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Some inference of the repeated measurements model

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Abstract

This paper is dedicated to studying the repeated measurements model (RMM). We investigate the analysis of variance (ANOVA) of the model and identify the confidence intervals of the variance components and identify the analytic form of the likelihood- ratio test for the model.

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1. Introduction

Repeated measurements model (RMM) is one of the most models widely used in the field of experimental design especially in agricultural researches, biomedical and epidemiology that is involved in this field. Repeated measurements is a term used to describe data in which the response variable for each experimental units is observed on multiple occasions and possibly under different experimental conditions [10]. Many literatures have been given to the univariate repeated measurements analysis of variance (RM ANOVA). The repeated measurements model has been investigated by many researchers. AL-Mouel and Wang [9] presented the sphericity test for the one-way multivariate repeated measurements analysis of variance model. Fakhir [2] studied a two-way multivariate repeated measurement analysis of covariance model. Abbas in [1] discussed a one-way multivariate repeated measurements analysis of covariance model. AL-Shmailawi and Hasan in [10] studied a one-way repeated measurements model. AL- Mouel and AL-Isawi in [4] gave best quadratic unbiased estimator of variance components for balanced data for linear one-way repeated measurements model. AL-Mouel and AL-Isawi [5] studied minimum-quadratic unbiased estimator of variance components for the repeated measurements model by a new approach. AL- Mouel and AL-Isawi [5] investigated the estimator of variance components of one-way repeated measurements model using MINQUE-

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principle. AL-Mouel and Jassim [7] proposed the lasso estimator for high dimensional repeated measurements model. In this work, the mathematical model is defined and the confidence intervals of the variance components are obtained and define the analytical form of the likelihood-ratio test for the model. The results of this paper are applied in many biological, industrial, agricultural and other applications.

2. Mathematical Model

We consider the model described by

$$\mathrm{y}_{ijk} = \mu + \tau_j + \beta_k + (\tau\beta)_{jk} + \delta_{i(j)} + \gamma_{i(k)} + \omega_{i(jk)} + \varepsilon_{ijk} \tag{2.1}$$

where
$i = 1, 2, \ldots, a$ is an index for experimental unit within group j,
$j = 1, 2, \ldots, b$ is an index for levels of the between-units factor (Group),
$k = 1, 2, \ldots, c$ is an index for levels of the within- units factor (Time),
y_{ijk} is the response measurement at time k for unit i within group j,
μ is the overall mean,
τ_j is the added effect for treatment group j,
β_k is the added effect for time k,
$(\tau\beta)_{jk}$ is the added effect for the group $j\times$ time k interaction,
$\delta_{i(j)}$ is the random effect for due to experimental unit i within treatment group j,
$\gamma_{i(k)}$ is the random effect for due to experimental unit i within time k,
$\omega_{i(jk)}$ is the random effect for due to experimental unit i within interaction jk,
ε_{ijk} is the random error on time k for unit i within group j,
For the parameterization to be of full rank, we imposed the following set of conditions:
$\sum_{j=1}^{b} \tau_j = 0 \quad , \quad \sum_{k=1}^{c} \beta_k = 0$
$\sum_{j=1}^{b} (\tau\beta)_{jk} = 0$, for each $k = 1, 2, \ldots, c$
$\sum_{k=1}^{c} (\tau\beta)_{jk} = 0$, for each $j = 1, 2, \ldots, b$
We assume also that ε_{ijk} 's, $\delta_{i(j)}$'s, $\gamma_{i(k)}$'s and $\omega_{i(jk)}$'s are independent with

$$\varepsilon_{ijk} \sim i.i.d.N\left(0, \sigma_\varepsilon^2\right) \tag{2.2}$$

$$\delta_{i(j)} \sim i.i.d.N\left(0, \sigma_\delta^2\right) \tag{2.3}$$

$$\gamma_{i(k)} \sim i.i.d.N\left(0, \sigma_\gamma^2\right) \tag{2.4}$$

$$\omega_{i(jk)} \sim i.i.d.N\left(0, \sigma_\omega^2\right) \tag{2.5}$$

Now we state the analysis of variance table for the repeated measurement model including sum of square terms for Group (SS_G), Time (SS_T) , (Group $\times$ Time) $(SS_{G\times T})$, unit (Group) $\left(SS_{U(G)}\right)$, unit (Time) $\left(SS_{U(T)}\right)$, unit (Group $\times$ Time) $\left(SS_{U(G\times T)}\right)$, and error (SS_E), also the degree of freedom (d.f), mean square (MS), and expectation of mean square $E(MS)$ as the following:

Table 1: ANOVA table for the repeated measurement model.

