



# On the degree of approximation of certain continuous bivariate functions by double matrix means of a double Fourier series

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## Abstract

In this paper, we have studied the degree of approximation of certain bivariate functions by double factorable matrix means of a double Fourier series. Four theorems are proved using single rest bounded variation sequences, single head bounded variation sequences, double rest bounded variation sequences, and two non-negative mediate functions. These results expressed in terms of two functions of modulus type and two non-negative mediate functions, imply many particular results as shown at last section of this paper.

*Keywords:* Double Fourier series, Lipschitz class, Factorable matrices, Cesàro means, Nörlund means, Riesz means.

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## 1. Introduction

Let  $f(x, y)$  be a complex-valued function,  $2\pi$ -periodic in each variable, and integrable over the two-dimensional torus  $-\pi < x, y \leq \pi$ ; in symbols  $f \in L_{2\pi \times 2\pi}$ . The double Fourier series of  $f$  is defined by

$$\mathcal{L}[f] := \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)},$$

where

$$c_{jk} := \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) e^{-i(js+kt)} ds dt, \quad j, k \in \mathbb{Z}. \quad (1.1)$$

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The double sequence of symmetric rectangular partial sums will be denoted by

$$s_{mn}(x, y) := \sum_{j=-m}^m \sum_{k=-n}^n c_{jk} e^{i(jx+ky)}, \quad m, n \in \mathbb{N} \cup \{0\}.$$

Let  $A := (a_{mnjk})$  denote a doubly infinite matrix with non-negative entries and row sums 1. Here and in the sequel we shall be concerned with positive rectangular matrices; i.e.,  $a_{mnjk} = 0$  for  $j > m$  or  $k > n$ , and  $a_{mnjk} \geq 0$  for each  $0 \leq j \leq m, 0 \leq k \leq n$ . For any  $\{s_{jk}\}$  any double sequence, we define

$$t_{mn} := \sum_{j=0}^m \sum_{k=0}^n a_{mnjk} s_{jk}, \quad m, n \in \mathbb{N} \cup \{0\}.$$

The double sequence  $\{s_{jk}\}$  is said to be summable by  $A$  if  $t_{mn}$  tends to a finite limit as  $m, n \rightarrow \infty$  (see [23]).

A doubly infinite matrix  $A$  is said to be regular if it sums every bounded convergent double sequence  $\{s_{jk}\}$  to the same limit. Necessary and sufficient conditions of regularity of a matrix  $A$  are known (see [30]) and are:

$$\lim_{m,n \rightarrow \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk} = 1,$$

$$\lim_{m,n \rightarrow \infty} \sum_{j=0}^{\infty} a_{mnjk} = 0,$$

$$\lim_{m,n \rightarrow \infty} \sum_{k=0}^{\infty} a_{mnjk} = 0,$$

and

$$\sup_{m,n \geq 0} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{mnjk}| < \infty.$$

The matrix  $A$  will be called factor-able if there exist sequences  $\{a_{mj}\}$  and  $\{b_{nk}\}$  so that  $a_{mnjk} = a_{mj} b_{nk}$  and the above condition of regularity are satisfied, and we focus only on this case below (see [29], [18], [19], [1], [22]).

The transformation  $t_{mn}(x, y)$  of the partial sums  $s_{mn}(x, y)$  is

$$t_{mn}(x, y) := \sum_{j=0}^m \sum_{k=0}^n a_{mj} b_{nk} s_{jk}(x, y), \quad m, n \in \mathbb{N} \cup \{0\},$$

and based on (1.1) we get

$$t_{mn}(x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + s, y + t) K_{mn}(s, t) ds dt, \quad m, n \in \mathbb{N} \cup \{0\}, \tag{1.2}$$

where

$$K_{mn}(s, t) ds dt = \sum_{j=0}^m \sum_{k=0}^n a_{mj} b_{nk} D_j(s) D_k(t), \quad m, n \in \mathbb{N} \cup \{0\}, \tag{1.3}$$

is the matrix kernel and  $D_j(s)$  and  $D_k(t)$  are the Dirichlet kernels in terms of  $s$  and  $t$ , respectively.

Moreover, as a consequence of (1.2), the properties of the Dirichlet kernel, and the assumptions on the matrix  $A$ , we easily obtain the equality

$$t_{mn}(x, y) - f(x, y) = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \phi_{xy}(s, t) K_{mn}(s, t) ds dt, \tag{1.4}$$

where

$$\begin{aligned} \phi_{xy}(s, t) := & f(x + s, y + t) + f(x - s, y + t) \\ & + f(x + s, y - t) + f(x - s, y - t) - 4f(x, y). \end{aligned}$$

The (total) modulus of continuity of a continuous function  $f(x, y)$ ,  $2\pi$ -periodic in each variable, in symbols  $f \in L_{2\pi \times 2\pi}$ , is defined by (see [24], page 283)

$$\omega_1(f, \delta_1, \delta_2) = \sup_{x, y} \sup_{|u| \leq \delta_1, |v| \leq \delta_2} |f(x + u, y + v) - f(x, y)|, \quad \delta_1, \delta_2 \geq 0.$$

The (total) modulus of symmetric smoothness of a function  $f \in L_{2\pi \times 2\pi}$  is defined (see [24], page 283) by

$$\omega_2(f, \delta_1, \delta_2) = \sup_{x, y} \sup_{|u| \leq \delta_1, |v| \leq \delta_2} |\phi_{xy}(u, v)|, \quad \delta_1, \delta_2 \geq 0.$$

It is clear that

$$\omega_2(f, \delta_1, \delta_2) \leq 4\omega_1(f, \delta_1, \delta_2).$$

Now we shall recall the following definitions introduced in [21], [20].

**Definition 1.1.** A sequence  $\mathbf{c} := \{c_k\}$  of non-negative numbers tending to zero is called of Rest Bounded Variation, or briefly  $\mathbf{c} \in RBVS$ , if it has the property:

$$\sum_{k=m}^\infty |c_k - c_{k+1}| \leq K(\mathbf{c})c_m, \quad \forall m \in \mathbb{N},$$

where  $K := K(\mathbf{c})$  is a positive bounded constant which depends only on the sequence  $\mathbf{c}$ .

**Definition 1.2.** A sequence  $\mathbf{c} := \{c_k\}$  of non-negative numbers will be called of Head Bounded Variation, or briefly  $\mathbf{c} \in HBVS$ , if it satisfies the following inequalities:

$$\sum_{k=0}^{m-1} |c_k - c_{k+1}| \leq K(\mathbf{c})c_m,$$

for all  $m \in \mathbb{N}$ , or only for all  $m \leq N$  if the sequence  $\mathbf{c}$  has only finite nonzero term, and the last non-zero term is  $c_N$ .

Now we can give the conditions to be used later on. We suppose that for all  $m, n$  and  $0 \leq s \leq m$ ,  $0 \leq r \leq n$ , the conditions

$$\sum_{j=s}^\infty |a_{mj} - a_{m,j+1}| \leq K a_{ms}, \tag{1.5}$$

$$\sum_{k=r}^\infty |b_{nk} - b_{n,k+1}| \leq K b_{nr}, \tag{1.6}$$

$$\sum_{j=0}^{s-1} |a_{mj} - a_{m,j+1}| \leq K a_{ms}, \quad (1.7)$$

and

$$\sum_{k=0}^{r-1} |b_{nk} - b_{n,k+1}| \leq K b_{nr}, \quad (1.8)$$

hold true.

Also, we assume the validity of conditions

$$a_{mj} \geq 0, \quad b_{nk} \geq 0, \quad (m, n, j, k \in \{0, 1, 2, \dots\}), \quad \sum_{j=0}^m \sum_{k=0}^n a_{mj} b_{nk} = 1, \quad (1.9)$$

$$\int_0^{r_i} H_i(z_i) dz_i = \mathcal{O}(r_i H_i(r_i)), \quad r_i \rightarrow 0^+, \quad (1.10)$$

and

$$\int_{r_i}^{\pi} z_i^{-2} \omega^{(i)}(z_i) dz_i = \mathcal{O}(H_i(r_i)), \quad r_i \rightarrow 0^+, \quad i = 1, 2, \quad (1.11)$$

where  $H_i(z_i) \geq 0$  are two mediate function of the variable  $z_i$ , ( $i = 1, 2$ ), respectively, and  $\omega^{(i)}(z_i)$  are two non-negative functions of modulus type, i.e. continues functions on the interval  $[0, \pi]$ , non-decreasing, and possess the property:  $\omega^{(i)}(0) = 0$ .

