [1 - (1 - b)\alpha_n]\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\leq \max\{\|x_n - p\|, \ \frac{\|f(p) - p\|}{1 - b}\}. \end{aligned}$$

By induction, we conclude that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \ \frac{\|f(p) - p\|}{1 - b}\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded. We observe that $P_\Gamma(f)$ is a contraction. Indeed, for all $x, y \in K$, we have

$$\begin{aligned} \|P_\Gamma f(x) - P_\Gamma f(y)\| &\leq \|f(x) - f(y)\| \\ &\leq b\|x - y\|. \end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $P_\Gamma f$ has a unique fixed point, say $x_1 \in H$. That is, $x_1 = P_\Gamma f(x_1)$. By using properties of metric projection, it is equivalent to the following variational inequality problem

$$\langle x_1 - f(x_1), x_1 - p\rangle \leq 0, \ \ \forall \ \ p \in \Gamma. \tag{3.3}$$

We show that the uniqueness of a solution of variational inequality (3.3).
Suppose both $x^* \in \Gamma$ and $x^{**} \in \Gamma$ are solutions to (3.3), then

$$\langle x^* - f(x^*), x^* - x^{**}\rangle \leq 0 \tag{3.4}$$

and

$$\langle x^{**} - f(x^{**}), x^{**} - x^*\rangle \leq 0. \tag{3.5}$$

Adding up (3.4) and (3.5) yields

$$\langle x^{**} - x^* + f(x^*) - f(x^{**}), x^{**} - x^* \rangle \leq 0. \tag{3.6}$$

Noticing that

$$\langle x^{**} - x^* + f(x^*) - f(x^{**}), x^{**} - x^* \rangle \geq (1-b)\|x^* - x^{**}\|^2,$$

which implies that $x^* = x^{**}$ and the uniqueness is proved. Below we use x^* to denote the unique solution of (3.3). From (3.1), inequality (3.2) and Lemma 2.4, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n z_n + (1-\beta_n)T_n z_n - p\|^2 \\
&= (1-\beta_n)\|z_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|T_n z_n - z_n\|^2. \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)\|T_n z_n - z_n\|^2.
\end{aligned}$$

Hence,

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - T_n z_n\|^2. \tag{3.7}$$

Therefore, by Lemma 2.4 and inequality (3.7), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)T_n y_n - p\|^2 \\
&\leq \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)\big(T_n y_n - p\big)\|^2 + 2\alpha_n\langle p - f(p), p - x_{n+1}\rangle \\
&\leq \alpha_n b^2\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 + 2\alpha_n\langle p - f(p), p - x_{n+1}\rangle \\
&\leq \alpha_n b\|x_n - p\|^2 + (1-\alpha_n)\Big[\|x_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - T_n z_n\|^2\Big] \\
&\quad +2\alpha_n\langle p - f(p), p - x_{n+1}\rangle \\
&\leq [1-(1-b)\alpha_n]\|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|z_n - T_n z_n\|^2 \\
&\quad +2\alpha_n\langle p - f(p), p - x_{n+1}\rangle.
\end{aligned}$$

Therefore,

$$(1-\alpha_n)\beta_n(1-\beta_n)\|z_n - T_n z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\langle p - f(p), p - x_{n+1}\rangle. \tag{3.8}$$

Since $\{x_n\}$ is bounded, then there exists a constant $B > 0$ sucht that

$$\langle p - f(p), p - x_{n+1}\rangle \leq B, \quad \text{for all } n \geq 0.$$

Hence,

$$(1-\alpha_n)\beta_n(1-\beta_n)\|z_n - T_n z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n B. \tag{3.9}$$

Now we prove that $\{x_n\}$ converges strongly to x^*.
We divide the proof into two cases.
Case 1. Assume that there is $n_0 \in N$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \geq n_0$. Since $\{\|x_n - x^*\|\}$ is monotonic and bounded, $\{\|x_n - x^*\|\}$ is convergent. Clearly, we have

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0. \tag{3.10}$$

It then implies from (3.9) that

$$\lim_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n)\|z_n - T_n z_n\|^2 = 0. \tag{3.11}$$

Using the fact that $\lim_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0. \tag{3.12}$$

We observe that,

$$\|z_n - Tz_n\| \leq \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\|. \tag{3.13}$$