Source of variance	d.f	SS	MS	E (MS)
Group	$(b-1)$	SS_G	$\frac{SS_G}{b-1}$	$\frac{ac}{(b-1)}\sum_{j=1}^{b}\tau_j^2+c\sigma_\delta^2+\sigma_\omega^2+\sigma_\varepsilon^2$
Time	$(c-1)$	SS_T	$\frac{SS_T}{c-1}$	$\frac{ab}{(c-1)}\sum_{k=1}^{c}\beta_k^2+b\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2$
Group $\times$ Time	$(b-1)(c-1)$	$SS_{G\times T}$	$\frac{SS_{G\times T}}{(b-1)(c-1)}$	$\frac{a}{(b-1)(c-1)}\sum_{j=1}^{b}\sum_{k=1}^{c}(\tau\beta)_{jk}^2+\sigma_\omega^2+\sigma_\varepsilon^2$
Unit (Group)	$b(a-1)$	$SS_{U(G)}$	$\frac{SS_{U(G)}}{b(a-1)}$	$c\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2$
Unit (Time)	$c(a-1)$	$SS_{U(T)}$	$\frac{SS_{U(T)}}{c(a-1)}$	$\sigma_\delta^2+b\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2$
Unit(Group Time	$bc(a-1)$	$SS_{U(G\times T)}$	$\frac{SS_{U(G\times T)}}{bc(a-1)}$	$\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2$
Residual	$(b+c)(1-a)$	SS_E	$\frac{SS_E}{(1-a)(b+c)}$	σ_ε^2
Total	$abc-1$	SS_{Total}		

Where

$$SS_G = ac\sum_{j=1}^{b}\left(\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{...}\right)^2$$

$$SS_T = ab\sum_{k=1}^{c}\left(\overline{\mathrm{y}}_{..k}-\overline{\mathrm{y}}_{...}\right)^2$$

$$SS_{G\times T} = a\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\overline{\mathrm{y}}_{.jk}-\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{..k}-\overline{\mathrm{y}}_{...}\right)^2$$

$$SS_{U(G)} = c\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\overline{\mathrm{y}}_{ij.}-\overline{\mathrm{y}}_{.j.}\right)^2$$

$$SS_{U(T)} = b\sum_{i=1}^{a}\sum_{k=1}^{c}\left(\overline{\mathrm{y}}_{i.k}-\overline{\mathrm{y}}_{..k}\right)^2$$

$$SS_{U(G\times T)} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2$$

$$SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\overline{\mathrm{y}}_{.j.}+\overline{\mathrm{y}}_{..k}-\overline{\mathrm{y}}_{ij.}-\overline{\mathrm{y}}_{i.k}\right)^2$$

and
$\overline{\mathrm{y}}_{...} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\mathrm{y}_{ijk}}{abc}$ is the overal mean
$\overline{\mathrm{y}}_{.j.} = \frac{\sum_{i=1}^{a}\sum_{k=1}^{c}\mathrm{y}_{ijk}}{ac}$ is the mean for group j
$\overline{\mathrm{y}}_{..k} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{1}\mathrm{y}_{ijk}}{ab}$ is the mean for time k
$\overline{\mathrm{y}}_{.jk} = \frac{\sum_{i=1}^{a}\mathrm{y}_{ijk}}{a}$ is the mean for group $j\times$ time k
$\overline{\mathrm{y}}_{ij.} = \frac{\sum_{k=1}^{c}\mathrm{y}_{ijk}}{c}$ is the mean for i^{th} subject in group j
$\overline{\mathrm{y}}_{i.k} = \frac{\sum_{j=1}^{b}\mathrm{y}_{ijk}}{b}$ is the mean for i^{th} subject in time k

3. Confidence intervals

Construct confidence intervals of $\sigma_\varepsilon^2, \sigma_\delta^2, \sigma_\gamma^2, \sigma_\omega^2$ and the ratios between them, as well as their sum, as follows:

3.1. Confidence intervals for $\sigma_\varepsilon^2, \sigma_\delta^2, \sigma_\gamma^2$ and σ_ω^2

We consider the confidence interval for σ_ε^2
Since $\frac{SS_E}{\sigma_\varepsilon^2} \sim \chi^2[(b+c)(1-a)]$

$$p\left(\chi^2[(b+c)(1-a), \propto/2] \leq \frac{SS_E}{\sigma_\varepsilon^2} \leq \chi^2[(b+c)(1-a), 1-\propto/2]\right) = 1-\propto$$

$$p\left(\frac{\chi^2[(b+c)(1-a), \propto/2]}{SS_E} \leq \frac{1}{\sigma_\varepsilon^2} \leq \frac{\chi^2[(b+c)(1-a), 1-\propto/2]}{SS_E}\right) = 1-\propto$$

$$p\left(\frac{SS_E}{\chi^2[(b+c)(1-a), 1-\propto/2]} \leq \sigma_\varepsilon^2 \leq \frac{SS_E}{\chi^2[(b+c)(1-a), \propto/2]}\right) = 1-\propto$$