The degree of approximation of a class of functions using matrix means (with monotone entries) of Fourier series has been studied by P. Chandra [5], and five years latter has been improved by the same author in [6]. Many years latter, was L. Leindler [21] who employed rest bounded variation sequences and head bounded variation sequences to prove again four theorem obtaining the same degree of approximation as in Chandra's theorem, which of course, contain them in the sense of using a broader class of matrices. In 2005, J. Nemeth [27] realized that, in first theorem of Chandra as well as in the first theorem of Leindler, the case  $\alpha = 0$  is missing. This case has been completely covered in [27]. In 2011, we have proved also four theorems of this kind, using the product of many transformations of partial sums of the Fourier series of a continuous function. Were B. Wei and D. Yu [34] who, in Chandra's and Leindler's theorems, removed the monotonicity and the rest (head) bounded variation properties in proving of their results. Further, the B. Wei and D. Yu's results has been generalized (as well as Chandra's and Leindler's results) by present author in [12]. Very recently, some particular results of the mentioned theorems are generalized in [13], using the so-called generalized deferred Voronoi-Nörlund means of partial sums of their Fourier series, and some other related results can be found within [14]-[17].

On the other hand, to our best knowledge, the degree of approximation of bivariate functions by double means of double Fourier series and conjugate double Fourier series has been obtained, for the first time by F. Móricz and X. L. Shi [25]), who studied the rate of uniform approximation of functions belonging to the Lipschitz class and for those belonging to the Zygmund class, by rectangular double Cesàro means of the rectangular partial sums of double Fourier series,  $2\pi$ -periodic in each variable. These results have been generalized by F. Móricz and B. E. Rhoades (see [26]) obtaining the rate of uniform approximation of functions belonging to the Lipschitz class and for those belonging to the Zygmund class, using double Nörlund means of the rectangular partial sums of double Fourier series,  $2\pi$ -periodic in each variable. All results obtained in [25] are special cases of those obtained in [26]. Again, were F. Móricz and B. E. Rhoades (see [24]) in 1987, who used double Nörlund means of rectangular partial sums of double Fourier series and double Nörlund means of rectangular

partial sums of conjugate double Fourier series. Their results are given in terms of the modulus of symmetric smoothness. Latter on, N. L. Mittal and B. E. Rhoades, see [23], have studied the rate of uniform approximation by rectangular double matrix means of the rectangular partial sums of double Fourier series of continuous functions,  $2\pi$ -periodic in each variable. Their results are given in terms of the modulus of symmetric smoothness as well. Also, in their results, are obtained the rate of uniform approximation for functions belonging to the Lipschitz class and for those belonging to the Zygmund class. As corollaries, they have obtained the previous work of F. Móricz and B. E. Rhoades. The degree of approximation of bivariate continuous functions, belonging Lipschitz classes, by almost Euler means of double Fourier series has been treated in [28]. They showed that the degree of approximation depends on the modulus of continuity associated with the functions. Also, they derived from their results some interesting corollaries. For completeness of the flow of these studies it worth to remind the readers that results obtained in [25] are generalized further in [32], using the so-called double submethod of the rectangular partial sums of double Fourier series of a bivariate function  $2\pi$ -periodic in each variable.

Let  $Q := [0, 2\pi] \times [0, 2\pi]$ . The purpose of this paper is to estimate the deviation

$$D_{mn} := \max_{(x,y) \in Q} |t_{mn}(x, y) - f(x, y)|,$$

in terms of two functions of modulus type, satisfying some specific conditions, and two non-negative mediate functions.

To do this we need to prove some helpful lemmas, which is the second section of this paper. In third section are given main results with their proofs, and in fourth section we give some special consequences of the main results which indeed will close the organizing of this study.

## 2. Auxiliary Lemmas

**Lemma 2.1** ([5]). *If (1.10) and (1.11) hold then*

$$\int_0^{\pi/n} \omega(t)dt = \mathcal{O}(n^{-2}H(\pi/n)).$$

**Lemma 2.2** ([6]). *If (1.10) and (1.11) hold then*

$$\int_0^r t^{-1}\omega(t)dt = \mathcal{O}(rH(r)), \quad (r \rightarrow +0).$$

**Lemma 2.3.** *If, for  $m, n$  fixed,  $\{a_{mj}\} \in RBVS$  and  $\{b_{nk}\} \in RBVS$ , then uniformly in  $0 < t_1 \leq \pi$  and  $0 < t_2 \leq \pi$ ,*

$$K_m(t_1) := \sum_{j=0}^m a_{mj} \sin\left(j + \frac{1}{2}\right) t_1 = \mathcal{O}(A_{m\tau_1}), \tag{2.1}$$

$$K_n(t_2) := \sum_{k=0}^n b_{nk} \sin\left(k + \frac{1}{2}\right) t_2 = \mathcal{O}(B_{n\tau_2}), \tag{2.2}$$

where  $A_{m\tau_1} := \sum_{r=0}^{\tau_1} a_{mr}$ ,  $B_{n\tau_2} := \sum_{s=0}^{\tau_2} b_{ns}$ , and  $\tau_1 \wedge \tau_2$  denotes the integer part of  $\frac{\pi}{t_1}$  and  $\frac{\pi}{t_2}$  respectively.

*If  $\{a_{mj}\} \in HBVS$  and  $\{b_{nk}\} \in HBVS$ , then*

$$K_m(t_1) = \sum_{j=0}^m a_{mj} \sin\left(j + \frac{1}{2}\right) t_1 = \mathcal{O}\left(\frac{a_{mm}}{t_1}\right), \tag{2.3}$$

$$K_n(t_2) = \sum_{k=0}^n b_{nk} \sin \left( k + \frac{1}{2} \right) t_2 = \mathcal{O} \left( \frac{b_{nn}}{t_2} \right), \tag{2.4}$$

$$K_m(t_1) = \sum_{j=0}^m a_{mj} \sin \left( j + \frac{1}{2} \right) t_1 = \mathcal{O} \left( \frac{a_{m0}}{t_1} \right), \tag{2.5}$$

$$K_n(t_2) = \sum_{k=0}^n b_{nk} \sin \left( k + \frac{1}{2} \right) t_2 = \mathcal{O} \left( \frac{b_{n0}}{t_2} \right), \tag{2.6}$$

**Proof .** The proof can be done in the same way as Lemma 3 in [21]. That is why we omit the details.  $\square$

For further discussion we need to prove an another Lemma. To do this we first recall a definition from [35] (see also [3]).

**Definition 2.4.** A double sequence  $\alpha_{mn} := \{a_{mnjk}\}$  is called DRBVS, briefly  $\{a_{mnjk}\} \in DRBVS$ , if there is a positive bounded constant  $K(\alpha_{mn})$  such that  $(j, k = 0, 1, 2, \dots)$ ,

$$\left. \begin{aligned} & \sum_{j=m}^{\infty} |\Delta_{10} a_{mnjn}| \\ & \sum_{k=n}^{\infty} |\Delta_{01} a_{mnmk}| \\ & \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{mnjk}| \end{aligned} \right\} \leq K a_{mnmn},$$

where

$$\begin{aligned} \Delta_{11} a_{mnjk} &= a_{mnjk} - a_{mnj+1k} - a_{mnjk+1} + a_{mnj+1k+1}, \\ \Delta_{10} a_{mnjk} &= a_{mnjk} - a_{mnj+1k}, \quad \Delta_{01} a_{mnjk} = a_{mnjk} - a_{mnjk+1}. \end{aligned}$$

Using the above definition we prove the following.

