



The approximate analytical solutions of nonlinear fractional ordinary differential equations

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Abstract

The Sumudu homotopy perturbation method (SHPM) is applied to solve fractional order nonlinear differential equations in this paper. The current technique incorporates two notable strategies in particular Sumudu transform (ST) and homotopy perturbation method (HPM). The proposed method's hybrid property decreases the number of the quantity of computations and materials needed. In this method, illustration examples evaluate the accuracy and applicability of the mentioned procedure. The outcomes got by FSHPM are in acceptable concurrence with the specific arrangement of the problem.

Keywords: Homotopy perturbation method, Fractional differential equation, Sumudu transform, Caputo fractional derivative.

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1. Introduction

The engineering and physical systems that are best represented by fractional differential equations are described by fractional calculus (FC). Unfortunately, traditional mathematical models of integer-order derivatives, like nonlinear models, do not perform well in many cases. FC has had a significant impact in a variety of fields, including chemistry, energy, control theory, groundwater problems, mechanics, signal image analysis, and biology. Previously, the analysis of non-linear physical processes relied heavily on the study of travelling-wave solutions for non-linear equations [1, 2, 3, 4].

Lately, Most approximate and empirical methodologies have been used to resolve ordinary and partial differential equations in the Caputo sense such as the fractional variational iteration method

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[5, 6, 7, 8, 9], fractional differential transform method [10, 11, 12], fractional series expansion method [13, 14], fractional Sumudu variational iteration method [15, 16], fractional Laplace transform method [17], fractional homotopy perturbation method [18], fractional Sumudu decomposition method [19, 20, 21], fractional Fourier series method [22], fractional reduced differential transform method [23, 24, 25], fractional Adomian decomposition method [26, 27, 28, 29], and another methods [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46].

Our aim is to present the coupling method of ST and HPM, which is called as the SHPM, and to used it to solve the nonlinear FRDE. The remainder of this work is divided into the following sections. In Section 2, some fractional calculus definitions are provided. In Section 3, the fractional SHPM analysis method is implemented. Applications of fractional SHPM are demonstrated in Section 4. Section 5 contains the conclusion of this paper.

2. Preliminaries

This section covers some fractional calculus concepts and notation that will be useful in this work [1, 2].

Definition 2.1. Consider $\psi(\ell)$, where $\psi(\ell) \in R, \ell > 0$, is called in the space $C_{\vartheta}, \vartheta \in R$ if

$$\{\exists \rho, (\rho > \vartheta), s.t. \psi(\ell) = \ell^{\rho} \psi_1(\ell), \text{ where } \psi_1(\ell) \in C[0, \infty)\}$$

and $\psi(\ell)$ is called in the space C_{ϑ}^n if $\psi^{(n)} \in C_{\vartheta}, n \in N$.

Definition 2.2. The Riemann Liouville fractional integral operator of order $\alpha \geq 0$ of a function $\psi(\ell) \in C_{\vartheta}, \vartheta \geq -1$ is defined as

$$I^{\alpha} \psi(\ell) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^{\ell} (\ell - \tau)^{\alpha-1} \psi(\tau) d\tau, & \alpha > 0, \ell > 0 \\ I^0 \psi(\ell) = \psi(\ell), & \alpha = 0 \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Properties of the operator I^{α} are as follows: For $\psi \in C_{\vartheta}, \vartheta \geq -1, \alpha, \beta \geq 0$, then

1. $I^{\alpha} I^{\beta} \psi(\ell) = I^{\alpha+\beta} \psi(\ell)$
2. $I^{\alpha} I^{\beta} \psi(\ell) = I^{\beta} I^{\alpha} \psi(\ell)$
3. $I^{\alpha} \ell^n = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} \ell^{\alpha+n}$

Definition 2.3. The fractional derivative of $\psi(\ell)$ in the Caputo sense is defined as

$$D^{\alpha} \psi(\ell) = I^{n-\alpha} D^n \psi(\ell) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\ell} (\ell - \tau)^{n-\alpha-1} \psi^{(n)}(\tau) d\tau, \tag{2.2}$$

for $n - 1 < \alpha \leq n, n \in N, \ell > 0$ and $\psi \in C_{-1}^n$.