Therefore, a $100(1-\propto)\%$ confidence interval for σ_ε^2 is $\left(\frac{SS_E}{\chi^2[(b+c)(1-a),1-\propto/2]}, \frac{SS_E}{\chi^2[(b+c)(1-a),\propto/2]}\right)$.
We consider the confidence interval for σ_δ^2
since $MS_G \sim \left(c\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right) \frac{\chi^2[b(a-1)]}{b(a-1)}$

$$MS_E \sim \sigma_\varepsilon^2 \frac{\chi^2[(b+c)(1-a)]}{(b+c)(1-a)}$$

and MS_G, MS_E are independent

$$\frac{MS_G}{MS_E} \sim \left(\frac{c\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2}{\sigma_\varepsilon^2}\right) F[b(a-1), (b+c)(1-a)]$$

$$\Rightarrow \frac{MS_G/MS_E}{1+c\theta} \sim F[b(a-1), (b+c)(1-a)] \text{ where } 1+c\theta = \frac{c\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2}{\sigma_\varepsilon^2}$$

$$p\left(F[b(a-1), (b+c)(1-a), \propto/2] \leq \frac{MS_G/MS_E}{1+c\theta} \leq F[b(a-1), (b+c)(1-a), 1-\propto/2]\right)$$
$$= 1-\propto$$

$$p\left(\frac{MS_E}{MS_G} F[b(a-1), (b+c)(1-a), \propto/2] \leq \frac{1}{1+c\theta} \leq \frac{MS_E}{MS_G} F[b(a-1), (b+c)(1-a), 1-\propto/2]\right)$$
$$= 1-\propto$$

$$p\left(\frac{1}{c}(\frac{MS_G}{MS_E F[b(a-1),(b+c)(1-a),1-\propto/2]}-1)\leq\theta\right.$$
$$\left.\leq\frac{1}{c}(\frac{MS_G}{MS_E F[b(a-1),(b+c)(1-a),\propto/2]}-1)\right)$$
$$=1-\propto$$

Therefore, a $100(1-\propto)\%$ confidence interval for θ is

$$\left[\frac{1}{c}\left(\frac{MS_G}{MS_E F[b(a-1),(b+c)(1-a),1-\propto/2]}-1\right),\frac{1}{c}\left(\frac{MS_G}{MS_E F[b(a-1),(b+c)(1-a),\propto/2]}-1\right)\right].$$

We consider the confidence interval for σ_γ^2
since $MS_T\sim\left(\sigma_\delta^2+b\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)\frac{\chi^2[c(a-1)]}{c(a-1)}$

$$MS_E\sim\sigma_\varepsilon^2\frac{\chi^2[(b+c)(1-a)]}{(b+c)(1-a)}$$

and MS_T, MS_E are independent

$$\frac{MS_T}{MS_E}\sim\left(\frac{\sigma_\delta^2+b\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}{\sigma_\varepsilon^2}\right)F[c(a-1),(b+c)(1-a)]$$

$$\Rightarrow\frac{MS_T/MS_E}{1+b\emptyset}\sim F[c(a-1),(b+c)(1-a)]\text{ where }1+b\emptyset=\frac{\sigma_\delta^2+b\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}{\sigma_\varepsilon^2}$$

$$p\left(F[c(a-1),(b+c)(1-a),\propto/2]\leq\frac{MS_T/MS_E}{1+b\emptyset}\leq F[c(a-1),(b+c)(1-a),1-\propto/2]\right)=1-\propto$$

$$p\left(\frac{MS_E}{MS_T}F[c(a-1),(b+c)(1-a),\propto/2]\leq\frac{1}{1+b\emptyset}\leq\frac{MS_E}{MS_T}F[c(a-1),(b+c)(1-a),1-\propto/2]\right)$$
$$=1-\propto$$

$$p\left(\frac{1}{b}(\frac{MS_T}{MS_E F[c(a-1),(b+c)(1-a),1-\propto/2]}-1)\leq\emptyset\right.$$
$$\left.\leq\frac{1}{b}(\frac{MS_T}{MS_E F[c(a-1),(b+c)(1-a),\propto/2]}-1)\right)$$
$$=1-\propto$$

Therefore, a $100(1-\propto)\%$ confidence interval for $\emptyset$ is

$$\left[\frac{1}{b}\left(\frac{MS_T}{MS_E F[c(a-1),(b+c)(1-a),1-\propto/2]}-1\right),\frac{1}{b}\left(\frac{MS_T}{MS_E F[c(a-1),(b+c)(1-a),\propto/2]}-1\right)\right].$$