**Lemma 2.5.** Let

$$T_{mn}(t_1, t_2) := \sum_{j=0}^m \sum_{k=0}^n a_{mnjk} \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( k + \frac{1}{2} \right) t_2.$$

If for  $m, n$  fixed numbers  $\{a_{mnjk}\} \in DRBVS$ , then uniformly in  $t_1, t_2 \in (0, \pi]$ ,

$$|T_{mn}(t_1, t_2)| = \begin{cases} \mathcal{O} (A_{mn\tau_1\tau_2}), \\ \mathcal{O} (A_{mnm\tau_2}), \\ \mathcal{O} (A_{mn\tau_1n}), \end{cases}$$

where

$$A_{mn\tau_1\tau_2} := \sum_{i_1=0}^{\tau_1} \sum_{i_2=0}^{\tau_2} a_{mni_1i_2}, \quad A_{mnm\tau_2} := \sum_{i_1=0}^m \sum_{i_2=0}^{\tau_2} a_{mni_1i_2}, \quad A_{mn\tau_1n} := \sum_{i_1=0}^{\tau_1} \sum_{i_2=0}^n a_{mni_1i_2},$$

and  $\tau_1$  and  $\tau_2$  denote the integral parts of  $\frac{\pi}{t_1}$  and  $\frac{\pi}{t_2}$ , respectively.

**Proof .** Using the elementary identity  $\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$  and any arbitrary numbers  $\lambda_{jk} \geq 0$ , for  $0 \leq m_1 \leq m$ ,  $0 \leq n_1 \leq n$ , we have

$$\begin{aligned}
 S_2 &:= \sum_{j=m_1}^m \sum_{k=n_1}^n \lambda_{jk} \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( \frac{t_1}{2} \right) \sin \left( k + \frac{1}{2} \right) t_2 \sin \left( \frac{t_2}{2} \right) \\
 &= \frac{1}{2} \sum_{j=m_1}^m \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( \frac{t_1}{2} \right) \left[ \lambda_{jn_1} \cos n_1 t_2 \right. \\
 &\quad \left. - \sum_{k=n_1}^{n-1} \Delta_{01} \lambda_{jk} \cos (k + 1) t_2 - \lambda_{jn} \cos (n + 1) t_2 \right] \\
 &= \frac{1}{2} \left[ \sum_{j=m_1}^m \lambda_{jn_1} \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( \frac{t_1}{2} \right) \right] \cos n_1 t_2 \\
 &\quad - \frac{1}{2} \sum_{k=n_1}^{n-1} \cos (k + 1) t_2 \sum_{j=m_1}^m \Delta_{01} \lambda_{jk} \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( \frac{t_1}{2} \right) \\
 &\quad - \frac{1}{2} \cos (n + 1) t_2 \sum_{j=m_1}^m \lambda_{jn} \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( \frac{t_1}{2} \right) \\
 &= \frac{1}{4} \left[ \lambda_{m_1 n_1} \cos m_1 t_1 - \sum_{j=m_1}^{m-1} \Delta_{10} \lambda_{jn_1} \cos (j + 1) t_1 \right. \\
 &\quad \left. - \lambda_{mn_1} \cos (m + 1) t_1 \right] \cos n_1 t_2 - \frac{1}{4} \sum_{k=n_1}^{n-1} \cos (k + 1) t_2 \\
 &\quad \times \left[ \Delta_{01} \lambda_{m_1 k} \cos m_1 t_1 - \sum_{j=m_1}^{m-1} \Delta_{11} \lambda_{jk} \cos (j + 1) t_1 - \Delta_{01} \lambda_{mk} \cos (m + 1) t_1 \right] - \frac{1}{4} \cos (n + 1) t_2 \\
 &\quad \times \left[ \lambda_{m_1 n} \cos m_1 t_1 - \sum_{j=m_1}^{m-1} \Delta_{10} \lambda_{jn} \cos (j + 1) t_1 - \lambda_{mn} \cos (m + 1) t_1 \right] \\
 &= \frac{1}{4} \left[ \lambda_{m_1 n_1} \cos m_1 t_1 - \sum_{j=m_1}^{m-1} \Delta_{10} \lambda_{jn_1} \cos (j + 1) t_1 - \lambda_{mn_1} \cos (m + 1) t_1 \right] \\
 &\quad \times \cos n_1 t_2 - \frac{1}{4} \left[ \sum_{k=n_1}^{n-1} \Delta_{01} \lambda_{m_1 k} \cos m_1 t_1 \cos (k + 1) t_2 \right. \\
 &\quad - \sum_{j=m_1}^{m-1} \sum_{k=n_1}^{n-1} \Delta_{11} \lambda_{jk} \cos (j + 1) t_1 \cos (k + 1) t_2 \\
 &\quad \left. - \sum_{k=n_1}^{n-1} \Delta_{01} \lambda_{mk} \cos (m + 1) t_1 \cos (k + 1) t_2 \right] - \frac{1}{4} \left[ \lambda_{m_1 n} \cos m_1 t_1 \right. \\
 &\quad \left. - \sum_{j=m_1}^{m-1} \Delta_{10} \lambda_{jn} \cos (j + 1) t_1 - \lambda_{mn} \cos (m + 1) t_1 \right] \cos (n + 1) t_2.
 \end{aligned}$$

Whence,

$$\begin{aligned}
 |S_2| &\leq \frac{1}{4} \left[ \lambda_{m_1 n_1} + \sum_{j=m_1}^{m-1} |\Delta_{10} \lambda_{j n_1}| + \lambda_{m n_1} \right] + \frac{1}{4} \left[ \sum_{k=n_1}^{n-1} |\Delta_{01} \lambda_{m_1 k}| \right. \\
 &\quad \left. + \sum_{j=m_1}^{m-1} \sum_{k=n_1}^{n-1} |\Delta_{11} \lambda_{jk}| + \sum_{k=n_1}^{n-1} |\Delta_{01} \lambda_{mk}| \right] \\
 &\quad + \frac{1}{4} \left[ \lambda_{m_1 n} + \sum_{j=m_1}^{m-1} |\Delta_{10} \lambda_{jn}| + \lambda_{mn} \right].
 \end{aligned}$$

Thus, since  $a_{mnjk} \geq 0$  and supposing that  $m \geq \tau_1, n \geq \tau_2$ , we get

$$\begin{aligned}
 |T_{mn}(t_1, t_2)| &\leq \sum_{k=0}^{\tau_1} \sum_{j=0}^{\tau_2} a_{mnkj} + \left| \sum_{j=\tau_1}^m \sum_{k=0}^{\tau_2} a_{mnjk} \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( k + \frac{1}{2} \right) t_2 \right| \\
 &\quad + \left| \sum_{j=0}^{\tau_1} \sum_{k=\tau_2}^n a_{mnjk} \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( k + \frac{1}{2} \right) t_2 \right| \\
 &\quad + \left| \sum_{j=\tau_1}^m \sum_{k=\tau_2}^n a_{mnjk} \sin \left( j + \frac{1}{2} \right) t_1 \sin \left( k + \frac{1}{2} \right) t_2 \right| \\
 &:= \sum_{k=0}^{\tau_1} \sum_{j=0}^{\tau_2} a_{mnkj} + G_1 + G_2 + G_3.
 \end{aligned}$$

For  $G_1$ , we have

$$\begin{aligned}
 G_1 &\leq \frac{1}{4} \left[ a_{mn\tau_1 0} + \sum_{j=\tau_1}^{m-1} |\Delta_{10} a_{mnj0}| + a_{mnm0} \right] \\
 &\quad + \frac{1}{4} \left[ \sum_{k=0}^{\tau_2-1} |\Delta_{01} a_{mn\tau_1 k}| + \sum_{j=\tau_1}^{m-1} \sum_{k=0}^{\tau_2-1} |\Delta_{11} a_{mnjk}| + \sum_{k=0}^{\tau_2-1} |\Delta_{01} a_{mnmk}| \right] \\
 &\quad + \frac{1}{4} \left[ a_{mn\tau_1 n} + \sum_{j=\tau_1}^{m-1} |\Delta_{10} a_{mnjn}| + a_{mnmn} \right] \\
 &\leq \frac{1}{4} \left[ a_{mn\tau_1 0} + \sum_{j=\tau_1}^{\infty} |\Delta_{10} a_{mnj0}| + a_{mnm0} \right] \\
 &\quad + \frac{1}{4} \left[ \sum_{k=0}^{\infty} |\Delta_{01} a_{mn\tau_1 k}| + \sum_{j=\tau_1}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{mnjk}| + \sum_{k=0}^{\infty} |\Delta_{01} a_{mnmk}| \right] \\
 &\quad + \frac{1}{4} \left[ a_{mn\tau_1 n} + \sum_{j=\tau_1}^{\infty} |\Delta_{10} a_{mnjn}| + a_{mnmn} \right] \\
 &\leq \frac{1}{4} [a_{mn\tau_1 0} + K a_{mn\tau_1 0} + a_{mnm0}] + \frac{1}{4} [K a_{mn\tau_1 0} + K a_{mn\tau_1 0} + K a_{mnm0}] \\
 &\quad + \frac{1}{4} [a_{mn\tau_1 n} + K a_{mn\tau_1 n} + a_{mnmn}]
 \end{aligned}$$