The fundamental properties of the operator D^{α} are given as follows:

1. $D^{\alpha} I^{\alpha} \psi(\ell) = \psi(\ell)$

$$2. D^\alpha I^\alpha \psi(\ell) = \psi(\ell) - \sum_{k=0}^{n-1} \psi^{(k)}(0) \frac{\ell^k}{k!}$$

Definition 2.4. The Mittag-Leffler function E_α when $\alpha > 0$ is given by the following formula:

$$E_\alpha(\ell) = \sum_{n=0}^{\infty} \frac{\ell^\alpha}{\Gamma(n\alpha + 1)} \tag{2.3}$$

Definition 2.5. The Sumudu transform is defined over the set of function

$$A = \left\{ \psi(\ell) / \exists M, \omega_1, \omega_2 > 0 \text{ s.t. } |\psi(\ell)| < M e^{\frac{|\ell|}{\omega_j}}, \text{ if } \ell \in (-1)^j \times [0, \infty) \right\}$$

by the following formula

$$S[\psi(\ell)](\omega) = \int_0^\infty e^{-\ell} \psi(\omega \ell) d\ell, \quad \omega \in (-\omega_1, \omega_2) \tag{2.4}$$

Definition 2.6. The Sumudu transform of the Caputo fractional derivative is defined as

$$S[D_\ell^\alpha \psi(\ell)] = \omega^{-\alpha} S[\psi(\ell)] - \sum_{k=0}^{n-1} \omega^{-\alpha+k} \psi^{(k)}(0), \quad n - 1 < \alpha \leq n$$

3. Fractional Sumudu Homotopy Perturbation Method (FSHPM)

Consider a general fractional nonlinear nonhomogeneous partial differential equation is given by the form:

$$D_\ell^{(\alpha)} \psi(\ell) + R[\psi(\ell)] + [N\psi(\ell)] = \eta(\ell), \quad 0 < \alpha \leq 1 \tag{3.1}$$

with the initial condition

$$\psi(0) = \zeta(\ell), \tag{3.2}$$

where $D_\ell^{(\alpha)} \psi(\ell)$ is the Caputo fractional derivative of the function $\psi(\ell)$ defined as:

$$D_\ell^{(\alpha)} \psi(\ell) = \frac{d^\alpha \psi(\ell)}{d\ell^\alpha} \begin{cases} \frac{1}{\Gamma(n - \alpha)} \int_0^\ell (\ell - \omega)^{n-\alpha-1} \frac{d^n \psi(\ell)}{d\ell^n} d\omega, & n - 1 < \alpha < n \\ \frac{d^n \psi(\ell)}{d\ell^n}, & \alpha = n \in N \end{cases} \tag{3.3}$$

and R is linear differential operator, N represents the general nonlinear differential operator, and $\eta(\ell)$ is the source term.

Applying Sumudu transform on both sides of (3.1), we get

$$S[D_\ell^{(\alpha)} \psi(\ell)] + S[R[\psi(\ell)]] + S[N[\psi(\ell)]] = S[\eta(\ell)], \tag{3.4}$$

Using the property of the ST, we obtain

$$S[\psi(\ell)] = \psi(0) + \omega^\alpha S[\eta(\ell)] - \omega^\alpha S[R[\psi(\ell)] + N[\psi(\ell)]], \tag{3.5}$$

Operating the inverse Sumudu transform on both sides of (3.5), we get

$$\psi(\ell) = S^{-1}(\zeta(\ell)) + S^{-1}(\omega^\alpha S[\eta(\ell)]) - S^{-1}(\omega^\alpha S[R[\psi(\ell)] + N[\psi(\ell)]]), \tag{3.6}$$