By inequalities (3.12), (3.13) and Lemma 2.6, we have

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.14}$$

From (3.1), convexity of $\|.\|^2$ and Lemma 2.8, it follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&= \alpha_n \|(\lambda_n x_n) - p\|^2 + (1-\alpha_n)\|P_K(I-\theta_n A)u_n - P_K(I-\theta_n A)p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\Big[\|u_n - p\|^2 + \theta_n(\theta_n - 2\alpha)\|Au_n - Ap\|^2\Big] \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)a(b-2\alpha)\|Au_n - Ap\|^2.
\end{aligned}$$

Therefore, we have

$$(1-\alpha_n)a(2\alpha - b)\|Au_n - Ap\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|f(x_n) - p\|^2.$$

Since, $\alpha_n \to 0$ as $n \to \infty$, inequality (3.10) and $\{x_n\}$ is bounded, we obtain

$$\lim_{n\to\infty} \|Au_n - Ap\|^2 = 0. \tag{3.15}$$

It follows from (3.1) that

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_K(I-\theta_n A)u_n - P_K(I-\theta_n A)p\|^2 \\
&\leq \langle z_n - p, (I-\theta_n A)u_n - (I-\theta_n A)p\rangle \\
&= \frac{1}{2}\Big[\|(I-\theta_n A)u_n - (I-\theta_n A)p\|^2 + \|z_n - p\|^2 - \|(I-\theta_n A)u_n - (I-\theta_n A)p - (z_n - p)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|u_n - p\|^2 + \|z_n - p\|^2 - \|u_n - z_n\|^2 + 2\theta_n\langle z_n - p, Au_n - Ap\rangle - {\theta_n}^2\|Au_n - Ap\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|x_n - p\|^2 + \|z_n - p\|^2 - \|u_n - z_n\|^2 + 2\theta_n\langle z_n - p, Au_n - Ap\rangle - {\theta_n}^2\|Au_n - Ap\|^2\Big].
\end{aligned}$$

So, we obtain

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - z_n\|^2 + 2\theta_n\langle z_n - p, Au_n - Ap\rangle - {\theta_n}^2\|Au_n - Ap\|^2,$$

and thus

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - (1-\alpha_n)\|u_n - z_n\|^2 - (1-\alpha_n){\theta_n}^2\|Au_n - Ap\|^2 \\
&+ 2\theta_n(1-\alpha_n)\langle z_n - p, Au_n - Ap\rangle.
\end{aligned}$$

Since, $\alpha_n \to 0$ as $n \to \infty$, inequalities (3.10) and (3.15), we obtain

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.16}$$

Let $p \in F$. Using Lemma 2.10 and since $g(p) \le g(u_n)$, we get

$$\|x_n - u_n\|^2 \le \|x_n - p\|^2 - \|u_n - p\|^2. \tag{3.17}$$

Therefore, from (3.1), Lemma 2.4 and inequality (3.17), we get that

$$\begin{array}{rcl}\|x_{n+1} - p\|^2 &=& \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\|^2 \\ &=& \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(y_n - p)\|^2 \\ &\le& (1-\alpha_n)^2\|y_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\ &\le& (1-\alpha_n)^2\|z_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\ &\le& (1-\alpha_n)^2(\|x_n - p\|^2 - \|x_n - u_n\|^2) + 2\alpha_n\|f(x_n) - p\|\|x_{n+1} - p\| \\ &\le& (1 - 2\alpha_n + \alpha_n^2)\|x_n - p\|^2 - (1-\alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n\|f(x_n) - p\|\|x_{n+1} - p\| \\ &\le& \|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 - (1-\alpha_n)^2\|x_n - u_n\|^2 + 2\alpha_n\|f(x_n) - p\|\|x_{n+1} - p\|,\end{array}$$

and hence

$$(1-\alpha_n)^2\|x_n - u_n\|^2 \;\; \le \;\; \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|x_n - p\|^2 + 2\alpha_n\|f(x_n) - p\|\|x_{n+1} - p\|.$$

Thanks inequality (3.10), $\{x_n\}$ is bounded and $\alpha_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.18}$$

Using inequalities (3.16) and (3.18), we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.19}$$

Now, we prove that $\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle \le 0$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to ω in K and

$$\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle = \lim_{k\to+\infty}\langle x^* - f(x^*), x^* - x_{n_k}\rangle.$$

