Int. J. Nonlinear Anal. Appl. 12 (2021) No. 12, 629-663
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.13960.1728

Using the infimum form of auxiliary functions to study the common coupled coincidence points in fuzzy semi-metric spaces

Hsien-Chung Wu*

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan

(Communicated by Shahram Saeidi)

Abstract

The common coupled coincidence points and common coupled fixed points in fuzzy semi-metric spaces are investigated in this paper. In fuzzy semi-metric space, the symmetric condition is not necessarily assumed to be satisfied. In this case, regarding the non-symmetry of metric, there are four kinds of triangle inequalities that can be considered. In order to investigate the common coupled coincidence points and common coupled fixed points, the fuzzy semi-metric space is further assumed to satisfy the so-called canonical condition that is inspired from the intuitive observations. The sufficient conditions for guaranteeing the common coupled coincidence points and common coupled fixed points will be different for the four different kinds of triangle inequalities.

Keywords: Common coupled coincidence points, Common coupled fixed points, Fuzzy semi-metric space, Canonical condition.
2010 MSC: 54E35, 54H25

1. Introduction

Probabilistic metric space was introduced by Schweizer and Sklar [15, 16, 17] in which the (conventional) metric space is associated with the probability theory. For more details on the theory of probabilistic metric space, we may also refer to Hadžić and Pap [8] and Chang et al. [2]. The Menger space is a special kind of probabilistic metric space. Inspired by the Menger space, Kramosil and Michalek [11] proposed the so-called fuzzy metric space that is described below.

*Corresponding author
Email address: hcwu@nknu.edu.tw (Hsien-Chung Wu)

Received: February 2018 *Accepted:* September 2020

Let X be a nonempty universal set, let $*$ be a t-norm, and let M be a mapping defined on $X \times X \times [0, \infty)$ into $[0, 1]$. The 3-tuple $(X, M, *)$ is called a *fuzzy metric space* when the following conditions are satisfied:

- for any $x, y \in X$, $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
- $M(x, y, 0) = 0$ for all $x, y \in X$;
- $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t \geq 0$;
- $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t \geq 0$ (the so-called triangle inequality).

The mapping M in fuzzy metric space $(X, M, *)$ can be regarded as a membership function of a fuzzy subset of $X \times X \times [0, \infty)$. Sometimes M is called a *fuzzy metric* of the space $(X, M, *)$. According to the first and second conditions of fuzzy metric space, the mapping $M(x, y, t)$ can be interpreted as the membership degree of the distance that is less than or equal to t between x and y. In this paper, we shall consider the so-called fuzzy semi-metric space in which the symmetric condition $M(x, y, t) = M(y, x, t)$ is not assumed to be satisfied. In this case, there are four kinds of triangle inequalities that should be considered. In order to investigate the the common coupled coincidence points and common coupled fixed points, the fuzzy semi-metric space is further assumed to satisfy the so-called canonical condition that is inspired from the intuitive observations.

The common coupled coincidence points and common coupled fixed points in fuzzy metric spaces have been studied by Hu et al. [9], Mohiuddine and Alotaibi [12], Qiu and Hong [13], and the references therein. In this paper, we shall study the common coupled coincidence points and common coupled fixed points in fuzzy semi-metric spaces that is endowed with four kinds of triangle inequalities. Although the common coupled fixed points are the special case of common coupled coincidence points, if the uniqueness is considered, then the sufficient conditions will be completely different. Therefore we shall separately study the common coupled fixed points and common coupled coincidence points regarding the uniqueness.

This paper is organized as follows. In Section 2, we propose the fuzzy semi-metric space that is endowed with four kinds of triangle inequalities. In Section 3, we introduce the auxiliary functions that will be used to study the Cauchy sequence in fuzzy semi-metric space. In Section 4, we study the Cauchy sequence in fuzzy semi-metric space by means of the auxiliary functions established in Section 3. In Section 5, we derive many kinds of common coupled coincidence points in fuzzy semi-metric spaces that can be endowed with the different types of triangle inequalities introduced in Section 2. Finally, in Section 6, we also study the common coupled fixed points in fuzzy semi-metric spaces.

2. Fuzzy Semi-Metric Spaces

The function $* : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the following axioms is called a *t-norm* (*triangular norm*):

- (boundary condition) $a * 1 = a$.
- (commutativity) $a * b = b * a$.
- (increasing property) If $b < c$, then $a * b \leq a * c$.

- (associativity) $(a * b) * c = a * (b * c)$.

By the commutativity of t-norm, if the t-norm is continuous with respect to the first component (resp. second component), then it is also continuous with respect to the second component (resp. first component). In other words, for any fixed $a \in [0, 1]$, if the function $f(x) = a * x$ (resp. $f(x) = x * a$) is continuous, then the function $g(x) = x * a$ (resp. $g(x) = a * x$) is continuous. Similarly, if the t-norm is left-continuous (resp. right-continuous) with respect to the first or second component, then it is also left-continuous (resp. right-continuous) with respect to each component. The following properties regarding t-norm will be used in the further study.

Proposition 2.1. (Wu [19]) *Suppose that the t-norm $*$ is left-continuous at 1 with respect to the first or second component. We have the following properties.*

(i) *For any $a, b \in (0, 1)$ with $a > b$, there exists $r \in (0, 1)$ such that $a * r \geq b$.*

(ii) *For any $a \in (0, 1)$ and any $p \in \mathbb{N}$, there exists $r \in (0, 1)$ such that $\overbrace{r * r * \cdots * r}^{p \text{ times}} > a$.*

Proposition 2.2. *We have the following properties.*

(i) *Given any fixed $a, b \in [0, 1]$, suppose that the t-norm $*$ is continuous at a and b with respect the first or second component. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences in $[0, 1]$ such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then $a_n * b_n \to a * b$ as $n \to \infty$.*

(ii) *Given any fixed $a, b \in (0, 1]$, suppose that the t-norm $*$ is left-continuous at a and b with respect the first or second component. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences in $[0, 1]$ such that $a_n \to a-$ and $b_n \to b-$ as $n \to \infty$, then $a_n * b_n \to a * b$ as $n \to \infty$.*

(iii) *Given any fixed $a, b \in [0, 1)$, suppose that the t-norm $*$ is right-continuous at a and b with respect the first or second component. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences in $[0, 1]$ such that $a_n \to a+$ and $b_n \to b+$ as $n \to \infty$, then $a_n * b_n \to a * b$ as $n \to \infty$.*

Definition 2.3. Let X be a nonempty universal set, and let M be a mapping defined on $X \times X \times [0, \infty)$ into $[0, 1]$. Then (X, M) is called a *fuzzy semi-metric space* when the following conditions are satisfied:

- for any $x, y \in X$, $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
- $M(x, y, 0) = 0$ for all $x, y \in X$ with $x \neq y$;

We say that M satisfies the *symmetric condition* when $M(x, y, t) = M(y, x, t)$ for any $x, y \in X$ and $t > 0$. We say that M satisfies the *strongly symmetric condition* when $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t \geq 0$.

In general, the fuzzy semi-metric space (X, M) does not necessarily satisfy the symmetric condition. Therefore four kinds of triangle inequalities are proposed below.

Definition 2.4. Let X be a nonempty universal set, let $*$ be a t-norm, and let M be a mapping defined on $X \times X \times [0, \infty)$ into $[0, 1]$.