By applying HPM, we represent the solution as an infinite series which is given below:

$$\psi(\ell) = \sum_{n=0}^{\infty} p^n \psi_n(\ell) \quad (3.7)$$

and the nonlinear term can be designed as

$$N[\psi(\ell)] = \sum_{n=0}^{\infty} p^n H_n(\psi_0, \psi_1, \dots, \psi_n) \quad (3.8)$$

where $H_n(\ell)$ is the He's polynomial and be computed using the following formula:

$$H_n(\psi_0, \psi_1, \dots, \psi_n) = \frac{1}{n!} \frac{d^n}{dp^n} \left[N \left(\sum_{i=0}^{\infty} p^i \psi_i(\ell) \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots$$

By applying (3.7) and (3.8) in (3.6), we get:

$$\sum_{n=0}^{\infty} p^n \psi_n(\ell) = S^{-1}(\zeta(\ell)) + S^{-1}(\omega^\alpha S[\eta(\ell)]) - p S^{-1} \left(\omega^\alpha S \left[R \left[\sum_{n=0}^{\infty} p^n \psi_n(\ell) \right] + \sum_{n=0}^{\infty} p^n H_n \right] \right) \quad (3.9)$$

When the coefficients of like powers of p are compared in (3.9), the series of equations is given by the following:

$$\begin{aligned} p^0 : \psi_0(\ell) &= S^{-1}(\zeta(\ell)) + S^{-1}(\omega^\alpha S[\eta(\ell)]), \\ p^1 : \psi_1(\ell) &= -S^{-1} \left(\omega^\alpha S \left[R[\psi_0(\ell)] + H_0 \right] \right), \\ p^2 : \psi_2(\ell) &= -S^{-1} \left(\omega^\alpha S \left[R[\psi_1(\ell)] + H_1 \right] \right), \\ &\vdots \\ p^n : \psi_n(\ell) &= -S^{-1} \left(\omega^\alpha S \left[R[\psi_{n-1}(\ell)] + H_{n-1} \right] \right), \end{aligned} \quad (3.10)$$

Subsequently, we use a truncated series to approximate the analytical solution of the (3.1):

$$\psi(\ell) = \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n \psi_n(\ell) \quad (3.11)$$

4. Applications

The proposed technique (HSPM) for solving fractional Riccati differential equations will be applied in this section.

4.1. Example

First, Consider the fractional Riccati differential equation

$$D_\ell^\alpha \psi(\ell) = -\psi^2(\ell) + 1, \quad 0 < \alpha \leq 1 \quad (4.1)$$

subject to initial condition

$$\psi(0) = 0 \quad (4.2)$$

Applying Sumudu transform on both sides of (4.1), and using the differential property of ST, we have

$$S[\psi(\ell)] = \psi(0) + \omega^\alpha S[-\psi^2(\ell) + 1] \tag{4.3}$$

When the inverse Sumudu transform is set to (4.3), it implies that

$$\psi(\ell) = \frac{\ell^\alpha}{\Gamma(\alpha + 1)} - S^{-1}(\omega^\alpha S[\psi^2(\ell)]) \tag{4.4}$$

According to the HPM, substituting (3.7) and

$$\begin{aligned} \psi(\ell) &= \sum_{n=0}^{\infty} p^n \psi_n(\ell) \\ \psi^2(\ell) &= \sum_{n=0}^{\infty} p^n H_n(\psi_0, \psi_1, \dots, \psi_n) \end{aligned} \tag{4.5}$$

in (4.4), we have

$$\sum_{n=0}^{\infty} p^n \psi_n(\ell) = \frac{\ell^\alpha}{\Gamma(\alpha + 1)} - p S^{-1}\left(\omega^\alpha S\left[\sum_{n=0}^{\infty} p^n H_n\right]\right) \tag{4.6}$$