We consider the confidence interval for σ_ω^2
since $MS_{G\times T} \sim \left(\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right) \frac{\chi^2[bc(a-1)]}{bc(a-1)}$

$$MS_E \sim \sigma_\varepsilon^2 \frac{\chi^2[(b+c)(1-a)]}{(b+c)(1-a)}$$

and $MS_{G\times T}, MS_E$ are independent

$$\frac{MS_{G\times T}}{MS_E} \sim \left(\frac{\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2}{\sigma_\varepsilon^2}\right) F[bc(a-1),(b+c)(1-a)]$$

$$\Rightarrow \frac{MS_{G\times T}/MS_E}{1+\varphi} \sim F[bc(a-1),(b+c)(1-a)] \text{ where } 1+\varphi = \frac{\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2}{\sigma_\varepsilon^2}$$

$$p\left(F[bc(a-1),(b+c)(1-a),\propto/2] \le \frac{MS_{G\times T}/MS_E}{1+\varphi} \le F[bc(a-1),(b+c)(1-a),1-\propto/2]\right) \\ = 1-\propto$$

$$p\left(\frac{MS_E}{MS_{G\times T}}F[bc(a-1),(b+c)(1-a),\propto/2] \le \frac{1}{1+\varphi} \le \frac{MS_E}{MS_{G\times T}}F[bc(a-1),(b+c)(1-a),1-\propto/2]\right) \\ = 1-\propto$$

$$p\left(\left(\frac{MS_{G\times T}}{MS_E F[bc(a-1),(b+c)(1-a),1-\propto/2]} - 1\right) \le \varphi \\ \le \left(\frac{MS_{G\times T}}{MS_E F[bc(a-1),(b+c)(1-a),\propto/2]} - 1\right)\right) \\ = 1-\propto$$

Therefore, a $100(1-\propto)\%$ confidence interval for φ is

$$\left[\left(\frac{MS_{G\times T}}{MS_E F[bc(a-1),(b+c)(1-a),1-\propto/2]} - 1\right), \left(\frac{MS_{G\times T}}{MS_E F[bc(a-1),(b+c)(1-a),\propto/2]} - 1\right)\right].$$

3.2. Simultaneous confidence intervals $\frac{\sigma_\delta^2}{\sigma_\varepsilon^2}, \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}$ *and* $\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}$

Using distribution laws in

$$SS_E \sim \sigma_\varepsilon^2 \chi^2[(b+c)(1-a)]$$
$$SS_{G\times T} \sim \left(\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right)\chi^2[bc(a-1)]$$
$$SS_G \sim \left(c\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right)\chi^2[b(a-1)]$$
$$SS_T \sim \left(\sigma_\delta^2 + b\sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right)\chi^2[c(a-1)]$$

we have

$$\frac{MS_{G\times T}/\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}{MS_E/\sigma_\varepsilon^2}\sim F[bc(a-1),(b+c)(1-a)]$$
$$\frac{MS_G/\left(c\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}{MS_E/\sigma_\varepsilon^2}\sim F[b(a-1),(b+c)(1-a)]$$
$$\frac{MS_T/\left(\sigma_\delta^2+b\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}{MS_E/\sigma_\varepsilon^2}\sim F[c(a-1),(b+c)(1-a)]$$

$$p\left(F[bc(a-1),(b+c)(1-a),\propto/2]\le\frac{\sigma_\varepsilon^2}{\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}\frac{MS_{G\times T}}{SS_E}\right.$$
$$\le F[bc(a-1),(b+c)(1-a),1-\propto/2])$$
$$=1-\propto_1$$

$$p\left(\frac{MS_{G\times T}/MS_E}{F[bc(a-1),(b+c)(1-a),1-\propto/2]}\le\frac{\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}{\sigma_\varepsilon^2}\le\frac{MS_{G\times T}/MS_E}{F[bc(a-1),(b+c)(1-a),\propto/2]}\right)$$
$$=1-\propto_1$$

$$p\left(\frac{MS_{G\times T}/MS_E}{F[bc(a-1),(b+c)(1-a),1-\propto/2]}\le 1+\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}+\frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}+\frac{\sigma_\delta^2}{\sigma_\varepsilon^2}\le\frac{MS_{G\times T}/MS_E}{F[bc(a-1)(b+c)(1-a),\propto/2]}\right)$$
$$=1-\propto_1 \tag{3.1}$$

$$p\left(F[b(a-1),(b+c)(1-a),\propto/2]\le\frac{\sigma_\varepsilon^2}{c\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}\frac{MS_G}{SS_E}\le F[b(a-1),(b+c)(1-a),1-\propto/2]\right)$$
$$=1-\propto_2$$