- We say that M satisfies the $\bowtie$-*triangle inequality* when the following inequality is satisfied:

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

- We say that M satisfies the $\triangleright$-*triangle inequality* when the following inequality is satisfied:

$$M(x,y,t)*M(z,y,s)\leq M(x,z,t+s) \text{ for all } x,y,z\in X \text{ and } s,t>0.$$

- We say that M satisfies the $\triangleleft$-*triangle inequality* when the following inequality is satisfied:

$$M(y,x,t)*M(y,z,s)\leq M(x,z,t+s) \text{ for all } x,y,z\in X \text{ and } s,t>0.$$

- We say that M satisfies the $\diamond$-*triangle inequality* when the following inequality is satisfied:

$$M(y,x,t)*M(z,y,s)\leq M(x,z,t+s) \text{ for all } x,y,z\in X \text{ and } s,t>0.$$

We say that M satisfies the *strong* $\circ$-*triangle inequality* for $\circ\in\{\bowtie,\triangleright,\triangleleft,\diamond\}$ when $s,t>0$ is replaced by $s,t\geq 0$.

Given a fuzzy semi-metric space (X,M), when we say that the mapping M satisfies some kinds of triangle inequalities, it implicitly means that the t-norm $*$ is considered in (X,M).

Remark 2.5. The following interesting observations will be used in the further study.

- Suppose that M satisfies the $\bowtie$-triangle inequality. Then

$$M(a,b,t_1)*M(b,c,t_2)*M(c,d,t_3)\leq M(a,c,t_1+t_2)*M(c,d,t_3)\leq M(a,d,t_1+t_2+t_3).$$

On the other hand, we also have

$$M(b,a,t_1)*M(c,b,t_2)=M(c,b,t_2)*M(b,a,t_1)\leq M(c,a,t_1+t_2),$$

which implies

$$M(b,a,t_1)*M(c,b,t_2)*M(d,c,t_3)\leq M(d,a,t_1+t_2+t_3).$$

In general, we have

$$M\left(x_1,x_2,t_1\right)*M\left(x_2,x_3,t_2\right)*\cdots*M\left(x_p,x_{p+1},t_p\right)\leq M\left(x_1,x_{p+1},t_1+t_2+\cdots+t_p\right)$$

and

$$M\left(x_2,x_1,t_1\right)*M\left(x_3,x_2,t_2\right)*\cdots*M\left(x_{p+1},x_p,t_{p+1}\right)\leq M\left(x_{p+1},x_1,t_1+t_2+\cdots+t_p\right).$$

- Suppose that M satisfies the $\triangleright$-triangle inequality. Since

$$M(c,b,t_2)*M(a,b,t_1)=M(a,b,t_1)*M(c,b,t_2)\leq\min\left\{M(a,c,t_1+t_2),M(c,a,t_1+t_2)\right\},$$

which implies

$$M(a,b,t_1)*M(c,b,t_2)*M(d,c,t_3)\leq\min\left\{M(a,d,t_1+t_2+t_3),M(d,a,t_1+t_2+t_3)\right\}. \quad (2.1)$$

In general, we have

$$\begin{aligned}&M\left(x_1,x_2,t_1\right)*M\left(x_3,x_2,t_2\right)*M\left(x_4,x_3,t_3\right)*\cdots*M\left(x_{p+1},x_p,t_p\right)\\&\quad\leq\min\left\{M\left(x_1,x_{p+1},t_1+t_2+\cdots+t_p\right),M\left(x_{p+1},x_1,t_1+t_2+\cdots+t_p\right)\right\}.\end{aligned}$$

- Suppose that M satisfies the $\triangleleft$-triangle inequality. Since

$$M(b,c,t_2)*M(b,a,t_1)=M(b,a,t_1)*M(b,c,t_2)\leq \min\left\{M(a,c,t_1+t_2),M(c,a,t_1+t_2)\right\},$$

which implies

$$M(b,a,t_1)*M(b,c,t_2)*M(c,d,t_3)\leq \min\left\{M(a,d,t_1+t_2+t_3),M(d,a,t_1+t_2+t_3)\right\}. \quad (2.2)$$

In general, we have

$$\begin{aligned} &M\left(x_2,x_1,t_1\right)*M\left(x_2,x_3,t_2\right)*M\left(x_3,x_4,t_3\right)*\cdots*M\left(x_p,x_{p+1}\right)\\ &\quad\leq \min\left\{M\left(x_1,x_{p+1},t_1+t_2+\cdots+t_p\right),M\left(x_{p+1},x_1,t_1+t_2+\cdots+t_p\right)\right\}. \end{aligned}$$

- Suppose that M satisfies the $\diamond$-triangle inequality. Then

$$\begin{aligned} &M(a,b,t_1)*M(b,c,t_2)*M(d,c,t_3)=M(b,c,t_1)*M(a,b,t_2)*M(d,c,t_3)\\ &\quad\leq M(c,a,t_1+t_2)*M(d,c,t_3)\leq M(a,d,t_1+t_2+t_3) \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} &M(b,a,t_1)*M(c,b,t_2)*M(c,d,t_3)\leq M(a,c,t_1+t_2)*M(c,d,t_3)\\ &\quad= M(c,d,t_3)*M(a,c,t_1+t_2)\leq M(d,a,t_1+t_2+t_3). \end{aligned} \quad (2.4)$$

In general, we have the following cases.

(a) If p is even, then

$$\begin{aligned} &M\left(x_1,x_2,t_1\right)*M\left(x_2,x_3,t_2\right)*M\left(x_4,x_3,t_3\right)*M\left(x_4,x_5,t_4\right)*M\left(x_6,x_5,t_5\right)\\ &\quad*M\left(x_6,x_7,t_6\right)*\cdots*M\left(x_p,x_{p+1},t_p\right)\leq M\left(x_{p+1},x_1,t_1+t_2+\cdots+t_p\right) \end{aligned}$$

and

$$\begin{aligned} &M\left(x_2,x_1,t_1\right)*M\left(x_3,x_2,t_2\right)*M\left(x_3,x_4,t_3\right)*M\left(x_5,x_4,t_4\right)*M\left(x_5,x_6,t_5\right)\\ &\quad*M\left(x_7,x_6,t_6\right)*\cdots*M\left(x_p,x_{p+1},t_p\right)\leq M\left(x_1,x_{p+1},t_1+t_2+\cdots+t_p\right). \end{aligned}$$

(b) If p is odd, then

$$\begin{aligned} &M\left(x_1,x_2,t_1\right)*M\left(x_2,x_3,t_2\right)*M\left(x_4,x_3,t_3\right)*M\left(x_4,x_5,t_4\right)*M\left(x_6,x_5,t_5\right)\\ &\quad*M\left(x_6,x_7,t_6\right)*\cdots*M\left(x_p,x_{p+1},t_p\right)\leq M\left(x_1,x_{p+1},t_1+t_2+\cdots+t_p\right) \end{aligned}$$

and

$$\begin{aligned} &M\left(x_2,x_1,t_1\right)*M\left(x_3,x_2,t_2\right)*M\left(x_3,x_4,t_3\right)*M\left(x_5,x_4,t_4\right)*M\left(x_5,x_6,t_5\right)\\ &\quad*M\left(x_7,x_6,t_6\right)*\cdots*M\left(x_{p+1},x_p\,t_p\right)\leq M\left(x_{p+1},x_1,t_1+t_2+\cdots+t_p\right). \end{aligned}$$

Definition 2.6. Let (X,M) be a fuzzy semi-metric space.