Comparing the coefficients of like powers of p , we get

$$\begin{aligned} p^0 : \psi_0(\ell) &= \frac{\ell^\alpha}{\Gamma(\alpha + 1)} \\ p^1 : \psi_1(\ell) &= -S^{-1}\left(\omega^\alpha S[H_0]\right) \\ &= -S^{-1}\left(\omega^\alpha S\left[\frac{\ell^\alpha}{\Gamma(\alpha + 1)}\right]\right) = -\frac{\Gamma(2\alpha + 1)}{\Gamma^2(\alpha + 1)} \frac{\ell^{3\alpha}}{\Gamma(3\alpha + 1)} \\ p^2 : \psi_2(\ell) &= -S^{-1}\left(\omega^\alpha S[H_1]\right) \\ &= -S^{-1}\left(\omega^\alpha S\left[\frac{2\ell^\alpha}{\Gamma(\alpha + 1)} \left(-\frac{\Gamma(2\alpha + 1)}{\Gamma^2(\alpha + 1)} \frac{\ell^{3\alpha}}{\Gamma(3\alpha + 1)}\right)\right]\right) \\ &= \frac{2\Gamma(2\alpha + 1)\Gamma(4\alpha + 1)}{\Gamma^3(\alpha + 1)\Gamma(3\alpha + 1)} \frac{\ell^{5\alpha}}{\Gamma(5\alpha + 1)} \\ p^3 : \psi_3(\ell) &= -S^{-1}\left(\omega^\alpha S[H_2]\right) \\ &= -S^{-1}\left(\omega^\alpha S\left[\frac{2\ell^\alpha}{\Gamma(\alpha + 1)} \left(\frac{2\Gamma(2\alpha + 1)\Gamma(4\alpha + 1)}{\Gamma^3(\alpha + 1)\Gamma(3\alpha + 1)} \frac{\ell^{5\alpha}}{\Gamma(5\alpha + 1)}\right)\right]\right) \\ &+ \left(-\frac{\Gamma(2\alpha + 1)}{\Gamma^2(\alpha + 1)} \frac{\ell^{3\alpha}}{\Gamma(3\alpha + 1)}\right)^2 \\ &= -\frac{\Gamma^2(2\alpha + 1)\Gamma(6\alpha + 1)}{\Gamma^4(\alpha + 1)\Gamma^2(3\alpha + 1)} \frac{\ell^{7\alpha}}{\Gamma(7\alpha + 1)} - \frac{4\Gamma(2\alpha + 1)\Gamma(4\alpha + 1)\Gamma(6\alpha + 1)}{\Gamma^4(\alpha + 1)\Gamma(3\alpha + 1)\Gamma(5\alpha + 1)} \frac{\ell^{7\alpha}}{\Gamma(7\alpha + 1)} \\ &\vdots \end{aligned}$$

Hence, the approximate series solution (3.11) is given by

$$\begin{aligned} \psi(\ell) &= \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n \psi_n(\ell) \\ &= \frac{\ell^\alpha}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{\ell^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} \frac{\ell^{5\alpha}}{\Gamma(5\alpha+1)} - \\ &\quad \frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)} \frac{\ell^{7\alpha}}{\Gamma(7\alpha+1)} - \frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)}{\Gamma^4(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \frac{\ell^{7\alpha}}{\Gamma(7\alpha+1)} \\ &\quad + \dots \end{aligned} \quad (4.7)$$

As $\alpha = 1$, the equation (4.7) becomes

$$\psi(\ell) = \ell - \frac{\ell^3}{3} + \frac{2\ell^5}{15} - \frac{17\ell^7}{63} + \dots \quad (4.8)$$

The FSHPM results are completely consistent with the FHPM results [47].