From (3.14), inequality (3.19) and $I - T$ is demiclosed, we obtain $\omega \in Fix(T)$. Using (3.1) and Lemma 2.9 we arrive at

$$\begin{array}{rcl}\|x_n - J^g_\lambda x_n\| &\le& \|u_n - J^g_\lambda x_n\| + \|u_n - x_n\| \\ &\le& \|J^g_{\lambda_n} x_n - J^g_\lambda x_n\| + \|u_n - x_n\| \\ &\le& \|u_n - x_n\| + \|J^g_\lambda\Big(\dfrac{\lambda_n - \lambda}{\lambda_n} J^g_{\lambda_n} x_n + \dfrac{\lambda}{\lambda_n} x_n\Big) - J^g_\lambda x_n\| \\ &\le& \|u_n - x_n\| + \|\dfrac{\lambda_n - \lambda}{\lambda_n} J^g_{\lambda_n} x_n + \dfrac{\lambda}{\lambda_n} x_n - x_n\| \\ &\le& \|u_n - x_n\| + \Big(1 - \dfrac{\lambda}{\lambda_n}\Big)\|u_n - x_n\| \\ &\le& \Big(2 - \dfrac{\lambda}{\lambda_n}\Big)\|u_n - x_n\|.\end{array}$$

Hence,

$$\lim_{n\to\infty} \|x_n - J_\lambda^g x_n\| = 0. \tag{3.20}$$

Since J_λ^g is single valued and nonexpasive, using (3.20) and Lemma 2.3, then $\omega \in Fix(J_\lambda^g) = \operatorname{argmin}_{x\in K} g(u)$. Let us show $\omega \in VI(A,K)$. Now, let us introduce the multivalued map $B : H \to 2^H$ defined by:

$$Bz = \begin{cases} Az + N_K(z) & \text{if} \quad z \in K, \\ \emptyset & \text{if} \quad z \notin K, \end{cases} \tag{3.21}$$

where $N_K(z)$ is the normal K at z and is defined as follows:

$$N_K(z) = \{w \in H \ : \ \langle w, z - v\rangle \geq 0 \ \ \forall\, v \in K\}.$$

From Lemma 2.7, we have that B is maximal monotone and $B^{-1}(0) = VI(A,K)$. Let $(u,v) \in G(A)$, where $G(A) := \{[x,u] : \ x \in D(A), \ u = Ax\}$. Since $v - Au \in N_K(u)$ and $z_n \in K$, we have

$$\langle u - z_n, v - Au\rangle \geq 0.$$

On other hand, from $z_n = P_K(I - \theta_n A)u_n$, we have, $\langle u - z_n, z_n - (I - \theta_n A)u_n\rangle \geq 0$ and hence

$$\langle u - z_n, \frac{z_n - u_n}{\theta_n} + Au_n\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle u - z_{n_k}, v\rangle &\geq \langle u - z_{n_k}, Au\rangle \\
&\geq \langle u - z_{n_k}, Au\rangle - \langle u - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\theta_{n_k}} + Au_{n_k}\rangle \\
&\geq \langle u - z_{n_k}, Au - Az_{n_k}\rangle + \langle u - z_{n_k}, Az_{n_k} - Au_{n_k}\rangle - \langle u - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\theta_{n_k}}\rangle \\
&\geq \langle u - z_{n_k}, Az_{n_k} - Au_{n_k}\rangle - \langle u - z_{n_k}, \frac{z_{n_k} - u_{n_k}}{\theta_{n_k}}\rangle.
\end{aligned}$$

By using A is $\dfrac{1}{\alpha}$ Lipschitz, we have

$$\langle u - z_{n_k}, v\rangle \geq -N\Big(\frac{\|z_{n_k} - u_{n_k}\|}{\alpha} + \frac{\|z_{n_k} - u_{n_k}\|}{a}\Big).$$

where N is a positive constant such that $\sup_{k\geq 1}\{\|u - z_{n_k}\|\} \leq M$. Since $z_{n_k} \rightharpoonup \omega$, it follows from (3.16) that $\langle u - \omega, v\rangle \geq 0$ as $k \to \infty$. Since B is maximal monotone, we have $a \in B^{-1}(0)$ and we obtain that $\omega \in VI(A,K)$. Therefore, $\omega \in \Gamma$.
Hence,

$$\begin{aligned}
\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle &= \lim_{k\to+\infty}\langle x^* - f(x^*), x^* - x_{n_k}\rangle \\
&= \langle x^* - f(x^*), x^* - \omega)\rangle \leq 0.
\end{aligned}$$

