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Coupled Systems of Equations with Entire and Polynomial Functions

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Abstract

We consider the coupled system $F(x,y) = G(x,y) = 0$, where

$$F(x,y) = 0m_1 A_k(y)x^{m_1-k} \text{ and } G(x,y) = 0m_2 B_k(y)x^{m_2-k}$$

with entire functions $A_k(y), B_k(y)$. We derive a priory estimates for the sums of the roots of the considered system and for the counting function of roots.

Keywords: Coupled Systems, Entire and Polynomial functions, A Priory Estimates, Resultant.
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1. Introduction and Statements of the Main Result

Let us consider the system

$$F(x,y) = G(x,y) = 0, \tag{1.1}$$

where

$$F(x,y) = \sum_{k=0}^{m_1} A_k(y)x^{m_1-k} \text{ and } G(x,y) = \sum_{k=0}^{m_2} B_k(y)x^{m_2-k} \quad (x, y \in \mathbb{C})$$

with the entire functions

$$A_k(y) = \sum_{j=0}^{\infty} a_{kj}y^j, \quad B_k(y) = \sum_{j=0}^{\infty} b_{kj}y^j \ , k \geq 1.$$

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Such systems arise in various applications. In particular, they describe stationary states of various systems of nonlinear differential equations [12] and functional-differential equations [8]. The basic methods for the investigations of systems of the type (1.1) are topological methods, in particular, the fixed point theorems [4, 10, 17]. The other approach for the problem of computing zeros of analytic mappings (in other words, for solving systems of analytic equations) is the logarithmic residue based approach. A multidimensional logarithmic residue formula is available in the literature, cf. [2, 9]. That formula involves the integral of a differential form, which can be transformed into a sum of certain Riemann integrals. The zeros and their respective multiplicities can be computed from these integrals by solving a generalized eigenvalue problem that has the Hankel structure. Besides, in the case, when $A_j(z)$ and $B_j(z)$ are polynomials, the literature is very rich, cf. [5, 14] and references therein.

A few words about the numerical methods in the coupled systems theory. The classical numerical methods can be found in [15]; recently, the Newton method was considerably developed [3, 16]. Besides the essential role is played the Adomian polynomials [1]. Note that for the application of the Newton method, the differentiability is required. For the applications of the topological methods and Newton one, a priory estimates for the roots are often required, however, to the best of our knowledge, such estimates for (1.1) were not enough considered in the available literature.

A pair of complex numbers (y, x) is a solution of (1.1) if $F(x, y) = G(x, y) = 0$. Besides x will be called an X-root coordinate (corresponding to y) and y a Y-root coordinate (corresponding to x). All the considered roots are counted with their multiplicities. In this paper we suggest the a priory estimates for the Y-coordinates of the roots of (1.1). Our approach is new and based on the recent results for matrix-valued functions.

For $m = m_1 + m_2$ introduce the $m \times m$-matrices

$$C_j = \begin{pmatrix} a_{0j} & a_{1j} & a_{2j} & \dots & a_{m_1-1,j} & a_{m_1,j} & 0 & 0 & \dots & 0 \\ 0 & a_{0j} & a_{1j} & \dots & a_{m_1-2,j} & a_{m_1-1,j} & a_{m_1,j} & 0 & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a_{0,j} & a_{1,j} & a_{2,j} & a_{3,j} & \dots & a_{m_1,j} \\ b_{0j} & b_{1j} & b_{2j} & \dots & b_{m_2-1,j} & b_{m_2,j} & 0 & 0 & \dots & 0 \\ 0 & b_{0j} & b_{1j} & \dots & b_{m_2-2,j} & b_{m_2-1,j} & b_{m_2,j} & 0 & \dots & 0 \\ . & . & . & . & . & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & b_{0j} & b_{1j} & b_{2j} & b_{3j} & \dots & b_{m_2,j} \end{pmatrix}$$

(j=0, 1, ...). It is supposed that C_0 is invertible. Put $D_j = C_0 C_j(j!)$, for a $\gamma \in (0, 1]$, and assume that the series

$$\Theta_0 := \left[\sum_{k=1}^{\infty} D_k D_k^*\right]^{1/2} \text{ converges.} \tag{1.2}$$