$$p\left(\frac{MS_G/MS_E}{F[b(a-1),(b+c)(1-a),1-\propto/2]}\le\frac{c\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}{\sigma_\varepsilon^2}\le\frac{MS_G/MS_E}{F[b(a-1),(b+c)(1-a),\propto/2]}\right)$$
$$=1-\propto_2$$

$$p\left(\frac{MS_G/MS_E}{F[b(a-1),(b+c)(1-a),1-\propto/2]}\le 1+\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}+\frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}+\frac{c\sigma_\delta^2}{\sigma_\varepsilon^2}\le\frac{MS_G/MS_E}{F(b(a-1),(b+c)(1-a),\propto/2)}\right)$$
$$=1-\propto_2 \tag{3.2}$$

and

$$p\left(F[c(a-1),(b+c)(1-a),\propto/2]\le\frac{\sigma_\varepsilon^2}{\sigma_\delta^2+b\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}\frac{MS_T}{SS_E}\le F[c(a-1),(b+c)(1-a),1-\propto/2]\right)$$
$$=1-\propto_3$$

$$p\left(\frac{MS_T/MS_E}{F[c(a-1),(b+c)(1-a),1-\propto/2]}\leq\frac{\sigma_\delta^2+b\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}{\sigma_\varepsilon^2}\leq\frac{MS_T/MS_E}{F[c(a-1),(b+c)(1-a),\propto/2]}\right)$$
$$=1-\propto_3$$

$$p\left(\frac{MS_T/MS_E}{F[c(a-1),(b+c)(1-a),1-\propto/2]}\leq 1+\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}+\frac{b\sigma_\gamma^2}{\sigma_\varepsilon^2}+\frac{\sigma_\delta^2}{\sigma_\varepsilon^2}\leq\frac{MS_T/MS_E}{F[c(a-1),(b+c)(1-a),\propto/2]}\right)$$
$$=1-\propto_3 \tag{3.3}$$

we can make the joint probability statements from (3.1), (3.2), and (3.3), i.e.

$$p\left(L_1\leq 1+\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}+\frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}+\frac{\sigma_\delta^2}{\sigma_\varepsilon^2}\leq U_1, L_2\leq 1+\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}+\frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}+\frac{c\sigma_\delta^2}{\sigma_\varepsilon^2}\leq U_2, L_3\leq 1+\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}+\frac{b\sigma_\gamma^2}{\sigma_\varepsilon^2}+\frac{\sigma_\delta^2}{\sigma_\varepsilon^2}\leq U_3\right)$$
$$=1-\varphi \tag{3.4}$$

$$p\left(L_\omega\leq\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}\leq U_\omega, L_\delta\leq\frac{\sigma_\delta^2}{\sigma_\varepsilon^2}\leq U_\delta, L_\gamma\leq\frac{\sigma_\gamma^2}{\sigma_\varepsilon^2}\leq U_\gamma\right)=1-\varphi$$

where

$$\begin{aligned}
&1-\varphi\geq(1-\propto_1)(1-\propto_2)(1-\propto_3)\\
&L_\omega=\min(L_1-1,L_2-1,L_3-1)\\
&U_\omega=\min(U_1-1,U_2-1,U_3-1)\\
&L_\delta=\frac{1}{c}(L_2-1), U_\delta=(U_2-1)\\
&L_\gamma=\frac{1}{b}(L_3-1), U_\gamma=(U_3-1)
\end{aligned}$$

3.3. Confidence intervals for $\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2$

$$p\left(\begin{array}{l}\hat\theta-\frac{1}{abc}\sqrt{\left[a^2D_1^2MS_G^2+b^2D_2^2MS_T^2+(bc(a-1)-1)^2D_3^2MS_{G\times T}^2+((b+c)(1-a))^2D_4^2MS_E\right]}\leq\sigma_\delta^2+\sigma_\gamma^2\\ +\sigma_\omega^2+\sigma_\varepsilon^2\leq\hat\theta+\frac{1}{abc}\sqrt{\left[a^2H_1^2MS_G^2+b^2H_2^2MS_T^2+(bc(a-1)-1)^2H_3^2MS_{G\times T}^2+((b+c)(1-a))^2H_4^2MS_E\right]}\end{array}\right)$$
$$=1-\propto$$

where

$$\hat\theta=\frac{1}{abc}aMS_G^2+bMS_T^2+(bc(a-1)-1)MS_{G\times T}^2+(b+c)(1-a)MS_E$$

$$\begin{array}{ll}
D_1=1-F^{-1}(c(a-1),\infty;1-\propto/2) & H_1=F^{-1}(c(a-1),\infty;\propto/2)-1\\
D_2=1-F^{-1}(b(a-1),\infty;1-\propto/2) & H_2=F^{-1}(b(a-1),\infty;\propto/2)-1\\
D_3=1-F^{-1}(bc(a-1),\infty;1-\propto/2) & H_3=F^{-1}(bc(a-1),\infty;\propto/2)-1\\
D_4=1-F^{-1}((b+c)(1-a),\infty;1-\propto/2) & H_4=F^{-1}((b+c)(1-a),\infty;\propto/2)-1
\end{array}$$