- We say that M is *nondecreasing* when, given any fixed $x,y\in X$, $M(x,y,t_1)\geq M(x,y,t_2)$ for $t_1>t_2>0$.

- We say that M is *symmetrically nondecreasing* when, given any fixed $x, y \in X$, $M(x, y, t_1) \geq M(y, x, t_2)$ for $t_1 > t_2 > 0$.

Proposition 2.7. *Let (X, M) be a fuzzy semi-metric space. Then we have the following properties.*

(i) *If M satisfies the $\bowtie$-triangle inequality, then M is nondecreasing.*
(ii) *If M satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality, then M is both nondecreasing and symmetrically nondecreasing.*
(iii) *If M satisfies the $\diamond$-triangle inequality, then M is symmetrically nondecreasing.*

Let (X, d) be a metric space. If the sequence $\{x_n\}_{n=1}^{\infty}$ in (X, d) converges to x, i.e., $d(x_n, x) \to 0$ as $n \to \infty$, then it is denoted by $x_n \xrightarrow{d} x$ as $n \to \infty$. In this case, we also say that x is a *d-limit* of the sequence $\{x_n\}_{n=1}^{\infty}$.

Definition 2.8. Let (X, M) be a fuzzy semi-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.

- We write $x_n \xrightarrow{M^{\triangleright}} x$ as $n \to \infty$ when
$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ for all } t > 0.$$
In this case, we call x a *$M^{\triangleright}$-limit* of the sequence $\{x_n\}_{n=1}^{\infty}$.

- We write $x_n \xrightarrow{M^{\triangleleft}} x$ as $n \to \infty$ when
$$\lim_{n\to\infty} M(x, x_n, t) = 1 \text{ for all } t > 0.$$
In this case, we call x a *$M^{\triangleleft}$-limit* of the sequence $\{x_n\}_{n=1}^{\infty}$.

- We write $x_n \xrightarrow{M} x$ as $n \to \infty$ when
$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(x, x_n, t) = 1 \text{ for all } t > 0.$$
In this case, we call x a *M-limit* of the sequence $\{x_n\}_{n=1}^{\infty}$.

The following interesting results will be used for the further study.

Proposition 2.9. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ that is left-continuous with respect to the first or second component, and let $\{(x_n, y_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $X \times X \times (0, \infty)$. Assume that the following inequality is satisfied*
$$\sup_n (a_n * b_n) \geq \left(\sup_n a_n\right) * \left(\sup_n b_n\right)$$
for any sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in $[0, 1]$.

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality, and that $t_n \to t^\circ$, $x_n \xrightarrow{M} x^\circ$ and $y_n \xrightarrow{M} y^\circ$ as $n \to \infty$. Then the following statements hold true.*
   - *If the mapping $M(x^\circ, y^\circ, \cdot) : (0, \infty) \to [0, 1]$ is continuous at t°, then*
$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x^\circ, y^\circ, t^\circ).$$

- *If the mapping* $M(y^\circ, x^\circ, \cdot) : (0, \infty) \to [0, 1]$ *is continuous at* t°, *then*

$$\lim_{n\to\infty} M(y_n, x_n, t_n) = M(y^\circ, x^\circ, t^\circ).$$

(ii) *Suppose that* M *satisfies the* $\triangleright$*-triangle inequality or the* $\triangleleft$*-triangle inequality, and that* $t_n \to t^\circ$, $x_n \xrightarrow{M} x^\circ$ *and* $y_n \xrightarrow{M} y^\circ$ *as* $n \to \infty$. *Then the following statements hold true.*

- *If the mapping* $M(x^\circ, y^\circ, \cdot) : (0, \infty) \to [0, 1]$ *is continuous at* t°, *then*

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = \lim_{n\to\infty} M(y_n, x_n, t_n) = M(x^\circ, y^\circ, t^\circ).$$

- *If the mapping* $M(y^\circ, x^\circ, \cdot) : (0, \infty) \to [0, 1]$ *is continuous at* t°, *then*

$$\lim_{n\to\infty} M(y_n, x_n, t_n) = \lim_{n\to\infty} M(y_n, x_n, t_n) = M(y^\circ, x^\circ, t^\circ).$$

(iii) *Suppose that* M *satisfies the* $\diamond$*-triangle inequality. Then the following statements hold true.*

- *Suppose that the mapping* $M(x^\circ, y^\circ, \cdot) : (0, \infty) \to [0, 1]$ *is continuous at* t°, *and that* $t_n \to t^\circ$ *as* $n \to \infty$. *If* $x_n \xrightarrow{M^\triangleright} x^\circ$ *and* $y_n \xrightarrow{M^\triangleright} y^\circ$ *as* $n \to \infty$ *simultaneously, or* $x_n \xrightarrow{M^\triangleleft} x^\circ$ *and* $y_n \xrightarrow{M^\triangleleft} y^\circ$ *as* $n \to \infty$ *simultaneously, then*

$$\lim_{n\to\infty} M(y_n, x_n, t_n) = M(x^\circ, y^\circ, t^\circ).$$

- *Suppose that the mapping* $M(y^\circ, x^\circ, \cdot) : (0, \infty) \to [0, 1]$ *is continuous at* t°, *and that* $t_n \to t^\circ$ *as* $n \to \infty$. *If* $x_n \xrightarrow{M^\triangleright} x^\circ$ *and* $y_n \xrightarrow{M^\triangleright} y^\circ$ *as* $n \to \infty$ *simultaneously, or* $x_n \xrightarrow{M^\triangleleft} x^\circ$ *and* $y_n \xrightarrow{M^\triangleleft} y^\circ$ *as* $n \to \infty$ *simultaneously, then*

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(y^\circ, x^\circ, t^\circ).$$

Definition 2.10. Let (X, M) be a fuzzy semi-metric space, and let $\{x_n\}_{n=1}^\infty$ be a sequence in X.

- We say that $\{x_n\}_{n=1}^\infty$ is a $>$-*Cauchy sequence* when, given any pair (r, t) with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - r$ for all pairs (m, n) of integers m and n with $m > n \geq n_{r,t}$.

- We say that $\{x_n\}_{n=1}^\infty$ is a $<$-*Cauchy sequence* when, given any pair (r, t) with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all pairs (m, n) of integers m and n with $m > n \geq n_{r,t}$.

- We say that $\{x_n\}_{n=1}^\infty$ is a *Cauchy sequence* when, given any pair (r, t) with $t > 0$ and $0 < r < 1$, there exists $n_{r,t} \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1-r$ and $M(x_n, x_m, t) > 1-r$ for all pairs (m, n) of integers m and n with $m, n \geq n_{r,t}$ and $m \neq n$.

Definition 2.11. Let (X, M) be a fuzzy semi-metric space.

- We say that (X, M) is $(>, \triangleright)$-*complete* when each $>$-Cauchy sequence is convergent in the sense of $x_n \xrightarrow{M^\triangleright} x$.