$$\begin{aligned} &\leq \frac{1}{4} [(3K + 1)a_{mn\tau_1 0} + (K + 1)a_{mn\tau_1 n} + (K + 1)a_{mnm 0} + a_{mnmn}] \\ &\leq Ka_{mn\tau_1 \tau_2}, \end{aligned}$$

taking into account that  $\{a_{mnjk}\} \in DHBVS$  implies

$$\begin{aligned} a_{mn\tau_1 0} &\leq \sum_{j=\tau_1}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{mnjk}| \leq \sum_{j=\tau_1}^{\infty} \sum_{k=\tau_2}^{\infty} |\Delta_{11} a_{mnjk}| \leq Ka_{mn\tau_1 \tau_2}, \\ a_{mn\tau_1 n} &\leq \sum_{j=\tau_1}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{mnjk}| \leq \sum_{j=\tau_1}^{\infty} \sum_{k=\tau_2}^{\infty} |\Delta_{11} a_{mnjk}| \leq Ka_{mn\tau_1 \tau_2}, \\ a_{mnm 0} &\leq \sum_{j=m}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{mnjk}| \leq \sum_{j=\tau_1}^{\infty} \sum_{k=\tau_2}^{\infty} |\Delta_{11} a_{mnjk}| \leq Ka_{mn\tau_1 \tau_2}, \end{aligned}$$

and

$$a_{mnmn} \leq \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{mnjk}| \leq \sum_{j=\tau_1}^{\infty} \sum_{k=\tau_2}^{\infty} |\Delta_{11} a_{mnjk}| \leq Ka_{mn\tau_1 \tau_2}.$$

Similarly, we have verified that

$$G_2 \leq Ka_{mn\tau_1 \tau_2}, \quad \text{and} \quad G_3 \leq Ka_{mn\tau_1 \tau_2}.$$

Thus, we obtain

$$\begin{aligned} |T_{mn}(t_1, t_2)| &\leq \sum_{j=0}^{\tau_1} \sum_{k=0}^{\tau_2} a_{mnkj} + G_1 + G_2 + G_3 \\ &\leq \sum_{j=0}^{\tau_1} \sum_{k=0}^{\tau_2} a_{mnkj} + Ka_{mn\tau_1 \tau_2} + Ka_{mn\tau_1 \tau_2} + Ka_{mn\tau_1 \tau_2} = \mathcal{O}(A_{mn\tau_1 \tau_2}), \end{aligned}$$

which is what we wanted to prove.

Similarly, since  $a_{mnjk} \geq 0$  and supposing that  $n \geq \tau_2$ , we get

$$\begin{aligned} |T_{mn}(t_1, t_2)| &\leq \sum_{j=0}^m \sum_{k=0}^{\tau_2} a_{mnkj} + \sum_{k=0}^m \left| \sum_{k=\tau_2}^n a_{mnjk} \sin \left( k + \frac{1}{2} \right) t_2 \right| \\ &\leq A_{mnm\tau_2} + \frac{1}{2} \sum_{j=0}^m \left[ a_{mnjn_1} + \sum_{k=\tau_2}^{n-1} |\Delta_{01} a_{mnjk}| + a_{mnjn} \right] \\ &\leq A_{mnm\tau_2} + \frac{1}{2} \sum_{j=0}^m \left[ a_{mnj\tau_2} + Ka_{mnj\tau_2} + a_{mnjn} \right] \\ &\leq A_{mnm\tau_2} + \frac{1}{2} \sum_{j=0}^m \left[ (1 + K)a_{mnj\tau_2} + Ka_{mnj\tau_2} \right] \\ &= \mathcal{O}(A_{mnm\tau_2}). \end{aligned}$$

With same reasoning, we have verified that for  $m \geq \tau_1$  and  $a_{mnjk} \geq 0$ ,

$$|T_{mn}(t_1, t_2)| = \mathcal{O}(A_{mn\tau_1 n})$$

holds true.

The proof is completed.  $\square$

### 3. Main Results

One of our first main result is the following.

**Theorem 3.1.** *Let  $f \in L_{2\pi \times 2\pi}$ , (1.10), and  $\omega_1(f, s, t) = \mathcal{O}(\omega^{(1)}(s)\omega^{(2)}(t))$  hold true, where  $\omega^{(1)}(s)$  and  $\omega^{(2)}(t)$  are two non-negative functions of modulus type satisfying conditions (1.11). If (1.9) holds true, and  $\{a_{mj}\}, \{b_{nk}\} \in HBVS$ , then*

$$\max_{(x,y) \in Q} |t_{mn}(x, y) - f(x, y)| = \mathcal{O} [a_{mm}b_{nn}H_1(a_{mm})H_2(b_{nn})]. \tag{3.1}$$

**Proof .** Using (1.2), (1.3), (1.9), and equality

$$D_i(z) = \frac{\sin\left(i + \frac{1}{2}\right)z}{2 \sin \frac{z}{2}}, \quad (i = 1, 2),$$

we have

$$\begin{aligned} t_{mn}(x, y) - f(x, y) &= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \frac{\phi_{xy}(s, t)}{4 \sin \frac{s}{2} \sin \frac{t}{2}} \sum_{j=0}^m \sum_{k=0}^n a_{mj}b_{nk} \sin\left(j + \frac{1}{2}\right) s \sin\left(k + \frac{1}{2}\right) t dsdt. \end{aligned} \tag{3.2}$$

Taking into account that

$$|\phi_{xy}(u, v)| \leq \omega_2(f, \delta_1, \delta_2) \leq 4\omega_1(f, \delta_1, \delta_2) \leq 4\omega^{(1)}(\delta_1)\omega^{(2)}(\delta_2),$$

we can write

$$\begin{aligned} D_{mn} &\leq \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \frac{\omega_1(f, s, t)}{\sin \frac{s}{2} \sin \frac{t}{2}} \left| \sum_{j=0}^m \sum_{k=0}^n a_{mj}b_{nk} \sin\left(j + \frac{1}{2}\right) s \sin\left(k + \frac{1}{2}\right) t \right| dsdt \\ &= \mathcal{O} \left( \int_0^{a_{mm}} \int_0^{b_{nn}} + \int_{a_{mm}}^\pi \int_0^{b_{nn}} + \int_0^{a_{mm}} \int_{b_{nn}}^\pi + \int_{a_{mm}}^\pi \int_{b_{nn}}^\pi \right) \\ &\quad \times \frac{\omega^{(1)}(s)\omega^{(2)}(t)}{st} |K_m(s)| |K_n(t)| dsdt := \sum_{r=1}^4 I_r, \quad \text{say.} \end{aligned} \tag{3.3}$$

Based on our assumptions it is clear that

$$\left| \sum_{j=0}^m \sum_{k=0}^n a_{mj}b_{nk} \sin\left(j + \frac{1}{2}\right) s \sin\left(k + \frac{1}{2}\right) t \right| \leq 1,$$

and subsequently using inequality

$$\frac{\sin \frac{z}{2}}{z} \geq \frac{1}{\pi}, \quad z \in (0, \pi],$$

we obtain

$$\begin{aligned} I_1 &= \mathcal{O}(1) \int_0^{a_{mm}} \int_0^{b_{nn}} s^{-1}t^{-1}\omega^{(1)}(s)\omega^{(2)}(t) dsdt \\ &= \mathcal{O}(1) \int_0^{a_{mm}} s^{-1}\omega^{(1)}(s) ds \int_0^{b_{nn}} t^{-1}\omega^{(2)}(t) dt = \mathcal{O} (a_{mm}b_{nn}H_1(a_{mm})H_2(b_{nn})), \end{aligned} \tag{3.4}$$

by Lemma 2.2.