Finally, We show that the sequence $\{x_n\}$ converges to the point x^*. Observe that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)y_n-x^*\|^2 \\
&\leq \|\alpha_n(f(x_n)-f(x^*))+(1-\alpha_n)(y_n-x^*)\|^2+2\alpha_n\langle x^*-f(x^*),x^*-x_{n+1}\rangle \\
&\leq \Big(\alpha_n\|f(x_n)-f(x^*)\|+\|(1-\alpha_n)(y_n-x^*)\|\Big)^2+2\alpha_n\langle x^*-f(x^*),x^*-x_{n+1}\rangle \\
&\leq \Big(\alpha_n b\|x_n-x^*\|+(1-\alpha_n)\|y_n-x^*\|\Big)^2+2\alpha_n\langle x^*-f(x^*),x^*-x_{n+1}\rangle \\
&\leq \Big((1-\alpha_n(1-b))\|x_n-x^*\|\Big)^2+2\alpha_n\langle x^*-f(x^*),x^*-x_{n+1}\rangle \\
&\leq (1-\alpha_n(1-b))\|x_n-x^*\|^2+2\alpha_n\langle x^*-f(x^*),x^*-x_{n+1}\rangle
\end{aligned}$$

Hence, by Lemma 2.5, we conclude that the sequence $\{x_n\}$ converges strongly to the point $x^*\in\Gamma$.
Case 2. Assume that the sequence $\{\|x_n-x^*\|\}$ is not monotonically decreasing sequence. Set $B_n=\|x_n-x^*\|$ and $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\geq n_0$ (for some n_0 large enough) by $\tau(n)=\max\{k\in\mathbb{N}:k\leq n,\ \ B_k\leq B_{k+1}\}$.
We have τ is a non-decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and $B_{\tau(n)}\leq B_{\tau(n)+1}$ for $n\geq n_0$. From (3.9), we have

$$(1-\alpha_{\tau(n)})\beta_{\tau(n)}(1-\beta_{\tau(n)})g(\|z_{\tau(n)}-T_{\tau(n)}z_{\tau(n)}\|)\leq 2\alpha_{\tau(n)}B\to 0 \ \text{ as } \ n\to\infty.$$

Hence,

$$\lim_{n\to\infty}\|z_{\tau(n)}-T_{\tau(n)}z_{\tau(n)}\|=0. \tag{3.22}$$

At the same time, we observe that

$$\|z_{\tau(n)}-Tz_{\tau(n)}\|\leq\|z_{\tau(n)}-T_{\tau(n)}z_{\tau(n)}\|+\|T_{\tau(n)}z_{\tau(n)}-Tz_{\tau(n)}\|. \tag{3.23}$$

Thanks inequalities (3.22), (3.23) and Lemma 2.6, we have

$$\lim_{n\to\infty}\|z_{\tau(n)}-Tz_{\tau(n)}\|=0.$$

Following similar to the argument as in Case 1, we can show that $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ are bounded in K and $\limsup_{\tau(n)\to+\infty}\langle x^*-f(x^*),x^*-x_{\tau(n)})\rangle\leq 0$. We have for all $n\geq n_0$,

$$0\leq\|x_{\tau(n)+1}-x^*\|^2-\|x_{\tau(n)}-x^*\|^2\leq\alpha_{\tau(n)}[-(1-b)\|x_{\tau(n)}-x^*\|^2+2\langle x^*-f(x^*),x^*-x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)}-x^*\|^2\leq\frac{2}{1-b}\langle x^*-f(x^*),x^*-x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)}-x^*\|^2=0.$$

Therefore,

$$\lim_{n\to\infty}B_{\tau(n)}=\lim_{n\to\infty}B_{\tau(n)+1}=0.$$

Furthermore, for all $n\geq n_0$, we have $B_{\tau(n)}\leq B_{\tau(n)+1}$ if $n\neq\tau(n)$ (that is, $n>\tau(n)$); because $B_j>B_{j+1}$ for $\tau(n)+1\leq j\leq n$. As consequence, we have for all $n\geq n_0$,

$$0\leq B_n\leq\max\{B_{\tau(n)},\ B_{\tau(n)+1}\}=B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty}B_n=0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. $\square$