Here and below the asterisk means the adjointness. So $_0$ is an $m \times m$-matrix and under (1.2) by the Hólder inequality, it follows that the pencil

$$H_0(y) := \sum_{k=1}^{\infty} \frac{y^k}{(k!)^\gamma} D_k \quad (y \in \mathbb{C})$$

satisfies the inequality

$$\|H_0(y)\| \le c_0 \sum_{k=1}^{\infty} \frac{|y|^k}{(k!)^\gamma} \le c_0 \left[\sum_{k=1}^{\infty} 2^{-p'k}\right]^{1/p'} \left[\sum_{k=1}^{\infty} \frac{|2y|^{k/\gamma}}{k!}\right]^\gamma \le c_1 e^{\gamma |2y|^{1/\gamma}},$$

where $\gamma + 1/p' = 1$, $\|.\|$ is the matrix spectral norm, that is the operator norm corresponding to the Euclidean norm of vectors,

$$c_0 = \sup_k \|D_k\|, \ c_1 = c_0 \ \left[02^{-kp'}\right]^{1/p'}.$$

So function H_0 has order no more than $1/\gamma$.

Put

$$\omega_k = \begin{cases} \lambda_k(\Theta_0) & \text{for } k = 1, ..., m, \\ 0 & \text{for } k = m+1, m+2, ... \end{cases}$$

Here and below $\lambda_k(A)$ are the eigenvalues of a matrix A counted with their multiplicities and enumerated in the decreasing way: $|\lambda_{k+1}(A)| \le |\lambda_k(A)|$. Now we are in a position to formulate our main result.

Theorem 1.1. *For a $\gamma \in (0,1]$, let condition (1.2) hold. Then the Y-roots y_k of (1.1) counted with their multiplicities and enumerated in the nondecreasing way: $|\tilde{y}_k| \le |\tilde{y}_{k+1}|$ $(k = 1, 2, ...)$ satisfy the inequalities*

$$\sum_1^j \frac{1}{|\tilde{y}_k|} < \sum_1^j \left[\omega_k + \frac{m^\gamma}{(k+m)^\gamma}\right] \quad (j = 1, 2, ...).$$

This theorem is proved in the next section.

Note that by Lemma 2.11.3 [6],

$$\|C_0\| \le \frac{N_2^{m-1}(C_0)}{(m-1)^{(m-1)/2}\ |det\ C_0|},$$

where

$$N_2(C_0) = \sqrt{Trace\ C_0 C_0^*}.$$

So $\|\Theta_0\| \le \theta_0$, where

$$\theta_0 := \frac{N_2^{m-1}(C_0)}{(m-1)^{(m-1)/2}\ |det\ C_0|} \left[\sum_{k=1}^\infty \|C_j\|^2\right]^{1/2}.$$

Thus,

$$\omega_k \le \theta_0 \text{ for } k = 1, ..., m \text{ and } \omega_k = 0 \text{ for } k \ge m+1.$$

2. Proof of Theorem 1.1

Let $T_k,\ k = 1, 2, ...$ be $n \times n$-matrices. Consider the entire matrix pencil

$$H(\lambda) := \sum k = 0^\infty \frac{\lambda^k}{(k!)^\gamma} T_k \quad (T_0 = I_n, \in \mathbb{C}), \tag{2.1}$$

where I_n is the unit $n \times n$-matrix. The characteristic values of H, that is the zeros of $det\ H(z)$, with their multiplicities are enumerated in the nondecreasing way are denoted by $z_k(H)$. Suppose that

$$\Theta_H := \left[\sum k = 1^\infty T_k T_k^*\right]^{1/2} \text{ converges }. \tag{2.2}$$

Furthermore, put

$$\hat{\omega}_k(H) = \lambda_k(\Theta_H) \text{ for } k = 1, ..., n \text{ and } \hat{\omega}_k(H) = 0 \text{ for } k \geq n + 1.$$

Let condition (2.2) hold. Then the characteristic values of the pencil H defined by (2.1) satisfy the inequalities

$$\sum_{k=1}^{j} \frac{1}{|z_k(H)|} < \sum_{k=1}^{j} \left[\hat{\omega}_k(H) + \frac{n^\gamma}{(k+n)^\gamma}\right] \quad (j = 1, 2, ...). \tag{2.3}$$