Since  $\{a_{mj}\}, \{b_{nk}\} \in HBVS$ , and  $|K_n(t)| \leq 1$ , then using Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned} I_2 &= \mathcal{O}(1) \int_{a_{mm}}^{\pi} \int_0^{b_{nn}} s^{-2}t^{-1}\omega^{(1)}(s)\omega^{(2)}(t)dsdt \\ &= \mathcal{O}(a_{mm}) \int_{a_{mm}}^{\pi} s^{-2}\omega^{(1)}(s)ds \int_0^{b_{nn}} t^{-1}\omega^{(2)}(t)dt = \mathcal{O}(a_{mm}b_{nn}H_1(a_{mm})H_2(b_{nn})). \end{aligned} \tag{3.5}$$

With similar reasoning, using once again Lemma 2.2 and Lemma 2.3, we also get

$$\begin{aligned} I_3 &= \mathcal{O}(1) \int_0^{a_{mm}} \int_{b_{nn}}^{\pi} s^{-1}t^{-2}\omega^{(1)}(s)\omega^{(2)}(t)dsdt \\ &= \mathcal{O}(b_{nn}) \int_{a_{mm}}^{\pi} s^{-1}\omega^{(1)}(s)ds \int_0^{b_{nn}} t^{-2}\omega^{(2)}(t)dt = \mathcal{O}(a_{mm}b_{nn}H_1(a_{mm})H_2(b_{nn})). \end{aligned} \tag{3.6}$$

Finally, using Lemma 2.3 and conditions (1.11), we obtain

$$\begin{aligned} I_4 &= \mathcal{O}(a_{mm}b_{nn}) \int_{b_{nn}}^{\pi} \int_{a_{mm}}^{\pi} s^{-2}t^{-2}\omega^{(1)}(s)\omega^{(2)}(t)dsdt \\ &= \mathcal{O}(a_{mm}b_{nn}) \int_{a_{mm}}^{\pi} s^{-2}\omega^{(1)}(s) \int_{b_{nn}}^{\pi} t^{-2}\omega^{(2)}(t)dsdt = \mathcal{O}(a_{mm}b_{nn}H_1(a_{mm})H_2(b_{nn})). \end{aligned} \tag{3.7}$$

Inserting (3.4)–(3.7) into (3.3) we obtain (3.1).

The proof is completed.  $\square$

**Theorem 3.2.** *Let  $f \in L_{2\pi \times 2\pi}$ , and  $\omega_1(f, s, t) = \mathcal{O}(\omega^{(1)}(s)\omega^{(2)}(t))$ , where  $\omega^{(1)}(s)$  and  $\omega^{(2)}(t)$  are two non-negative functions of modulus type satisfying conditions (1.11). If  $\{a_{mj}\}, \{b_{nk}\} \in HBVS$ , then*

$$\begin{aligned} D_{mn} &= \mathcal{O}\left(\omega^{(1)}\left(\frac{\pi}{m}\right)\omega^{(2)}\left(\frac{\pi}{n}\right) + b_{nn}\omega^{(1)}\left(\frac{\pi}{m}\right)H_2\left(\frac{\pi}{n}\right)\right. \\ &\quad \left.+ a_{mm}\omega^{(2)}\left(\frac{\pi}{m}\right)H_1\left(\frac{\pi}{m}\right) + a_{mm}b_{nn}H_1\left(\frac{\pi}{m}\right)H_2\left(\frac{\pi}{n}\right)\right). \end{aligned} \tag{3.8}$$

If, in addition,  $\omega^{(i)}(z)$ , ( $i = 1, 2$ ), satisfy (1.10), then

$$D_{mn} = \mathcal{O}\left(\left(1 + \sum_{k=0}^n b_{nk} + \sum_{j=0}^m a_{mj}\right)a_{mm}b_{nn}H_1\left(\frac{\pi}{m}\right)H_2\left(\frac{\pi}{n}\right)\right). \tag{3.9}$$

**Proof .** Following (3.3) we write

$$\begin{aligned} D_{mn} &= \mathcal{O}\left(\int_0^{\frac{\pi}{m}} \int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{m}}^{\pi} \int_0^{\frac{\pi}{n}} + \int_0^{\frac{\pi}{m}} \int_{\frac{\pi}{n}}^{\pi} + \int_{\frac{\pi}{m}}^{\pi} \int_{\frac{\pi}{n}}^{\pi}\right) \\ &\quad \times \frac{\omega^{(1)}(s)\omega^{(2)}(t)}{\sin \frac{s}{2} \sin \frac{t}{2}} |K_m(s)||K_n(t)| dsdt := \sum_{r_1=1}^4 J_{r_1}, \quad \text{say.} \end{aligned} \tag{3.10}$$

At first we estimate  $J_1$ . Using the inequality  $|\sin(d + \frac{1}{2})z| \leq (d + \frac{1}{2})z$  and our assumptions, we have

$$|J_1| = \mathcal{O}(mn) \int_0^{\frac{\pi}{m}} \int_0^{\frac{\pi}{n}} \omega^{(1)}(s)\omega^{(2)}(t)dsdt = \mathcal{O}\left(\omega^{(1)}\left(\frac{\pi}{m}\right)\omega^{(2)}\left(\frac{\pi}{n}\right)\right). \tag{3.11}$$

To estimate  $J_2$  we use (2.4) and (1.11). Indeed,

$$|J_2| = \mathcal{O}(mb_{nn}) \int_0^{\frac{\pi}{m}} \int_{\frac{\pi}{n}}^{\pi} \omega^{(1)}(s)t^{-2}\omega^{(2)}(t)dsdt = \mathcal{O}\left(b_{nn}\omega^{(1)}\left(\frac{\pi}{m}\right)H_2\left(\frac{\pi}{n}\right)\right). \tag{3.12}$$

In analogy, we use (2.3) and (1.11) to obtain

$$|J_3| = \mathcal{O}(na_{mm}) \int_{\frac{\pi}{m}}^{\pi} \int_0^{\frac{\pi}{n}} s^{-2}\omega^{(1)}(s)\omega^{(2)}(t)dsdt = \mathcal{O}\left(a_{mm}\omega^{(2)}\left(\frac{\pi}{m}\right)H_1\left(\frac{\pi}{m}\right)\right). \tag{3.13}$$

Finally, using (2.3), (2.4), and (1.11), we get

$$|J_4| = \mathcal{O}(a_{mm}b_{nn}) \int_{\frac{\pi}{m}}^{\pi} \int_{\frac{\pi}{n}}^{\pi} s^{-2}\omega^{(1)}(s)t^{-2}\omega^{(2)}(t)dsdt = \mathcal{O}\left(a_{mm}b_{nn}H_1\left(\frac{\pi}{m}\right)H_2\left(\frac{\pi}{n}\right)\right). \tag{3.14}$$

Putting (3.11)–(3.14) into (3.10) we get

$$D_{mn} = \mathcal{O}\left(\omega^{(1)}\left(\frac{\pi}{m}\right)\omega^{(2)}\left(\frac{\pi}{n}\right) + \omega^{(1)}\left(\frac{\pi}{m}\right)H_2\left(\frac{\pi}{n}\right) + \omega^{(2)}\left(\frac{\pi}{m}\right)H_1\left(\frac{\pi}{m}\right) + a_{mm}b_{nn}H_1\left(\frac{\pi}{m}\right)H_2\left(\frac{\pi}{n}\right)\right), \tag{3.15}$$

which proves (3.8).