Remark 3.2. *let $\{T_n\}_{n\geq 0}$ be a sequence of nonexpansive mappings of K into K, let $\{\lambda_n\}_{n\geq 0}$ be a sequence of real number such that and $0 \leq \lambda_n \leq 1$. For each $n \geq 0$, we define a mapping W_n of K into K as follows:*

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ &\vdots \\ U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ &\vdots \\ U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{aligned} \tag{3.24}$$

Such that is called W_n is the so called W-mapping generated by an countable infinite family of nonexpansive mappings $T_1, T_2, ..., T_n, ...$ and scalars $\lambda_1, \lambda_2, ..., \lambda_n, ...$ such that the common fixed points set $F := \bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$, see for example [30]. Clearly, W_n is nonexpansive and from [30], we know that $\bigcap\limits_{n=1}^{\infty} F(T_n) = F(W_n)$. Furthermore, from [27], we have the sequence $\{W_n\}_{n\geq 1}$ satisfies the condition $\sum\limits_{n=0}^{\infty} \sup\Big\{\|W_{n+1}x - W_n x\| : \ x \in B\Big\} < \infty$ for any bounded subset B of K imposed in Theorem 3.1.

By above remark, Lemma 2.3 and the fact that nonexpansive mapping is quasi-nonexpansive. We obtain the following result.

Theorem 3.3. *Let K be a nonempty, closed and convex subset of a real Hilbert H and $A : K \to H$ be an α-inverse strongly monotone operator. Let $f : K \to K$ be a contraction with coefficient b and $g : K \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. For each $n = 1, ...,$ let $T_n : K \to K$ be a nonexpansive mapping such that $\Gamma := \bigcap\limits_{n=1}^{\infty} Fix(T_n) \cap VI(A,K) \cap argmin_{u\in K}\, g(u) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} u_n = argmin_{u\in K}\Big[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big], \\ z_n = P_K(I - \theta_n A)u_n, \\ y_n = \beta_n z_n + (1-\beta_n)W_n z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)W_n y_n, \end{cases} \tag{3.25}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\liminf\limits_{n\to\infty} \beta_n(1 - \beta_n) > 0$, $\theta_n \in [a,b] \subset \big(0, min\{1, 2\alpha\}\big)$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ. Then, the sequence $\{x_n\}$ generated by (3.25) converges strongly to an element of Γ.

We apply Theorem 3.1 to approximate fixed points of quasi-nonexpansive mapping.

Corollary 3.4. *Let K be a nonempty, closed and convex subset of a real Hilbert H and $A : K \to H$ be an α-inverse strongly monotone operator. Let $f : K \to K$ be a contraction with coefficient b and $g : K \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $T : K \to K$ be a* *quasi-nonexpansive mapping such that* $\Gamma := Fix(T) \cap VI(A,K) \cap argmin_{u\in K}\, g(u) \neq \emptyset$ *and* $I - T$ *is demiclosed at origin. Let* $\{x_n\}$ *be a sequence defined iteratively from arbitrary* $x_0 \in K$ *by:*

$$\begin{cases} u_n = argmin_{u\in K}\Big[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big], \\ z_n = P_K(I - \theta_n A)u_n, \\ y_n = \beta_n z_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n, \end{cases} \tag{3.26}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\lim\limits_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$, $\theta_n \in [a,b] \subset \big(0, \, min\{1, \, 2\alpha\}\big)$ *and* $\{\lambda_n\}$ *is a sequence such that* $\lambda_n \geq \lambda > 0$ *for all* $n \geq 1$ *and some* λ. *Then, the sequence* $\{x_n\}$ *generated by* (3.26) *converges strongly to an element of* Γ.

4. Application

In this section, we study the problem of finding a common element of the set of solution of convex minimization problem, the set of solution of variational inequality problem and the set of zeros of monotone operator in real Hilbert spaces.