- We say that (X, M) is $(>, \triangleleft)$-*complete* when each $>$-Cauchy sequence is convergent in the sense of $x_n \xrightarrow{M^\triangleleft} x$.

- We say that (X, M) is $(<, \triangleright)$-*complete* when each $<$-Cauchy sequence is convergent in the sense of $x_n \xrightarrow{M^{\triangleright}} x$.
- We say that (X, M) is $(<, \triangleleft)$-*complete* when each $<$-Cauchy sequence is convergent in the sense of $x_n \xrightarrow{M^{\triangleleft}} x$.

Definition 2.12. Let (X, M) be a fuzzy semi-metric space.

- We say that the function $f : X \to X$ is $(\triangleright, \triangleright)$-*continuous with respect to* M when, given any sequence $\{x_n\}_{n=1}^{\infty}$ in X, $x_n \xrightarrow{M^{\triangleright}} x$ as $n \to \infty$ implies $f(x_n) \xrightarrow{M^{\triangleright}} f(x)$ as $n \to \infty$.
- We say that the function $f : X \to X$ is $(\triangleright, \triangleleft)$-*continuous with respect to* M when, given any sequence $\{x_n\}_{n=1}^{\infty}$ in X, $x_n \xrightarrow{M^{\triangleright}} x$ as $n \to \infty$ implies $f(x_n) \xrightarrow{M^{\triangleleft}} f(x)$ as $n \to \infty$.
- We say that the function $f : X \to X$ is $(\triangleleft, \triangleright)$-*continuous with respect to* M when, given any sequence $\{x_n\}_{n=1}^{\infty}$ in X, $x_n \xrightarrow{M^{\triangleleft}} x$ as $n \to \infty$ implies $f(x_n) \xrightarrow{M^{\triangleright}} f(x)$ as $n \to \infty$.
- We say that the function $f : X \to X$ is $(\triangleleft, \triangleleft)$-*continuous with respect to* M when, given any sequence $\{x_n\}_{n=1}^{\infty}$ in X, $x_n \xrightarrow{M^{\triangleleft}} x$ as $n \to \infty$ implies $f(x_n) \xrightarrow{M^{\triangleleft}} f(x)$ as $n \to \infty$.

Based on the intuitive concept of the value $M(x, y, t)$, we see that $M(x, y, t) = 1$ means the distance between x and y that is surely less than or equal to t, and $M(x, y, t) = 0$ means the distance between x and y that is surely greater than t. Therefore if $x \neq y$ with distance $t_{xy} \neq 0$ between x and y, then it is impossible for $M(x, y, t) = 0$ and for all $t > 0$. In other words, there exists $t_0 > 0$ with $t_{xy} < t_0$ satisfying $M(x, y, t_0) \neq 0$. We propose the following definition.

3. Auxiliary Functions Based on the Infimum

Since $M(x, y, t)$ is the membership degree of the distance between x and y that is less than or equal to t, it is natural to see that the mapping $M(x, y, \cdot)$ is nondecreasing or symmetrically nondecreasing as shown in Proposition 2.7. On the other hand, since the distance will always be less than or equal to a large t, it is also reasonable to argue that if t is sufficiently large, then the membership degree $M(x, y, t)$ is close to 1. Therefore we propose the following definition.

Definition 3.1. Let (X, M) be a fuzzy semi-metric space. We say that M satisfies the *canonical condition* when

$$\lim_{t \to +\infty} M(x, y, t) = 1 \text{ for any fixed } x, y \in X.$$

Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$. We define the mapping $\eta : X^4 \times [0, +\infty) \to [0, 1]$ by

$$\eta(x, y, u, v, t) = M(x, y, t) * M(u, v, t).$$

The following properties will be used to define the auxiliary functions.

Proposition 3.2. *Let (X, M) be a fuzzy semi-metric space. Suppose that M satisfies the canonical condition. If the t-norm $*$ is left-continuous at* 1 *with respect to the first or second component, then*

$$\lim_{t \to +\infty} \eta(x, y, u, v, t) = 1. \tag{3.1}$$

The theorems of common coupled coincidence points and common coupled fixed points should be based on the Cauchy sequences. Therefore we introduce the auxiliary functions to obtain the useful properties regarding the Cauchy sequences. .

Definition 3.3. Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition, and that the t-norm $*$ is left-continuous at 1 with respect to the first or second component. Given any fixed $x, y, u, v \in X$ and any fixed $\lambda \in (0, 1]$, we define the set

$$\Pi(\lambda, x, y, u, v) = \{t > 0 : \eta(x, y, u, v, t) \geq 1 - \lambda\}.$$

and the function $\Psi(\lambda, \cdot, \cdot, \cdot, \cdot) : X^4 \to [0, +\infty)$ by

$$\Psi(\lambda, x, y, u, v) = \inf \Pi(\lambda, x, y, u, v) = \inf \{t > 0 : \eta(x, y, u, v, t) \geq 1 - \lambda\}.$$

We need to claim $\Pi(\lambda, x, y, u, v) \neq \emptyset$. In this case, we have $\Psi(\lambda, x, y, u, v) < +\infty$. Suppose that $\Pi(\lambda, x, y, u, v) = \emptyset$. By definition, we must have $\eta(x, y, u, v, t) < 1 - \lambda$ for all $t > 0$. This says that

$$\lim_{t \to +\infty} \eta(x, y, u, v, t) \leq 1 - \lambda < 1,$$

which contradicts (3.1). Therefore we indeed have $\Pi(\lambda, x, y, u, v) \neq \emptyset$. This says that Definition 3.3 is well-defined.

Proposition 3.4. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition, and that the t-norm $*$ is left-continuous at 1 with respect to the first or second component. Given any fixed $x, y, u, v \in X$ and $\lambda \in (0, 1)$, we have the following properties.*

(i) *If $\epsilon > 0$ is sufficiently small such that $\Psi(\lambda, x, y, u, v) > \epsilon$, then*

$$\eta\left(x, y, u, v, \Psi(\lambda, x, y, u, v) - \epsilon\right) < 1 - \lambda.$$

(ii) *Suppose that M satisfies the $\bowtie$-triangle inequality or the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality. For any $\epsilon > 0$, we have*

$$\eta\left(x, y, u, v, \Psi(\lambda, x, y, u, v) + \epsilon\right) \geq 1 - \lambda \tag{3.2}$$

(iii) *Suppose that M satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality. For any $\epsilon > 0$, we have*

$$\eta\left(x, y, u, v, \Psi(\lambda, y, x, u, v) + \epsilon\right) \geq 1 - \lambda$$

and

$$\eta\left(x, y, u, v, \Psi(\lambda, x, y, v, u) + \epsilon\right) \geq 1 - \lambda.$$

(iv) *Suppose that M satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality or the $\diamond$-triangle inequality. For any $\epsilon > 0$, we have*

$$\eta\left(x, y, u, v, \Psi(\lambda, y, x, v, u) + \epsilon\right) \geq 1 - \lambda$$

Proposition 3.5. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition, and that the t-norm $*$ is left-continuous at 1 with respect to the first or second component.*