Further, applying Lemma 2.1 we have

$$|J_1| = \mathcal{O}(mn) \int_0^{\frac{\pi}{m}} \int_0^{\frac{\pi}{n}} \omega^{(1)}(s)\omega^{(2)}(t)dsdt = \mathcal{O}\left((mn)^{-1}H_1\left(\frac{\pi}{m}\right)H_2\left(\frac{\pi}{n}\right)\right). \tag{3.16}$$

However for  $m \geq \mu \geq 0, n \geq \nu \geq 0$ , since  $\{a_{mj}\}, \{b_{nk}\} \in HBVS$ , we have

$$\begin{aligned} |a_{m\mu} - a_{mm}| &\leq \sum_{j=\mu}^{m-1} |a_{mj} - a_{mj+1}| \leq \sum_{j=0}^{m-1} |a_{mj} - a_{mj+1}| \\ &\leq Ka_{mm} \implies a_{m\mu} \leq (K + 1)a_{mm}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} |b_{n\nu} - b_{nn}| &\leq \sum_{k=\nu}^{n-1} |b_{nk} - b_{nk+1}| \leq \sum_{k=0}^{n-1} |b_{nk} - b_{nk+1}| \\ &\leq Kb_{nn} \implies b_{n\nu} \leq (K + 1)b_{nn}, \end{aligned} \tag{3.18}$$

and thus

$$1 = \sum_{\mu=0}^m \sum_{\nu=0}^n a_{m\mu}b_{n\nu} \leq K(m + 1)(n + 1)a_{mm}b_{nn} \implies \frac{1}{mn} = \mathcal{O}(a_{mm}b_{nn}).$$

Whence, (3.16) becomes

$$|J_1| = \mathcal{O} \left( a_{mm} b_{nn} H_1 \left( \frac{\pi}{m} \right) H_2 \left( \frac{\pi}{n} \right) \right). \tag{3.19}$$

To estimate the quantity  $|J_2|$  we use (2.4). Namely,

$$|J_2| = \mathcal{O} (mb_{nn}) \int_0^{\frac{\pi}{m}} \int_{\frac{\pi}{n}}^{\pi} \omega^{(1)}(s)t^{-2}\omega^{(2)}(t)dsdt = \mathcal{O} \left( \frac{b_{nn}}{m} H_1 \left( \frac{\pi}{m} \right) H_2 \left( \frac{\pi}{n} \right) \right). \tag{3.20}$$

However, for  $m \geq \mu \geq 0$  and  $\{a_{mj}\} \in HBVS$ , we get

$$1 = \sum_{\mu=0}^m \sum_{\nu=0}^n a_{m\mu} b_{n\nu} \leq K(m+1)a_{mm} \sum_{\nu=0}^n b_{n\nu} \implies \frac{1}{m} = \mathcal{O} \left( a_{mm} \sum_{\nu=0}^n b_{n\nu} \right).$$

Therefore,

$$|J_2| = \mathcal{O} \left( a_{mm} b_{nn} \sum_{k=0}^n b_{nk} H_1 \left( \frac{\pi}{m} \right) H_2 \left( \frac{\pi}{n} \right) \right). \tag{3.21}$$

Similarly, we have found that

$$|J_3| = \mathcal{O} \left( a_{mm} b_{nn} \sum_{j=0}^m a_{mj} H_1 \left( \frac{\pi}{m} \right) H_2 \left( \frac{\pi}{n} \right) \right). \tag{3.22}$$

Finally, inserting (3.19),(3.21), (3.22) and (3.14) into (3.10), we obtain

$$D_{mn} = \mathcal{O} \left( \left( a_{mm} b_{nn} + a_{mm} b_{nn} \sum_{k=0}^n b_{nk} + a_{mm} b_{nn} \sum_{j=0}^m a_{mj} \right) H_1 \left( \frac{\pi}{m} \right) H_2 \left( \frac{\pi}{n} \right) \right),$$

which proves (3.9).

The proof is completed.  $\square$

**Remark 3.3.** Note that if, in Theorem 3.2, the additional conditions  $\sum_{j=0}^m a_{mj} = 1$  and  $\sum_{k=0}^n b_{nk} = 1$ , ( $m, n \in \{0, 1, 2, \dots\}$ ) are satisfied, then (3.9) takes the following simpler form

$$D_{mn} = \mathcal{O} \left( a_{mm} b_{nn} H_1 \left( \frac{\pi}{m} \right) H_2 \left( \frac{\pi}{n} \right) \right).$$

Next statement holds true not only for factorable but also for non-factorable matrices (see [2]).

**Theorem 3.4.** Let  $f \in L_{2\pi \times 2\pi}$  and  $\omega_1(f, s, t) = \mathcal{O} (\omega^{(1)}(s)\omega^{(2)}(t))$ , where  $\omega^{(1)}(s)$  and  $\omega^{(2)}(t)$  are two non-negative functions of modulus type. If

$$a_{mnjk} \geq 0, \quad (m, n, j, k \in \{0, 1, 2, \dots\}), \quad \sum_{j=0}^m \sum_{k=0}^n a_{mnjk} = 1,$$

and  $\{a_{mnjk}\} \in DRBVS$ , then

$$\begin{aligned}
 D_{mn} &= \mathcal{O}\left(\omega^{(1)}\left(\frac{\pi}{m}\right)\omega^{(2)}\left(\frac{\pi}{n}\right)\right. \\
 &\quad + \sum_{k=1}^{n-1} k^{-1}\omega^{(1)}\left(\frac{\pi}{m}\right)\omega^{(1)}\left(\frac{\pi}{k}\right)A_{mnmk+1} \\
 &\quad + \sum_{j=1}^{m-1} j^{-1}\omega^{(1)}\left(\frac{\pi}{j}\right)\omega^{(1)}\left(\frac{\pi}{n}\right)A_{mnj+1k} \\
 &\quad \left. + \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} j^{-1}k^{-1}\omega^{(1)}\left(\frac{\pi}{j}\right)\omega^{(1)}\left(\frac{\pi}{k}\right)A_{mnj+1k+1}\right). \tag{3.23}
 \end{aligned}$$

**Proof .** As in Theorem 3.2 we have

$$D_{mn} \leq J_1 + J_2 + J_3 + J_4, \tag{3.24}$$

and

$$J_1 = \mathcal{O}\left(\omega^{(1)}\left(\frac{\pi}{m}\right)\omega^{(2)}\left(\frac{\pi}{n}\right)\right). \tag{3.25}$$

The use of the Lemma 2.5 implies

$$\begin{aligned}
 J_2 &= \mathcal{O}(m) \int_0^{\frac{\pi}{m}} \int_{\frac{\pi}{n}}^{\pi} \omega^{(1)}(s)t^{-1}\omega^{(2)}(t)A_{mnm\tau_2}dsdt \\
 &= \mathcal{O}(m) \int_0^{\frac{\pi}{m}} \omega^{(1)}(s)ds \int_{\frac{\pi}{n}}^{\pi} t^{-1}\omega^{(2)}(t)A_{mnm\tau_2}dt \\
 &= \mathcal{O}\left(\omega^{(1)}\left(\frac{\pi}{m}\right)\right) \sum_{k=1}^{n-1} \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} t^{-1}\omega^{(2)}(t)A_{mnm\tau_2}dt \\
 &= \mathcal{O}\left(\sum_{k=1}^{n-1} k^{-1}\omega^{(1)}\left(\frac{\pi}{m}\right)\omega^{(1)}\left(\frac{\pi}{k}\right)A_{mnmk+1}\right). \tag{3.26}
 \end{aligned}$$

Similarly, using Lemma 2.5, we have obtained

$$J_3 = \mathcal{O}\left(\sum_{j=1}^{m-1} j^{-1}\omega^{(1)}\left(\frac{\pi}{j}\right)\omega^{(1)}\left(\frac{\pi}{n}\right)A_{mnj+1k}\right). \tag{3.27}$$

Moreover, using Lemma 2.5 once again, we have

$$\begin{aligned}
 J_4 &= \mathcal{O}(1) \int_{\frac{\pi}{m}}^{\pi} \int_{\frac{\pi}{n}}^{\pi} s^{-1}t^{-1}\omega^{(1)}(s)\omega^{(2)}(t)A_{mn\tau_1\tau_2}dsdt \\
 &= \mathcal{O}(1) \sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \int_{\frac{\pi}{j+1}}^{\frac{\pi}{j}} \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} s^{-1}t^{-1}\omega^{(2)}(t)A_{mn\tau_1\tau_2}dsdt \\
 &= \mathcal{O}\left(\sum_{j=1}^{m-1}\sum_{k=1}^{n-1} j^{-1}k^{-1}\omega^{(1)}\left(\frac{\pi}{j}\right)\omega^{(1)}\left(\frac{\pi}{k}\right)A_{mnj+1k+1}\right). \tag{3.28}
 \end{aligned}$$

Inserting (3.25)-(3.28) into (3.24) we clearly obtain (3.23).