Lemma 4.1. *[3] Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H *and let* B *be an monotone operator on* H *such that such that* $B^{-1}(0) \neq \emptyset$ *and* $\overline{D(B)} \subset C \subset R(I + rB)$, *for all* $r > 0$. *Suppose that* $\{r_n\}$ *is a sequence of* $(0,\infty)$ *such* $\inf\{r_n : \ n \in \mathbb{N}\} > 0$ *and* $\sum\limits_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$, *then (a)* $\sum\limits_{n=0}^{\infty} \sup\Big\{\|J_{r_{n+1}}x - J_{r_n}x\| : \ \ x \in B\Big\} < \infty$ *for any bounded subset* B *of* C *and* $Fix(J_r) = B^{-1}(0) = \bigcap\limits_{n=0}^{\infty} Fix(J_{r_n})$ *where* J_r *be a mapping of* C *into itself defined by* $J_r x = \lim\limits_{n\to\infty} J_{r_n}x$, *for all* $x \in C$.

Hence, one has the following result.

Theorem 4.2. *Let* K *be a nonempty, closed and convex subset of a real Hilbert* H *and* $A : K \to H$ *be an* α*-inverse strongly monotone operator. Let* $f : K \to K$ *be a contraction with coefficient* b *and* $g : K \to (-\infty, \, +\infty]$ *be a proper convex and lower semi-continuous function. Let* B *be an monotone operator on* H *such that* $\Gamma := B^{-1}(0) \cap VI(A,K) \cap argmin_{u\in K}\, g(u) \neq \emptyset$, $\overline{D(B)} \subset K \subset R(I + rB)$, *for all* $r > 0$. *Let* $\{x_n\}$ *be a sequence defined iteratively from arbitrary* $x_0 \in K$ *by:*

$$\begin{cases} u_n = argmin_{u\in K}\Big[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big], \\ z_n = P_K(I - \theta_n A)u_n, \\ y_n = \beta_n z_n + (1-\beta_n)J_{r_n}z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{r_n}y_n, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $[0,1]$ *and* r_n *is a sequence of* $(0,\infty)$ *such* $\inf\{r_n : \ n \in \mathbb{N}\} > 0$ *and* $\sum\limits_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\lim\limits_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$, $\theta_n \in [a,b] \subset \big(0, \, min\{1, \, 2\alpha\}\big)$ *and* $\{\lambda_n\}$ *is a sequence such that* $\lambda_n \geq \lambda > 0$ *for all* $n \geq 1$ *and some* λ. *Then, the sequence* $\{x_n\}$ *generated by* (4.1) *converges strongly to an element of* Γ.

Proof . Letting $T_n = J_{r_n} = (I + r_n B)^{-1}$ in Theorem 3.1 and the fact that resolvent of B is nonexpansive mapping. The proof follows Theorem 3.1 and Lemma 4.1. □

5. Conclusion

In this work, we introduce and analyze a new iterative algorithm which is a combination of viscosity approximation method and proximal point algorithm for approximating a common element of the set of minimizers of a convex function, the set of solution of variational inequality problem and the set of common fixed points of an infinite family of quasi-nonexpansive mappings in real Hilbert spaces. Moreover, compactness assumption does not need to get strong convergnce. All the results in this paper hold for nonexpansive mappings in real Hilbert spaces.

References

[1] P.N. Anh and L.T. H. An, *The subgradient extragradient method extended to equilibrium problems*, Optim. 64 (2015) 225–248.

[2] L. Ambrosio, N. Gigli and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Second edition, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[3] K. Aoyama, I. H. Koji and W. Takahashi, *Weak convergence of an iterative sequence for accretive operators in Banach spaces*, Fixed Point Theory Appl. (2006), Art. ID 35390, 13 pp.

[4] K. O. Aremu, C. Izuchukwu, G. C. Ugwunnadi and O. T. Mewomo, *On the proximal point algorithm and demimetric mappings in CAT (0) spaces*, Demonstr. Math. 51(1) (2018) 277–294.

[5] F. E. Browder, *Convergenge theorem for sequence of nonlinear operator in Banach spaces,* Math. Z. 100 (1967) 201–225.

[6] L. C Ceng, N. Hadjisavvas and N. C. Wong, *Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems*, J. Glob. Optim. 46 (2010) 635–646.

[7] S.S. Chang, H.W.J. Lee and C.K. Chan, *A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization*, Nonlinear Anal. 70 (2009) 3307–3319.

[8] W. Cholamjiak, *Shrinking projection methods for a split equilibrium problem and a nonspreading-type multivalued mapping*, J. Nonlinear Sci. Appl., (in press).

[9] C. E. Chidume, *Geometric Properties of Banach spaces and Nonlinear Iterations*, Springer Verlag Series: Lecture Notes in Mathematics, Vol. 1965, 2009.