4. The likelihood-ratio test

Consider $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$
Let

$$\Omega = \left\{ \left(\mu, \tau_j, \beta_k, (\tau\beta)_{jk}, \sigma_\delta^2, \sigma_\gamma^2, \sigma_\omega^2, \sigma_\varepsilon^2\right), j = 1, \ldots, b, k = 1, \ldots, c : \mu, \tau_j, \beta_k, (\tau\beta)_{jk} \in R \text{ and } \sigma_\delta^2 > 0, \sigma_\gamma^2 > 0, \sigma_\omega^2 > 0, \sigma_\varepsilon^2 > 0 \right\}$$

$v = \left\{ \left(\mu, \tau_j, \beta_k, (\tau\beta)_{jk}, \sigma_\delta^2, \sigma_\gamma^2, \sigma_\omega^2, \sigma_\varepsilon^2\right) \in \Omega : \mu = 0, j = 1, \ldots, b, k = 1, \ldots, c \right\}$
The likelihood functions denoted by $L(v)$ and $L(\Omega)$ are the following:

$$L(v) = \left[2\pi\left(\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right)\right]^{-abc/2} \exp\left[\frac{-\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \tau_j - \beta_k - (\tau\beta)_{jk}\right)^2}{2\left(\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right)}\right]$$

$$L(\Omega) = \left[2\pi\left(\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right)\right]^{-abc/2} \exp\left[\frac{-\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \mu - \tau_j - \beta_k - (\tau\beta)_{jk}\right)^2}{2\left(\sigma_\delta^2 + \sigma_\gamma^2 + \sigma_\omega^2 + \sigma_\varepsilon^2\right)}\right]$$

Now we consider the problem of maximizing $L(v)$ and $L(\Omega)$. Recall that

$$\bar{\mathrm{y}}_{\ldots} = \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \mathrm{y}_{ijk}}{abc}, \bar{\mathrm{y}}_{.j.} = \frac{\sum_{i=1}^a \sum_{k=1}^c \mathrm{y}_{ijk}}{ac}, \bar{\mathrm{y}}_{..k} = \frac{\sum_{i=1}^a \sum_{j=1}^b \mathrm{y}_{ijk}}{ab}, \bar{\mathrm{y}}_{.jk} = \frac{\sum_{i=1}^a \mathrm{y}_{ijk}}{a}$$

$$\hat{\mu} = \bar{\mathrm{y}}_{\ldots}, \hat{\tau}_j = \bar{\mathrm{y}}_{.j.} - \bar{\mathrm{y}}_{\ldots}, \hat{\beta}_k = \bar{\mathrm{y}}_{..k} - \bar{\mathrm{y}}_{\ldots}, \widehat{(\tau\beta)}_{jk} = \bar{\mathrm{y}}_{.jk} - \bar{\mathrm{y}}_{.j.} - \bar{\mathrm{y}}_{.k} + \bar{\mathrm{y}}$$

and $\left(\widehat{\sigma_\delta^2} + \widehat{\sigma_\gamma^2} + \widehat{\sigma_\omega^2} + \widehat{\sigma_\varepsilon^2}\right) = \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{jk}\right)^2}{abc}$

$$\begin{aligned}
\text{Let} \lambda =& \frac{L(\hat{v})}{L(\hat{\Omega})} \\
=& \frac{\exp\left[\frac{-\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{.jk} - \bar{\mathrm{y}}_{\ldots}\right)^2}{2\left\{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}{\exp\left[\frac{-\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{.jk}\right)^2}{2\left\{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]} \\
=& \frac{\exp\left[\frac{-\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{.jk}\right)^2 - abc\bar{\mathrm{y}}_{\ldots}^2}{2\left\{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}{\exp\left[\frac{-abc}{2}\right]}
\end{aligned}$$

$$\Rightarrow \lambda = \exp\left[\frac{-abc}{2}\left(1 + \frac{abc\bar{\mathrm{y}}_{\ldots}^2}{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{.jk}\right)^2}\right) + \frac{abc}{2}\right]$$

$$\Rightarrow \ln \lambda = \frac{-\left(abc\bar{\mathrm{y}}_{\ldots}\right)^2}{2\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c \left(\mathrm{y}_{ijk} - \bar{\mathrm{y}}_{.jk}\right)^2}$$