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that*

$$\Psi(\mu, x_1, x_p, y_1, y_p) \leq \Psi(\lambda, x_1, x_2, y_1, y_2) + \Psi(\lambda, x_2, x_3, y_2, y_3) + \cdots + \Psi(\lambda, x_{p-2}, x_{p-1}, y_{p-2}, y_{p-1}) + \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) \tag{3.3}$$

$$\Psi(\mu, x_1, x_p, y_p, y_1) \leq \Psi(\lambda, x_1, x_2, y_2, y_1) + \Psi(\lambda, x_2, x_3, y_3, y_2) + \cdots + \Psi(\lambda, x_{p-2}, x_{p-1}, y_{p-1}, y_{p-2}) + \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) \tag{3.4}$$

$$\Psi(\mu, x_p, x_1, y_1, y_p) \leq \Psi(\lambda, x_p, x_{p-1}, y_{p-1}, y_p) + \Psi(\lambda, x_{p-1}, x_{p-2}, y_{p-2}, y_{p-1}) + \cdots + \Psi(\lambda, x_3, x_2, y_2, y_3) + \Psi(\lambda, x_2, x_1, y_1, y_2)$$

$$\Psi(\mu, x_p, x_1, y_p, y_1) \leq \Psi(\lambda, x_p, x_{p-1}, y_p, y_{p-1}) + \Psi(\lambda, x_{p-1}, x_{p-2}, y_{p-1}, y_{p-2}) + \cdots + \Psi(\lambda, x_3, x_2, y_3, y_2) + \Psi(\lambda, x_2, x_1, y_2, y_1).$$

(ii) *Suppose that M satisfies the $\triangleright$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that*

$$\begin{aligned}&\max\{\Psi(\mu, x_1, x_p, y_1, y_p), \Psi(\mu, x_1, x_p, y_p, y_1), \Psi(\mu, x_p, x_1, y_1, y_p), \Psi(\mu, x_p, x_1, y_p, y_1)\}\\ &\leq \Psi(\lambda, x_1, x_2, y_1, y_2) + \Psi(\lambda, x_3, x_2, y_3, y_2) + \Psi(\lambda, x_4, x_3, y_4, y_3)\\ &\quad + \cdots + \Psi(\lambda, x_p, x_{p-1}, y_p, y_{p-1})\end{aligned}$$

(iii) *Suppose that M satisfies the $\triangleleft$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that*

$$\begin{aligned}&\max\{\Psi(\mu, x_1, x_p, y_1, y_p), \Psi(\mu, x_1, x_p, y_p, y_1), \Psi(\mu, x_p, x_1, y_1, y_p), \Psi(\mu, x_p, x_1, y_p, y_1)\}\\ &\leq \Psi(\lambda, x_2, x_1, y_2, y_1) + \Psi(\lambda, x_2, x_3, y_2, y_3) + \Psi(\lambda, x_3, x_4, y_3, y_4)\\ &\quad + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p)\end{aligned}$$

(iv) *Suppose that M satisfies the $\diamond$-triangle inequality. Given any fixed $x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_p \in X$ and any fixed $\mu \in (0,1]$, there exists $\lambda \in (0,1)$ such that the following inequalities are satisfied.*

- *If p is even, then*

$$\begin{aligned}\Psi(\mu, x_1, x_p, y_1, y_p) \leq\ & \Psi(\lambda, x_1, x_2, y_1, y_2) + \Psi(\lambda, x_2, x_3, y_2, y_3) + \Psi(\lambda, x_4, x_3, y_4, y_3)\\ &+ \Psi(\lambda, x_4, x_5, y_4, y_5) + \Psi(\lambda, x_6, x_5, y_6, y_5) + \Psi(\lambda, x_6, x_7, y_6, y_7)\\ &+ \cdots + \Psi(\lambda, x_p, x_{p-1}, y_p, y_{p-1})\end{aligned} \tag{3.5}$$

$$\begin{aligned}\Psi(\mu, x_1, x_p, y_p, y_1) \leq\ & \Psi(\lambda, x_1, x_2, y_2, y_1) + \Psi(\lambda, x_2, x_3, y_3, y_2) + \Psi(\lambda, x_4, x_3, y_3, y_4)\\ &+ \Psi(\lambda, x_4, x_5, y_5, y_4) + \Psi(\lambda, x_6, x_5, y_5, y_6) + \Psi(\lambda, x_6, x_7, y_7, y_6)\\ &+ \cdots + \Psi(\lambda, x_p, x_{p-1}, y_{p-1}, y_p)\end{aligned} \tag{3.6}$$

$$\begin{aligned}\Psi(\mu, x_p, x_1, y_1, y_p) \leq\ & \Psi(\lambda, x_2, x_1, y_1, y_2) + \Psi(\lambda, x_3, x_2, y_2, y_3) + \Psi(\lambda, x_3, x_4, y_4, y_3)\\ &+ \Psi(\lambda, x_5, x_4, y_4, y_5) + \Psi(\lambda, x_5, x_6, y_6, y_5) + \Psi(\lambda, x_7, x_6, y_6, y_7)\\ &+ \cdots + \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1})\end{aligned} \tag{3.7}$$

$$\begin{aligned}\Psi(\mu, x_p, x_1, y_p, y_1) \leq\ & \Psi(\lambda, x_2, x_1, y_2, y_1) + \Psi(\lambda, x_3, x_2, y_3, y_2) + \Psi(\lambda, x_3, x_4, y_3, y_4)\\ &+ \Psi(\lambda, x_5, x_4, y_5, y_4) + \Psi(\lambda, x_5, x_6, y_5, y_6) + \Psi(\lambda, x_7, x_6, y_7, y_6)\\ &+ \cdots + \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p)\end{aligned} \tag{3.8}$$

- *If p is odd, then*

$$\begin{aligned}\Psi(\mu, x_1, x_p, y_1, y_p) \leq\ & \Psi(\lambda, x_2, x_1, y_2, y_1) + \Psi(\lambda, x_3, x_2, y_3, y_2) + \Psi(\lambda, x_3, x_4, y_3, y_4)\\ &+ \Psi(\lambda, x_5, x_4, y_5, y_4) + \Psi(\lambda, x_5, x_6, y_5, y_6) + \Psi(\lambda, x_7, x_6, y_7, y_6)\\ &+ \cdots + \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) \qquad (3.9)\\ \Psi(\mu, x_1, x_p, y_p, y_1) \leq\ & \Psi(\lambda, x_2, x_1, y_1, y_2) + \Psi(\lambda, x_3, x_2, y_2, y_3) + \Psi(\lambda, x_3, x_4, y_4, y_3)\\ &+ \Psi(\lambda, x_5, x_4, y_4, y_5) + \Psi(\lambda, x_5, x_6, y_6, y_5) + \Psi(\lambda, x_7, x_6, y_6, y_7)\\ &+ \cdots + \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) \qquad (3.10)\\ \Psi(\mu, x_p, x_1, y_1, y_p) \leq\ & \Psi(\lambda, x_1, x_2, y_2, y_1) + \Psi(\lambda, x_2, x_3, y_3, y_2) + \Psi(\lambda, x_4, x_3, y_3, y_4)\\ &+ \Psi(\lambda, x_4, x_5, y_5, y_4) + \Psi(\lambda, x_6, x_5, y_5, y_6) + \Psi(\lambda, x_6, x_7, y_7, y_6)\\ &+ \cdots + \Psi(\lambda, x_p, x_{p-1}, y_{p-1}, y_p) \qquad (3.11)\\ \Psi(\mu, x_p, x_1, y_p, y_1) \leq\ & \Psi(\lambda, x_1, x_2, y_1, y_2) + \Psi(\lambda, x_2, x_3, y_2, y_3) + \Psi(\lambda, x_4, x_3, y_4, y_3)\\ &+ \Psi(\lambda, x_4, x_5, y_4, y_5) + \Psi(\lambda, x_6, x_5, y_6, y_5) + \Psi(\lambda, x_6, x_7, y_6, y_7)\\ &+ \cdots + \Psi(\lambda, x_p, x_{p-1}, y_p, y_{p-1}) \qquad (3.12)\end{aligned}$$