[10] C. E. Chidume and N. Djitte, *Strong convergence theorems for zeros of bounded maximal monotone nonlinear operators*, J. Abst. Appl. Anal. Volume 2012, Article ID 681348, 19 pages.

[11] H. Dehghan, C. Izuchukwu, O. T. Mewomo, D. A. Taba and G. C. Ugwunnadi, *Iterative algorithm for a family of monotone inclusion problems in CAT(0) spaces*, Quaest. Math., (2019), doi.org/10.2928/16073606.2019.1593255.

[12] H. Iiduka, W. Takahashi and M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, PanAmer. Math. J. 14 (2004) 49–61.

[13] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. 29 (1991) 403–419.

[14] C. Izuchukwu, H. A. Abass and O. T. Mewomo, *Viscosity approximation method for solving minimization problem and fixed point problem for nonexpansive multivalued mapping in CAT(0) spaces*, Ann. Acad. Rom. Sci. Ser. Math. Appl. 11(1) (2019) 131–158.

[15] C. Izuchukwu, A. A. Mebawondu, K. O. Aremu, H. A. Abass and O. T. Mewomo, *Viscosity iterative techniques for approximating a common zero of monotone operators in an Hadamard space*, Rend. Circ. Mat. Palermo II, (2019). doi.org/10.1007/s12215-019-00415-2 in p-uniformly convex metric space, Numer. Algorithms., (2018). DOI: 10.1007/s11075-018-0633-9.

[16] C. Izuchukwu, K.O. Aremu, A.A. Mebawondu and O.T. Mewomo, *A viscosity iterative technique for equilibrium and fixed point problems in a Hadamard space*, Appl. Gen. Topol. 20(1) (2019) 193–210.

[17] L.O. Jolaoso, T.O. Alakoya, A. Taiwo and O.T. Mewomo, *A parallel combination extragradient method with Armijo line searching for finding common solution of finite families of equilibrium and fixed point problems*, Rend. Circ. Mat. Palermo II, (2019), DOI:10.1007/s12215-019-00431-2

[18] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. 70 (1995) 659–673.

[19] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. 13(3) (5003) 938–945.

[20] H. He, S. Liu and H. Zhou, *An explicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of an infinite family of nonexpansive mappings*, Nonlinear Anal. 72 (2010) 3124–3135.

[21] P. E. Mainge, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim. 47 (2008) 1499–1515.

[22] P.E. Maingé, *Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints*, Eur. J. Oper. Res. 205 (2010) 501–506.

[23] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, (French) Rev. Franaise Informat. Recherche Opérationnelle, 4 (1970) 154–158.

[24] A. Moudafi, *Viscosity approximation methods for fixed point problems*, J. Math. Anal. Appl. 241 (2000) 46–55.

[25] I. Miyadera, *Nonlinear Semigroups*, Translations of Mathematical Monographs, American Mathematical Society, Providence, 1992.

[26] W. Nilsrakoo and S. Saejung, *Weak and strong convergence theorems for countable Lipschitzian mappings and its applications*,, Nonlinear Anal. 69 (2008) 2695–2708.

[27] J. W. Peng and J.C. Yao, *Strong convergence theorems of an iterative scheme based on extragradient method for mixed equilibruim problem and fixed point problems*, Math. Com. Model. 49 (2009) 1816–1828.

[28] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. 149 (1970) 7588.

[29] Y. Shehu and O.T. Mewomo, *Further investigation into split common fixed point problem for demicontractive operators*, Acta Math, Sin. (Engl. Ser.), 32(11) (2016) 1357–1376.

[30] K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math. 5 (2001).

[31] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, *General alternative regularization method for solvng split equality common fixed point problem for quasi-pseudocontractive mappings in Hilbert spaces*, Ricerche Mat. (2019), DOI: 10.1007/s11587-019-00460-0.

[32] A. Taiwo, L. O. Jolaoso and O. T. Mewomo, *Parallel hybrid algorithm for solving pseudomonotone equilibrium and Split Common Fixed point problems*, Bull. Malays. Math. Sci. Soc., (2019), DOI: 10.1007/s40840-019-00781-1.

[33] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. 66(2) (2002) 240–256.

[34] L.-C. Zeng and J.-C. Yao, *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems*, Taiwanese J. Math. 10(5) (2006) 1293-1303.