The proof is completed.  $\square$

**Theorem 3.5.** *Let  $f \in L_{2\pi \times 2\pi}$ , (1.10), and  $\omega_1(f, s, t) = \mathcal{O}(\omega^{(1)}(s)\omega^{(2)}(t))$  hold true, where  $\omega^{(1)}(s)$  and  $\omega^{(2)}(t)$  are two non-negative functions of modulus type satisfying conditions (1.11). If (1.9) holds true, and  $\{a_{mj}\}, \{b_{nk}\} \in RBVS$ , then*

$$\max_{(x,y) \in Q} |t_{mn}(x, y) - f(x, y)| = \mathcal{O}(a_{m0}b_{n0}H_1(a_{m0})H_2(b_{n0})). \tag{3.29}$$

**Proof .** We do the same reasoning as in the proof of Theorem 3.1. Namely, we decompose the integral  $\int_0^\pi \int_0^\pi$  as follows

$$\begin{aligned} D_{mn} = \mathcal{O} & \left( \int_0^{a_{m0}} \int_0^{b_{n0}} + \int_{a_{m0}}^\pi \int_0^{b_{n0}} + \int_0^{a_{m0}} \int_{b_{n0}}^\pi + \int_{a_{m0}}^\pi \int_{b_{n0}}^\pi \right) \\ & \times \frac{\omega^{(1)}(s)\omega^{(2)}(t)}{\sin \frac{s}{2} \sin \frac{t}{2}} |K_m(s)K_n(t)| dsdt := \sum_{\ell=1}^4 I_\ell, \quad \text{say.} \end{aligned} \tag{3.30}$$

As (3.6) we obtain

$$I_1 = \mathcal{O}(1) \int_0^{a_{m0}} \int_0^{b_{n0}} s^{-1}t^{-1}\omega^{(1)}(s)\omega^{(2)}(t)dsdt = \mathcal{O}(a_{m0}b_{n0}H_1(a_{m0})H_2(b_{n0})), \tag{3.31}$$

by Lemma 2.2.

Since  $\{a_{mj}\}, \{b_{nk}\} \in RBVS$ , then using (2.5)–(2.6) of Lemma 2.3, and Lemma 2.2 we get

$$I_2 = \mathcal{O}(a_{m0}) \int_{a_{m0}}^\pi \int_0^{b_{n0}} s^{-2}t^{-1}\omega^{(1)}(s)\omega^{(2)}(t)dsdt = \mathcal{O}(a_{m0}b_{n0}H_1(a_{m0})H_2(b_{n0})), \tag{3.32}$$

$$I_3 = \mathcal{O}(b_{n0}) \int_0^{a_{m0}} \int_{b_{n0}}^\pi s^{-1}t^{-2}\omega^{(1)}(s)\omega^{(2)}(t)dsdt = \mathcal{O}(a_{m0}b_{n0}H_1(a_{m0})H_2(b_{n0})), \tag{3.33}$$

and

$$I_4 = \mathcal{O}(a_{m0}b_{n0}) \int_{a_{m0}}^\pi \int_{b_{n0}}^\pi s^{-2}t^{-2}\omega^{(1)}(s)\omega^{(2)}(t)dsdt = \mathcal{O}(a_{m0}b_{n0}H_1(a_{m0})H_2(b_{n0})). \tag{3.34}$$

Inserting (3.31)-(3.34) into (3.30) we obtain (3.29).

The proof is completed.  $\square$

#### 4. Corollaries

Suppose that  $A := (a_{mnjk})$  is a doubly matrix defined as follows (see [8], [9]):

$$a_{mnjk} = \begin{cases} \frac{p_j q_k}{P_m Q_n}, & 0 \leq j \leq m; 0 \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

where  $\{p_j\}$  and  $\{q_k\}$   $j, k = 0, 1, \dots$ , are sequences of non-negative numbers with  $p_0, q_0 > 0$  and  $P_m := \sum_{j=0}^m p_j, Q_k := \sum_{k=0}^n q_k, (m, n = 0, 1, \dots)$ . In this case, the matrix  $A := (a_{mnjk})$  is called factorable double Riesz matrix and the means  $R_{mn}(x, y)$  are called factorable double Riesz means i.e., we write  $R_{mn}(x, y)$  for  $t_{mn}(x, y)$ .

Whence, we obtain the following corollary from Theorem 3.1.

**Corollary 4.1.** *Let  $f \in L_{2\pi \times 2\pi}$ , (1.10), and  $\omega_1(f, s, t) = \mathcal{O}(\omega^{(1)}(s)\omega^{(2)}(t))$  hold true, where  $\omega^{(1)}(s)$  and  $\omega^{(2)}(t)$  are two non-negative functions of modulus type satisfying conditions (1.11). If (1.9) holds true, and  $\{p_j\}, \{q_k\} \in HBVS$ , then*

$$\max_{(x,y) \in Q} |R_{mn}(x, y) - f(x, y)| = \mathcal{O} \left[ \left( \frac{p_m q_n}{P_m Q_n} \right) H_1 \left( \frac{p_m}{P_m} \right) H_2 \left( \frac{q_n}{Q_n} \right) \right]. \tag{4.1}$$

We say that the function  $f$  belongs to the generalized Lipschitz class, briefly  $f \in \text{Lip}(\omega^{(1)}, \omega^{(2)})$ , if

$$\text{Lip}(\omega^{(1)}, \omega^{(2)}) = \left\{ f \in L_{2\pi \times 2\pi} : |f(x + u, y + v) - f(x, y)| = \mathcal{O}(\omega^{(1)}(u)\omega^{(2)}(v)) \right\},$$

where  $\omega^{(1)}(u)$  and  $\omega^{(2)}(v)$  are two non-negative functions of modulus type for  $u \geq 0, v \geq 0$ .

In particular, if  $\omega^{(1)}(u) = u^\alpha, 0 < \alpha \leq 1, \omega^{(2)}(v) = v^\beta, 0 < \beta \leq 1$ , and  $f \in \text{Lip}(\omega^{(1)}, \omega^{(2)}) := \text{Lip}(\alpha, \beta)$ , then

$$\omega_1(f, s, t) = \mathcal{O}(\delta_1^\alpha \delta_2^\beta), \quad \delta_1, \delta_2 \geq 0.$$

Thus, if  $f \in \text{Lip}(\gamma_1, \gamma_2)$  and

$$H_i(u) = \begin{cases} u^{\gamma_i-1}, & 0 < \gamma_i < 1, (i = 1, 2); \\ \log\left(\frac{\pi}{u}\right) & \gamma_i = 1, i = 1, 2, \end{cases}$$

then from Corollary 4.1 we get a two-dimensional version of a theorem proved in [4].

**Corollary 4.2.** *Assume that  $f \in C_{2\pi \times 2\pi}$  and  $f \in \text{Lip}(\gamma_1, \gamma_2), 0 < \gamma_i \leq 1, (i = 1, 2)$ . If  $\{p_j\}, \{q_k\} \in HBVS$ , then*

$$\max_{(x,y) \in Q} |R_{mn}(x, y) - f(x, y)| = \begin{cases} \mathcal{O} \left[ \left( \frac{p_m}{P_m} \right)^{\gamma_1} \left( \frac{q_n}{Q_n} \right)^{\gamma_2} \right], & 0 < \gamma_i < 1; \\ \mathcal{O} \left[ \frac{p_m q_n}{P_m Q_n} \log\left(\frac{\pi P_m}{p_m}\right) \log\left(\frac{\pi Q_n}{q_n}\right) \right], & \gamma_i = 1. \end{cases}$$

If we consider the particular case, when that  $A := (a_{mnjk})$  is a doubly matrix defined by (see [7]):

$$a_{mnjk} = \begin{cases} \frac{1}{(m+1)(n+1)}, & 0 \leq j \leq m; 0 \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

or simply we put  $p_j = 1, 0 \leq j \leq m$ , and  $q_k = 1, 0 \leq k \leq n$ , then we obtain:

**Corollary 4.3.** *Assume that  $f \in C_{2\pi \times 2\pi}$  and  $f \in \text{Lip}(\gamma_1, \gamma_2), 0 < \gamma_i \leq 1, (i = 1, 2)$ . Then*

$$\max_{(x,y) \in Q} |C_{mn}(x, y) - f(x, y)| = \begin{cases} \mathcal{O} \left( \frac{1}{(m+1)^{\gamma_1} (n+1)^{\gamma_2}} \right), & 0 < \gamma_i < 1; \\ \mathcal{O} \left[ \frac{\log(\pi(m+1)) \log(\pi(n+1))}{(m+1)(n+1)} \right], & \gamma_i = 1, \end{cases}$$

where

$$C_{mn}(x, y) := \frac{1}{(m+1)(n+1)} \sum_{j=0}^m \sum_{k=0}^n s_{jk}(x, y), \quad m, n \in \mathbb{N} \cup \{0\}.$$