Proof . We just prove part (i). If $\mu = 1$, then $\Psi(1, x_1, x_p, y_1, y_p) = 0$. The result is obvious. Therefore we assume $\mu \in (0, 1)$. According to part (ii) of Proposition 2.1, there exists $\lambda \in (0, 1)$ such that

$$(1-\lambda) * \cdots * (1-\lambda) > 1 - \mu. \qquad (3.13)$$

Given any $\epsilon > 0$, by the first observation of Remark 2.5, we have

$$\begin{aligned}&M\left(x_1, x_p, \Psi(\lambda, x_1, x_2, y_1, y_2) + \Psi(\lambda, x_2, x_3, y_2, y_3) + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) + (p-1)\epsilon\right)\\ &\quad \geq M\left(x_1, x_2, \Psi(\lambda, x_1, x_2, y_1, y_2) + \epsilon\right) * \cdots * M\left(x_{p-1}, x_p, \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) + \epsilon\right) \qquad (3.14)\end{aligned}$$

and

$$\begin{aligned}&M\left(y_1, y_p, \Psi(\lambda, x_1, x_2, y_1, y_2) + \Psi(\lambda, x_2, x_3, y_2, y_3) + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) + (p-1)\epsilon\right)\\ &\quad \geq M\left(y_1, y_2, \Psi(\lambda, x_1, x_2, y_1, y_2) + \epsilon\right) * \cdots * M\left(y_{p-1}, y_p, \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) + \epsilon\right) \qquad (3.15)\end{aligned}$$

Applying the increasing property and commutativity of t-norm to (3.14) and (3.15), we obtain

$$\begin{aligned}&\eta\left(x_1, x_p, y_1, y_p, \Psi(\lambda, x_1, x_2, y_1, y_2) + \Psi(\lambda, x_2, x_3, y_2, y_3) + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) + (p-1)\epsilon\right)\\ &\quad \geq \eta\left(x_1, x_2, y_1, y_2, \Psi(\lambda, x_1, x_2, y_1, y_2) + \epsilon\right) * \cdots * \eta\left(x_{p-1}, x_p, y_{p-1}, y_p, \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) + \epsilon\right)\\ &\quad \geq (1-\lambda) * \cdots * (1-\lambda) \text{ (by (3.2) and the increasing property of t-norm)}\\ &\quad > 1 - \mu \text{ (by (3.13)).}\end{aligned}$$

By the definition of Ψ_λ, we obtain

$$\Psi(\lambda, x_1, x_2, y_1, y_2) + \Psi(\lambda, x_2, x_3, y_2, y_3) + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_{p-1}, y_p) + (p-1)\epsilon \geq \Psi(\mu, x_1, x_p, y_1, y_p).$$

Taking $\epsilon \to 0+$, we obtain the inequality (3.3).

On the other hand, we also have

$$\begin{aligned}&M\left(x_1, x_p, \Psi(\lambda, x_1, x_2, y_2, y_1) + \Psi(\lambda, x_2, x_3, y_3, y_2) + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) + (p-1)\epsilon\right)\\ &\quad \geq M\left(x_1, x_2, \Psi(\lambda, x_1, x_2, y_2, y_1) + \epsilon\right) * \cdots * M\left(x_{p-1}, x_p, \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) + \epsilon\right) \qquad (3.16)\end{aligned}$$

and

$$
\begin{aligned}
&M\left(y_p, y_1, \Psi(\lambda, x_1, x_2, y_2, y_1) + \Psi(\lambda, x_2, x_3, y_3, y_2) + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) + (p-1)\epsilon\right) \\
&\quad \geq M\left(y_2, y_1, \Psi(\lambda, x_1, x_2, y_2, y_1) + \epsilon\right) * \cdots * M\left(y_p, y_{p-1}, \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) + \epsilon\right) \qquad (3.17)
\end{aligned}
$$

Applying the increasing property and commutativity of t-norm to (3.16) and (3.17), we obtain

$$
\begin{aligned}
&\eta\left(x_1, x_p, y_p, y_1, \Psi(\lambda, x_1, x_2, y_2, y_1) + \Psi(\lambda, x_2, x_3, y_3, y_2) + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) + (p-1)\epsilon\right) \\
&\quad \geq \eta\left(x_1, x_2, y_2, y_1, \Psi(\lambda, x_1, x_2, y_2, y_1) + \epsilon\right) * \cdots * \eta\left(x_{p-1}, x_p, y_p, y_{p-1}, \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) + \epsilon\right) \\
&\quad \geq (1-\lambda) * \cdots * (1-\lambda) \text{ (by (3.2) and the increasing property of t-norm)} \\
&\quad > 1-\mu \text{ (by (3.13))}.
\end{aligned}
$$

By the definition of Ψ_λ, we obtain

$$\Psi(\lambda, x_1, x_2, y_2, y_1) + \Psi(\lambda, x_2, x_3, y_3, y_2) + \cdots + \Psi(\lambda, x_{p-1}, x_p, y_p, y_{p-1}) + (p-1)\epsilon \geq \Psi(\mu, x_1, x_p, y_p, y_1).$$

Taking $\epsilon \to 0+$, we obtain the inequality (3.4). The other inequalities can be similarly obtained. This completes the proof. □

Proposition 3.6. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition, and that the t-norm $*$ is left-continuous at 1 with respect to the first or second component. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two sequences in X. Then the following statements hold true.*

(i) *$\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two $>$-Cauchy sequences if and only if, given any $\epsilon > 0$ and $\lambda \in (0,1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon,\lambda}$ implies $\Psi(\lambda, x_m, x_n, y_m, y_n) < \epsilon$.*

(ii) *$\{x_n\}_{n=1}^\infty$ is a $>$-Cauchy sequences and $\{y_n\}_{n=1}^\infty$ is a $<$-Cauchy sequences if and only if, given any $\epsilon > 0$ and $\lambda \in (0,1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon,\lambda}$ implies $\Psi(\lambda, x_m, x_n, y_n, y_m) < \epsilon$.*

(iii) *$\{x_n\}_{n=1}^\infty$ is a $<$-Cauchy sequences and $\{y_n\}_{n=1}^\infty$ is a $>$-Cauchy sequences if and only if, given any $\epsilon > 0$ and $\lambda \in (0,1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon,\lambda}$ implies $\Psi(\lambda, x_n, x_m, y_m, y_n) < \epsilon$.*

(iv) *$\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are two $<$-Cauchy sequences if and only if, given any $\epsilon > 0$ and $\lambda \in (0,1)$, there exists $n_{\epsilon,\lambda} \in \mathbb{N}$ such that $m > n \geq n_{\epsilon,\lambda}$ implies $\Psi(\lambda, x_n, x_m, y_n, y_m) < \epsilon$.*

4. Cauchy Sequences

We shall present many kinds of situations that can guarantee the Cauchy sequence in order to derive the theorems of common coupled coincidence points and common coupled fixed points.