For the proof see [7, Theorem 12.2.1]. Furthermore, for $m = m_1 + m_2$ introduce the $m \times m$ Sylvester matrix

$$S(y) = \begin{pmatrix} A_0 & A_1 & A_2 & \dots & A_{m_1-1} & A_{m_1} & 0 & 0 & \dots & 0 \\ 0 & A_0 & A_1 & \dots & A_{m_1-2} & A_{m_1-1} & A_{m_1} & 0 & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & A_0 & A_1 & A_2 & A_3 & \dots & A_{m_1} \\ B_0 & B_1 & B_2 & \dots & B_{m_2-1} & B_{m_2} & 0 & 0 & \dots & 0 \\ 0 & B_0 & B_1 & \dots & B_{m_2-2} & B_{m_2-1} & B_{m_2} & 0 & \dots & 0 \\ . & . & . & . & . & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & B_0 & B_1 & B_2 & B_3 & \dots & B_{m_2} \end{pmatrix}$$

with $A_j = A_j(y)$ and $B_j = B_j(y)$. So

$$S(y) = \sum_{k=0}^{\infty} C_j y^j = C_0 \sum_{k=0}^{\infty} D_j y^j = H_0(y).$$

As it is well known [11], the Y-roots of (1.1) are the characteristic values of $S(y)$ that is the zeros of the resultant $det\ S(y)$. Take into account that

$$det\ S(y) = det\ C_0\ \ det\ H_0(y).$$

Now the required result is due to (2.3).

3. The Counting Function for Roots

Put

$$\chi_k = \omega_k + \frac{m^\gamma}{(k+m)^\gamma} \quad (k = 1, 2, ...).$$

The following result is due to the well-known Lemma 1.2.1 [7] and Theorem 1.1.

Corollary 3.1. *Let $\phi(t)$ $(0 \leq t < \infty)$ be a continuous convex scalar-valued function, such that $\phi(0) = 0$. Then under condition (1.2), the inequalities*

$$\sum_{k=1}^{j} \phi\left(1|\tilde{y}_k|\right) < \sum_{k=1}^{j} \phi(\chi_k) \quad (j = 1, 2, ...)$$

are valid. In particular, for any $r \geq 1$,

$$\sum_{k=1}^{j} \frac{1}{|\tilde{y}_k|^r} < \sum_{k=1}^{j} \chi_k^r$$

and thus

$$\left[\sum_{k=1}^{j}\frac{1}{|\tilde{y}_k|^r}\right]^{1/r} < \left[\sum_{k=1}^{j}\omega_k^r\right]^{1/r} + m^{\gamma}\left[\sum_{k=1}^{j}\frac{1}{(k+m)^{r\gamma}}\right]^{1/r} \quad (j=1,2,...).$$

Furthermore, assume that

$$r\gamma > 1,\ r \geq 1. \tag{3.1}$$

Then

$$\zeta_m(\gamma r) := \sum_{k=1} \infty \frac{1}{(k+m)^{r\gamma}} < \infty.$$

Relation (3.1) with the notation

$$N_r(\Theta_0) = [\sum_{k=1} m\lambda_k^r(\Theta_0)]^{1/r}$$

yields our next result:

Corollary 3.2. *Let conditions (1.2) and (3.1) hold. Then*

$$(\sum_{k=1}^{\infty}\frac{1}{|\tilde{y}_k|^r})^{1/r} < N_r(\Theta_0) + m^{\gamma}\zeta_m^{1/r}(\gamma r).$$

Since $\tilde{y}_j \leq \tilde{y}_{j+1}$, from (1.3), it follows that $|y_j| > \eta_j$ where

$$\eta_j := \frac{j}{\sum_{k=1}^{j}[\omega_k + \frac{m^{\gamma}}{(k+m)^{\gamma}}]}.$$

We thus get our next result.

Corollary 3.3. *Under the hypothesis of Theorem 1.1, system (1.1) has in $|z| \leq \eta_j$ no more than $j-1$ Y-root coordinates; in particular, in $|z| \leq \eta_1$ system (1.1) does have Y-root coordinates.*

Let $\nu_Y(r)$ be the counting function of the Y-roots (1.1) in $|z| \leq r$. We thus get the inequality $\nu_Y(r) \leq j-1$ for any $r \leq \eta_j$.

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