When  $A := (a_{mnjk})$  is a doubly matrix defined by (see [33]):

$$a_{mnjk} = \begin{cases} \frac{1}{(m-j+1) \log m (n-k+1) \log n}, & 0 \leq j \leq m; 0 \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

then we have the following:

**Corollary 4.4.** *Assume that  $f \in C_{2\pi \times 2\pi}$  and  $f \in Lip(\gamma_1, \gamma_2)$ ,  $0 < \gamma_i \leq 1$ , ( $i = 1, 2$ ). Then*

$$\max_{(x,y) \in Q} |H_{mn}(x, y) - f(x, y)| = \begin{cases} \mathcal{O}\left(\frac{1}{(\log m)^{\gamma_1} (\log n)^{\gamma_2}}\right), & 0 < \gamma_i < 1; \\ \mathcal{O}\left[\frac{\log(\pi \log(m)) \log(\pi \log(n))}{\log m \log n}\right], & \gamma_i = 1, \end{cases}$$

where

$$H_{mn}(x, y) := \frac{1}{\log m \log n} \sum_{j=0}^m \sum_{k=0}^n \frac{1}{(m-j+1)(n-k+1)} s_{jk}(x, y)$$

are double harmonic means.

Also, if

$$a_{mnjk} = \begin{cases} \frac{p_{m-j} q_{n-k}}{P_m Q_n}, & 0 \leq j \leq m; 0 \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

then the matrix  $A := (a_{mnjk})$  is called factorable double Nörlund matrix and the means  $N_{mn}(x, y)$  are called double Nörlund means i.e., we write  $N_{mn}(x, y)$  for  $t_{mn}(x, y)$ . For this factorable matrix the Theorem 3.5 reduces to:

**Corollary 4.5.** *Let  $f \in L_{2\pi \times 2\pi}$ , (1.10), and  $\omega_1(f, s, t) = \mathcal{O}(\omega^{(1)}(s)\omega^{(2)}(t))$  hold true, where  $\omega^{(1)}(s)$  and  $\omega^{(2)}(t)$  are two non-negative functions of modulus type satisfying conditions (1.11). If (1.9) holds true, and  $\{p_j\}, \{q_k\} \in RBVS$ , then*

$$\max_{(x,y) \in Q} |N_{mn}(x, y) - f(x, y)| = \mathcal{O}\left[\left(\frac{P_m Q_n}{P_m Q_n}\right) H_1\left(\frac{P_m}{P_m}\right) H_2\left(\frac{Q_n}{Q_n}\right)\right]. \tag{4.2}$$

**Remark 4.6.** *From (4.1) and (4.2) we note that we have obtained the same degree of approximation even if we have used the different conditions on the entries of the matrices on the different means  $R_{mn}(x, y)$  and  $N_{mn}(x, y)$ .*

Now we give the following corollary from Theorem 3.1, which indeed is a two-dimensional version of a result proved in [10] for single Nörlund means of a Fourier series.

**Corollary 4.7.** *Let  $f \in C_{2\pi \times 2\pi}$  and  $\omega_1(f, s, t) = \mathcal{O}(\omega^{(1)}(s)\omega^{(2)}(t))$ , where  $\omega^{(1)}(s)$  and  $\omega^{(2)}(t)$  are two non-negative functions of modulus type. If  $\{p_{jk}\} \in DRBVS$ , then*

$$\max_{(x,y) \in Q} |R_{mn}(x, y) - f(x, y)| = \mathcal{O}\left(\frac{1}{P_{mn}} \sum_{j=1}^m \sum_{k=1}^n (jk)^{-1} \omega^{(1)}\left(\frac{\pi}{j}\right) \omega^{(1)}\left(\frac{\pi}{k}\right) P_{jk}\right).$$

**Proof .** Since  $p_{jk} \in DRBVS$ , then for  $n_2 \geq n_1$  and  $m_2 \geq m_1$ , we have

$$p_{mn_2} = |p_{mn_2}| \leq \sum_{k=n_2}^{\infty} |\Delta_{01}p_{mk}| \leq \sum_{k=n_1}^{\infty} |\Delta_{01}p_{mk}| \leq Kp_{mn_1},$$

$$p_{m_2n} = |p_{m_2n}| \leq \sum_{j=m_2}^{\infty} |\Delta_{10}p_{jn}| \leq \sum_{j=m_1}^{\infty} |\Delta_{10}p_{jn}| \leq Kp_{m_1n},$$

and

$$p_{m_2n_2} = |p_{m_2n_2}| \leq \sum_{j=m_2}^{\infty} \sum_{k=n_2}^{\infty} |\Delta_{11}p_{jk}| \leq \sum_{j=m_1}^{\infty} \sum_{k=n_1}^{\infty} |\Delta_{11}p_{jk}| \leq Kp_{m_1n_1}.$$

Thus, we obtain

$$A_{mnmk+1} = \frac{1}{P_{mn}} \sum_{i_1=0}^m \sum_{i_2=0}^{k+1} p_{mi_2} = \frac{P_{mk+1}}{P_{mn}} = \mathcal{O} \left( \frac{P_{mk}}{P_{mn}} \right),$$

$$A_{mnj+1n} = \frac{1}{P_{mn}} \sum_{i_1=0}^{j+1} \sum_{i_2=0}^n p_{i_1n} = \frac{P_{j+1n}}{P_{mn}} = \mathcal{O} \left( \frac{P_{jn}}{P_{mn}} \right),$$

and

$$A_{mnj+1k+1} = \frac{1}{P_{mn}} \sum_{i_1=0}^{j+1} \sum_{i_2=0}^{k+1} p_{i_1i_2} = \frac{P_{j+1k+1}}{P_{mn}} = \mathcal{O} \left( \frac{P_{jk}}{P_{mn}} \right).$$

which mean that sequence  $\{k^{-1}P_{mk}\}$  is non-increasing with respect to  $k$ , the sequence  $\{j^{-1}P_{jn}\}$  is non-increasing with respect to  $j$ , and the sequence  $\{j^{-1}k^{-1}P_{jk}\}$  is non-increasing with respect to both  $j$  and  $k$ .

Whence, we get

$$\omega^{(1)} \left( \frac{\pi}{m} \right) \omega^{(2)} \left( \frac{\pi}{n} \right) \leq \frac{1}{P_{mn}} \sum_{j=1}^m \sum_{k=1}^n \omega^{(1)} \left( \frac{\pi}{j} \right) \omega^{(2)} \left( \frac{\pi}{k} \right) (jk)^{-1}P_{jk},$$

$$\omega^{(1)} \left( \frac{\pi}{m} \right) A_{mnmk+1} = \mathcal{O} \left( \frac{1}{P_{mn}} \sum_{j=1}^m \omega^{(1)} \left( \frac{\pi}{j} \right) j^{-1}P_{jk} \right),$$

and

$$\omega^{(2)} \left( \frac{\pi}{n} \right) A_{mnm+1k} = \mathcal{O} \left( \frac{1}{P_{mn}} \sum_{k=1}^n \omega^{(2)} \left( \frac{\pi}{k} \right) k^{-1}P_{jk} \right).$$

Consequently, (3.23) takes the form

$$D_{mn} = \mathcal{O} \left( \frac{1}{P_{mn}} \sum_{j=1}^m \sum_{k=1}^n (jk)^{-1} \omega^{(1)} \left( \frac{\pi}{j} \right) \omega^{(1)} \left( \frac{\pi}{k} \right) P_{jk} \right).$$

The proof is completed.  $\square$

Now we give the following result from Corollary 4.7.

**Corollary 4.8.** Let  $f \in C_{2\pi \times 2\pi}$  and  $f \in Lip(\alpha, \beta)$ ,  $0 < \alpha, \beta \leq 1$ . If  $\{p_{jk}\} \in DRBVS$ , then

$$\max_{(x,y) \in Q} |R_{mn}(x, y) - f(x, y)| = \mathcal{O} \left( \frac{1}{P_{mn}} \sum_{j=1}^m \sum_{k=1}^n j^{-1-\alpha} k^{-1-\beta} P_{jk} \right).$$

Once more, Corollary 4.8 shows a two-dimensional version of a result proved in [31] for single Nörlund means of a Fourier series.

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