$$\Rightarrow -2\ln\lambda = \frac{(abc\overline{\mathrm{y}}_{...})^2}{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}$$

since the function $-2\ln\lambda$ is a decreasing function, it follows that the critical region of the likelihood-ratio test can also be expressed in the form
$d_1 = \{x : -2\ln\lambda \geq d\}$, writing $\Lambda(x) = -2\ln\lambda \Rightarrow d_1 = \{x : \Lambda(x) \geq d\}$
since $\overline{\mathrm{y}}_{...} \sim N\left(\mu, \frac{\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}{abc}\right)$ then

$$\frac{\overline{\mathrm{y}}_{...}-0}{\sqrt{\frac{\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2}{abc}}} \sim N(0,1) \Rightarrow \frac{abc\overline{\mathrm{y}}_{...}^2}{\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2} \sim \chi^2(1)$$

$$\Rightarrow \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}{\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2} \sim \chi^2((b-1)(c-1))$$

The test of the composite hypothesis based on an F statistic with 1 and $(b-1)(c-1)$ degrees of freedom.
Consider $H_0 : \tau_j = 0$ versus $H_1 : \tau_j \neq 0, j = 1, 2, \ldots, b$
The likelihood functions denoted by $L(v)$ and $L(\Omega)$ are the following:

$$L(v) = \left[2\pi\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)\right]^{-abc/2}\exp\left[\frac{-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\mu-\beta_k-(\tau\beta)_{jk}\right)^2}{2\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}\right]$$

$$L(\Omega) = \left[2\pi\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)\right]^{-abc/2}\exp\left[\frac{-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\mu-\tau_j-\beta_k-(\tau\beta)_{jk}\right)^2}{2\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}\right]$$

$$\begin{aligned}\text{Let}\lambda =&\frac{L(\hat{v})}{L(\hat{\Omega})}\\ =&\frac{\exp\left[\frac{-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}+\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{...}\right)^2}{2\left\{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}{\exp\left[\frac{-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}{2\left\{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}\\ =&\frac{\exp\left[\frac{-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2-\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{...}\right)^2}{2\left\{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}{\exp\left[\frac{-abc}{2}\right]}\end{aligned}$$

$$\Rightarrow \lambda = \exp\left[\frac{-abc}{2}\left(1+\frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{...}\right)^2}{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}\right)+\frac{abc}{2}\right]$$

$$\Rightarrow -2\ln\lambda = \frac{abc\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{...}\right)^2}{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}$$

$$\Rightarrow -2\ln\lambda = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left[\sqrt{abc}\left(\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{...}\right)\right]^2}{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}$$

since the function $-2\ln\lambda$ is a decreasing function, it follows that the critical region of the likelihood-ratio test can also be expressed in the form
$d_1=\{x:-2\ln\lambda\geq d\}$, writing $\Lambda(x)=-2\ln\lambda\Rightarrow d_1=\{x:\Lambda(x)\geq d\}$
The test of the composite hypothesis based on an F statistic with $(b-1)$ and $(b-1)(c-1)$ degrees of freedom.
Consider $H_0:\beta_k=0$ versus $H_1:\beta_k\neq 0, k=1,2,\ldots,c$
The likelihood functions denoted by $L(v)$ and $L(\Omega)$ are the following:

$$L(v)=\left[2\pi\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)\right]^{-abc/2}\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\mu-\tau_j-(\tau\beta)_{jk}\right)^2}{2\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}\right]$$

$$L(\Omega)=\left[2\pi\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)\right]^{-abc/2}\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\mu-\tau_j-\beta_k-(\tau\beta)_{jk}\right)^2}{2\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}\right]$$

$$\text{Let}\lambda=\frac{L(\hat{v})}{L(\widehat{\Omega})}$$

$$=\frac{\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}+\bar{\mathrm{y}}_{..k}-\bar{\mathrm{y}}_{...}\right)^2}{2\left\{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}{\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}\right)^2}{2\left\{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}$$

$$=\frac{\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}\right)^2-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c(\bar{\mathrm{y}}_{..k}-\bar{\mathrm{y}}_{...})^2}{2\left\{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}{\exp\left[\frac{-abc}{2}\right]}$$

$$\Rightarrow\lambda=\exp\left[\frac{-abc}{2}\left(1+\frac{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\bar{\mathrm{y}}_{..k}-\bar{\mathrm{y}}_{...}\right)^2}{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}\right)^2}\right)+\frac{abc}{2}\right]$$

$$\Rightarrow-2\ln\lambda=\frac{abc\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\bar{\mathrm{y}}_{..k}-\bar{\mathrm{y}}_{...}\right)^2}{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}\right)^2}$$