Given any $a \in [0,1]$, for convenience, we write

$$(*a)^n = \overbrace{a * a * \cdots * a}^{n \text{ times}}$$

and

$$\left[*\eta\left(a, b, c, d, \frac{t}{k^n}\right)\right]^{2^n} = \overbrace{\eta\left(a, b, c, d, \frac{t}{k^n}\right) * \eta\left(a, b, c, d, \frac{t}{k^n}\right) * \cdots * \eta\left(a, b, c, d, \frac{t}{k^n}\right)}^{2^n \text{ times}}.$$

Proposition 4.1. *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition, and that the t-norm is left-continuous at 1 in the first or second component. Let $0 < k < 1$ be any fixed constant, and let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences in X.*

(i) *Suppose that M satisfies the $\bowtie$-triangle inequality. Then the following statements hold true.*

- *Assume that there exist fixed elements $a_1, b_1, c_1, d_1 \in X$ such that*

$$\sup_{\lambda \in (0,1)} \Psi(\lambda, a_1, b_1, c_1, d_1) < \infty, \tag{4.1}$$

and that

$$\eta(x_n, x_{n+1}, y_n, y_{n+1}, t) \geq \left[*\eta\left(a_1, b_1, c_1, d_1, \frac{t}{k^n}\right)\right]^{2^n} \text{ for each } n \in \mathbb{N}. \tag{4.2}$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $<$-Cauchy sequences.

- *Assume that there exist fixed elements $a_2, b_2, c_2, d_2 \in X$ such that*

$$\sup_{\lambda \in (0,1)} \Psi(\lambda, a_2, b_2, c_2, d_2) < \infty, \tag{4.3}$$

and that

$$\eta(x_n, x_{n+1}, y_{n+1}, y_n, t) \geq \left[*\eta\left(a_2, b_2, c_2, d_2, \frac{t}{k^n}\right)\right]^{2^n} \text{ for each } n \in \mathbb{N}, \tag{4.4}$$

Then $\{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence and $\{y_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence.

- *Assume that there exist fixed elements $a_3, b_3, c_3, d_3 \in X$ such that*

$$\sup_{\lambda \in (0,1)} \Psi(\lambda, a_3, b_3, c_3, d_3) < \infty, \tag{4.5}$$

and that

$$\eta(x_{n+1}, x_n, y_n, y_{n+1}, t) \geq \left[*\eta\left(a_3, b_3, c_3, d_3, \frac{t}{k^n}\right)\right]^{2^n} \text{ for each } n \in \mathbb{N}, \tag{4.6}$$

then $\{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence and $\{y_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence.

- *Assume that there exist fixed elements $a_4, b_4, c_4, d_4 \in X$ such that*

$$\sup_{\lambda \in (0,1)} \Psi(\lambda, a_4, b_4, c_4, d_4) < \infty, \tag{4.7}$$

and that

$$\eta(x_{n+1}, x_n, y_{n+1}, y_n, t) \geq \left[*\eta\left(a_4, b_4, c_4, d_4, \frac{t}{k^n}\right)\right]^{2^n} \text{ for each } n \in \mathbb{N}, \tag{4.8}$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are $>$-Cauchy sequences.

(ii) *Suppose that M satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality. We also Assume that (4.1), (4.2), (4.7) and (4.8) are satisfied. Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both $>$-Cauchy and $<$-Cauchy sequences; that is, $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences.*

(iii) *Suppose that M satisfies the $\diamond$-triangle inequality, and that any one of the following two conditions is satisfied:*

- *conditions* (4.1), (4.2), (4.7) *and* (4.8) *are satisfied;*
- *conditions* (4.3), (4.4), (4.5) *and* (4.6) *are satisfied.*

Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both $>$-Cauchy and $<$-Cauchy sequences.

Proof . To prove part (i), it suffices to prove the first case. Given any $\lambda \in (0,1)$, from (4.2), we have the following inclusions

$$\begin{aligned}\left\{t>0:\left[*\eta\left(a_1,b_1,c_1,d_1,\frac{t}{k^n}\right)\right]^{2^n}\geq 1-\lambda\right\}&\\ \subseteq \left\{t>0:\eta\left(x_n,x_{n+1},y_n,y_{n+1},t\right)\geq 1-\lambda\right\}.&\end{aligned}\tag{4.9}$$

Using part (ii) of Proposition 2.1, there exists $\lambda_0 \in (0,1)$ such that

$$[*(1-\lambda_0)]^{2n} > 1-\lambda.$$

Therefore, if

$$\eta\left(a_1,b_1,c_1,d_1,\frac{t}{k^n}\right)\geq 1-\lambda_0$$

then, using the increasing property for t-norm,

$$\left[*\eta\left(a_1,b_1,c_1,d_1,\frac{t}{k^n}\right)\right]^{2^n}\geq [*(1-\lambda_0)]^{2n} > 1-\lambda,$$

which also says that

$$\begin{aligned}&\left\{t>0:\eta\left(a_1,b_1,c_1,d_1,\frac{t}{k^n}\right)\geq 1-\lambda_0\right\}\\ &\quad\subseteq\left\{t>0:\left[*\eta\left(a_1,b_1,c_1,d_1,\frac{t}{k^n}\right)\right]^{2^n}> 1-\lambda\right\}\\ &\quad\subseteq\left\{t>0:\left[*\eta\left(a_1,b_1,c_1,d_1,\frac{t}{k^n}\right)\right]^{2^n}\geq 1-\lambda\right\}.\end{aligned}\tag{4.10}$$

Therefore we obtain

$$\begin{aligned}&\Psi(\lambda,x_n,x_{n+1},y_n,y_{n+1})\\ &\quad=\inf\left\{t>0:\eta\left(x_n,x_{n+1},y_n,y_{n+1},t\right)\geq 1-\lambda\right\}\\ &\quad\leq\inf\left\{t>0:\left[*\eta\left(a_1,b_1,c_1,d_1,\frac{t}{k^n}\right)\right]^{2^n}\geq 1-\lambda\right\}\ \text{(by (4.9))}\\ &\quad\leq\inf\left\{t>0:\eta\left(a_1,b_1,c_1,d_1,\frac{t}{k^n}\right)\geq 1-\lambda_0\right\}\ \text{(by (4.10))}\\ &\quad=\inf\left\{k^n\cdot t>0:\eta\left(a_1,b_1,c_1,d_1,t\right)\geq 1-\lambda_0\right\}\\ &\quad=k^n\cdot\inf\left\{t>0:\eta\left(a_1,b_1,c_1,d_1,t\right)\geq 1-\lambda_0\right\}\\ &\quad=k^n\cdot\Psi(\lambda_0,a_1,b_1,c_1,d_1).\end{aligned}\tag{4.11}$$