4.2. Example

Consider the fractional Riccati differential equation

$$D_\ell^\alpha \psi(\ell) = 2\psi(\ell) - \psi^2(\ell) + 1, \quad 0 < \alpha \leq 1 \quad (4.9)$$

subject to the initial condition

$$\psi(0) = 0 \quad (4.10)$$

Applying Sumudu transform to the both sides of (4.9), and using the differential property of ST, we have

$$S[\psi(\ell)] = \psi(0) + \omega^\alpha S[2\psi(\ell) - \psi^2(\ell) + 1] \quad (4.11)$$

Using the inverse Sumudu transform on both sides of (4.11), we obtain

$$\psi(\ell) = \frac{\ell^\alpha}{\Gamma(\alpha+1)} - S^{-1}(\omega^\alpha S[2\psi(\ell) - \psi^2(\ell)]) \quad (4.12)$$

According to the HPM, substituting (3.7) and (4.5) in (4.12), we have

$$\sum_{n=0}^{\infty} p^n \psi_n(\ell) = \frac{\ell^\alpha}{\Gamma(\alpha+1)} + pS^{-1} \left(\omega^\alpha S \left[2 \sum_{n=0}^{\infty} p^n \psi_n - \sum_{n=0}^{\infty} p^n H_n \right] \right) \quad (4.13)$$

Comparing the coefficients of like powers of p , we obtain

$$\begin{aligned} p^0 : \psi_0(\ell) &= \frac{\ell^\alpha}{\Gamma(\alpha+1)} \\ p^1 : \psi_1(\ell) &= S^{-1}(\omega^\alpha S[2\psi_0 - H_0]) \\ &= S^{-1} \left(\omega^\alpha S \left[\frac{2\ell^\alpha}{\Gamma(\alpha+1)} - \frac{\ell^{2\alpha}}{\Gamma^2(\alpha+1)} \right] \right) \\ &= \frac{2\ell^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{\ell^{3\alpha}}{\Gamma(3\alpha+1)} \end{aligned} \quad (4.14)$$

$$\begin{aligned}
p^2 : \psi_2(\ell) &= S^{-1}(\omega^\alpha S[2\psi_1 - H_1]) \\
&= S^{-1}\left(\omega^\alpha S\left[\frac{4\ell^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right.\right. \\
&\quad \left.\left.+ \frac{4\ell^{3\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} - \frac{2\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)} \frac{\ell^{4\alpha}}{\Gamma(3\alpha+1)}\right]\right) \\
&= \frac{4\ell^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \frac{\ell^{4\alpha}}{\Gamma(4\alpha+1)} - \\
&\quad \frac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{\ell^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} \frac{\ell^{5\alpha}}{\Gamma(5\alpha+1)} \\
&\quad \vdots
\end{aligned}$$

Hence, the approximate series solution (3.11) is given by

$$\begin{aligned}
\psi(\ell) &= \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n \psi_n(\ell) \\
&= \frac{\ell^\alpha}{\Gamma(\alpha+1)} - \frac{2\ell^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{\ell^{3\alpha}}{\Gamma(3\alpha+1)} + \\
&\quad \frac{4\ell^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} \frac{\ell^{4\alpha}}{\Gamma(4\alpha+1)} - \\
&\quad \frac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{\ell^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} \frac{\ell^{5\alpha}}{\Gamma(5\alpha+1)} \\
&\quad + \dots
\end{aligned} \tag{4.15}$$

As $\alpha = 1$, the equation (4.15) becomes

$$\psi(\ell) = \ell - \frac{\ell^3}{3} + \frac{2\ell^5}{15} - \frac{\ell^7}{63} + \dots \tag{4.16}$$

The FSHPM results are completely consistent with the FHPM results [47].

5. Conclusion

In this work, the SHPM has been executed effectively to solve the fractional differential equations and find the approximate solutions of it. The analytical technique provides a series solution that fast approaches the exact solution. The achieved results demonstrate that the proposed method is effective in solving nonlinear fractional differential equations. FSHPM's findings are in excellent agreement with the results obtained by FHPM, as shown by two examples.

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