$$\Rightarrow-2\ln\lambda=\frac{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left[\sqrt{abc}\left(\bar{\mathrm{y}}_{..k}-\bar{\mathrm{y}}_{...}\right)\right]^2}{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\bar{\mathrm{y}}_{.jk}\right)^2}$$

since the function $-2\ln\lambda$ is a decreasing function, it follows that the critical region of the likelihood-ratio test can also be expressed in the form
$d_1=\{x:-2\ln\lambda\geq d\}$, writing $\Lambda(x)=-2\ln\lambda\Rightarrow d_1=\{x:\Lambda(x)\geq d\}$
The test of the composite hypothesis based on an F statistic with $(c-1)$ and $(b-1)(c-1)$ degrees of freedom.
Consider $H_0:(\tau\beta)_{jk}=0$ versus $H_1:(\tau\beta)_{jk}\neq 0, j=1,2,\ldots,b, k=1,2,\ldots,c$
The likelihood functions denoted by $L(v)$ and $L(\Omega)$ are the following:

$$L(v)=\left[2\pi\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)\right]^{-abc/2}\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\mu-\tau_j-\beta_k\right)^2}{2\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}\right]$$

$$L(\Omega)=\left[2\pi\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)\right]^{-abc/2}\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\mu-\tau_j-\beta_k-(\tau\beta)_{jk}\right)^2}{2\left(\sigma_\delta^2+\sigma_\gamma^2+\sigma_\omega^2+\sigma_\varepsilon^2\right)}\right]$$

$$\text{Let }\lambda=\frac{L(\hat{v})}{L(\hat{\Omega})}$$

$$=\frac{\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{..k}+\overline{\mathrm{y}}_{...}\right)^2}{2\left\{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}{\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}{2\left\{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}$$

$$=\frac{\exp\left[\frac{-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2-\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\overline{\mathrm{y}}_{.jk}-\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{..k}+\overline{\mathrm{y}}_{...}\right)^2}{2\left\{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2\right\}/abc}\right]}{\exp\left[\frac{-abc}{2}\right]}$$

$$\Rightarrow\lambda=\exp\left[\frac{-abc}{2}\left(1+\frac{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\overline{\mathrm{y}}_{.jk}-\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{..k}+\overline{\mathrm{y}}_{...}\right)^2}{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}\right)+\frac{abc}{2}\right]$$

$$\Rightarrow-2\ln\lambda=\frac{abc\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\overline{\mathrm{y}}_{.jk}-\overline{\mathrm{y}}_{.j.}-\overline{\mathrm{y}}_{..k}+\overline{\mathrm{y}}_{...}\right)^2}{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}$$

$$\Rightarrow-2\ln\lambda=\frac{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left[\sqrt{abc}\left(\overline{\mathrm{y}}_{.jk}-\overline{\mathrm{y}}_{.j.}-\bar{y}_{..k}+\overline{\mathrm{y}}_{....}\right)\right]^2}{\sum_{i=1}^a\sum_{j=1}^b\sum_{k=1}^c\left(\mathrm{y}_{ijk}-\overline{\mathrm{y}}_{.jk}\right)^2}$$

since the function $-2\ln\lambda$ is a decreasing function, it follows that the critical region of the likelihood-ratio test can also be expressed in the form
$d_1=\{x:-2\ln\lambda\geq d\}$, writing $\Lambda(x)=-2\ln\lambda\Rightarrow d_1=\{x:\Lambda(x)\geq d\}$
The test of the composite hypothesis based on an F statistic with $(b-1)(c-1)$ and $(b-1)(c-1)$ degrees of freedom.

5. Conclusions

The conclusions which are obtained throughout this work are given as follows

1- Confidence intervals were obtained for the variance components $\boldsymbol{\sigma}_\varepsilon^2,\boldsymbol{\sigma}_\delta^2,\boldsymbol{\sigma}_\gamma^2,\boldsymbol{\sigma}_\omega^2,\frac{\sigma_\delta^2}{\sigma_\varepsilon^2},\frac{\sigma_\gamma^2}{\sigma_\varepsilon^2},\frac{\sigma_\omega^2}{\sigma_\varepsilon^2}$ and $\boldsymbol{\sigma}_\delta^2+\boldsymbol{\sigma}_\gamma^2+\boldsymbol{\sigma}_\omega^2+\sigma_\varepsilon^2$ for the repeated measurements model (RMM).

2- The likelihood-ratio test of the parameters μ,τ_j,β_k and $(\tau\beta)_{jk}$ for the repeated measurements model (RMM) are obtained.

3- The results of this paper are widely applied in many fields of knowledge, including epidemiology, medical sciences, life sciences and others.

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