Now we assume that $m,n \in \mathbb{N}$ with $m > n$. Given any $\mu \in (0,1]$, by part (i) of Proposition 3.5, there exists $\lambda \in (0,1)$ (which depends on m and n) such that

$$\begin{aligned}
&\Psi(\mu, x_n, x_m, y_n, y_m)\\
&\quad\le \Psi(\lambda, x_n, x_{n+1}, y_n, y_{n+1}) + \Psi(\lambda, x_{n+1}, x_{n+2}, y_{n+1}, y_{n+2})\\
&\qquad + \cdots + \Psi(\lambda, x_{m-1}, x_m, y_{m-1}, y_m)\\
&\quad\le k^n \cdot \Psi(\lambda_0, a_1, b_1, c_1, d_1) + k^{n+1} \cdot \Psi(\lambda_0, a_1, b_1, c_1, d_1)\\
&\qquad + \cdots + k^{m-1} \cdot \Psi(\lambda_0, a_1, b_1, c_1, d_1) \text{ (by (4.11))}\\
&\quad= \Psi(\lambda_0, a_1, b_1, c_1, d_1) \cdot \frac{k^n \cdot (1-k^{m-n})}{1-k} \le \Psi(\lambda_0, a_1, b_1, c_1, d_1) \cdot \frac{k^n}{1-k}\\
&\quad\le \left[\sup_{\lambda\in(0,1)} \Psi(\lambda, a_1, b_1, c_1, d_1)\right] \cdot \frac{k^n}{1-k} \to 0 \text{ as } n \to \infty,
\end{aligned} \tag{4.12}$$

which also says that, given any $\epsilon \in (0,1)$ and $\mu \in (0,1)$, there exists $n_{\mu,\epsilon} \in \mathbb{N}$ such that $m > n \ge n_{\mu,\epsilon}$ implies $\Psi(\mu, x_n, x_m, y_n, y_m) < \epsilon$. By the fourth case of Proposition 3.6, it follows that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are $<$-Cauchy sequences. The other results can be similarly obtained by using the corresponding cases of Proposition 3.6 and part (i) of Proposition 3.5.

To prove part (ii), we consider the following cases.

- Suppose that M satisfies the $\triangleright$-triangle inequality. Using part (ii) of Proposition 3.5, we have

$$\begin{aligned}
&\max\{\Psi(\mu, x_n, x_m, y_n, y_m), \Psi(\mu, x_m, x_n, y_n, y_m), \Psi(\mu, x_n, x_m, y_m, y_n), \Psi(\mu, x_m, x_n, y_m, y_n)\}\\
&\quad\le \Psi(\lambda, x_m, x_{m-1}, y_m, y_{m-1}) + \Psi(\lambda, x_{m-1}, x_{m-2}, y_{m-1}, y_{m-2})\\
&\qquad + \cdots + \Psi(\lambda, x_{n+2}, x_{n+1}, y_{n+2}, y_{n+1}) + \Psi(\lambda, x_n, x_{n+1}, y_n, y_{n+1}).
\end{aligned} \tag{4.13}$$

By referring to (4.11), we can similarly obtain

$$\Psi(\lambda, x_{n+1}, x_n, y_{n+1}, y_n) \le k^n \cdot \Psi(\lambda_0, a_4, b_4, c_4, d_4). \tag{4.14}$$

By using (4.11), (4.14), (4.13) and referring to (4.12), we have

$$\begin{aligned}
&\max\{\Psi(\mu, x_n, x_m, y_n, y_m), \Psi(\mu, x_m, x_n, y_n, y_m), \Psi(\mu, x_n, x_m, y_m, y_n), \Psi(\mu, x_m, x_n, y_m, y_n)\}\\
&\quad\le \max\left\{\left[\sup_{\lambda\in(0,1)} \Psi(\lambda, a_1, b_1, c_1, d_1)\right], \left[\sup_{\lambda\in(0,1)} \Psi(\lambda, a_4, b_4, c_4, d_4)\right]\right\} \cdot \frac{k^n}{1-k} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Using the above argument, we can show that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are both $>$-Cauchy and $<$-Cauchy sequences.

- Suppose that M satisfies the $\triangleleft$-triangle inequality. Using part (iii) of Proposition 3.5, we have

$$\begin{aligned}
&\max\{\Psi(\mu, x_n, x_m, y_n, y_m), \Psi(\mu, x_m, x_n, y_n, y_m), \Psi(\mu, x_n, x_m, y_m, y_n), \Psi(\mu, x_m, x_n, y_m, y_n)\}\\
&\quad\le \Psi(\lambda, x_{n+1}, x_n, y_{n+1}, y_n) + \Psi(\lambda, x_{n+1}, x_{n+2}, y_{n+1}, y_{n+2})\\
&\qquad + \cdots + \Psi(\lambda, x_{m-2}, x_{m-1}, y_{m-2}, y_{m-1}) + \Psi(\lambda, x_{m-1}, x_m, y_{m-1}, y_m).
\end{aligned}$$

Using the above argument, we can show that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are both $>$-Cauchy and $<$-Cauchy sequences.

To prove part (iii), we consider the following cases.

- Assume that (4.1), (4.2), (4.7) and (4.8) are satisfied. If p is even, then, using (3.5) and (3.8) in part (iv) of Proposition 3.5, we can similarly show that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both $>$-Cauchy and $<$-Cauchy sequences. If p is odd, then, using (3.9) and (3.12) in Proposition 3.5, we can similarly obtain the desired results.

- Assume that (4.3), (4.4), (4.5) and (4.6) are satisfied. If p is even, then, using (3.6) and (3.7) in part (iv) of Proposition 3.5, we can similarly show that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both $>$-Cauchy and $<$-Cauchy sequences. If p is odd, then, using (3.10) and (3.11) in Proposition 3.5, we can similarly obtain the desired results.

This completes the proof. □

Remark 4.2. We shall present the sufficient conditions to guarantee the finiteness (4.1).

- Suppose that M satisfies the $\bowtie$-triangle inequality or the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality. Then the mapping $M(x,y,\cdot)$ is nondecreasing by parts (i) and (ii) of Proposition 2.7. Now we also assume that a and b have a finite distance beginning from a to b, and that c and d have a finite distance beginning from c to d. Then, by definition, there exists $t^*_{ab} > 0$ and $t^*_{cd} > 0$ such that $M(a,b,t^*_{ab}) = 1$ and $M(c,d,t^*_{cd}) = 1$. Let $t^* = \max\{t^*_{ab}, t^*_{cd}\}$. Since the mapping $M(x,y,\cdot)$ is nondecreasing, it follows that
$$\eta\left(a,b,c,d,t^*\right) = M\left(a,b,t^*\right) * M\left(c,d,t^*\right) = 1 \geq 1-\lambda \text{ for all } \lambda \in (0,1),$$
which says that
$$\Psi\left(\lambda,a,b,c,d\right) \leq t^* < +\infty \text{ for all } \lambda \in (0,1).$$
Therefore we conclude that
$$\sup_{\lambda\in(0,1)} \Psi\left(\lambda,a,b,c,d\right) \leq t^* < +\infty.$$

- Suppose that M satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality or the $\diamond$-triangle inequality. Then the mapping $M(x,y,\cdot)$ is symmetrically nondecreasing by parts (ii) and (iii) of Proposition 2.7. Now we also assume that a and b have a finite distance beginning from b to a, and that c and d have a finite distance beginning from d to c. Then, by definition, there exists $t^*_{ba} > 0$ and $t^*_{dc} > 0$ such that $M(b,a,t^*_{ba}) = 1$ and $M(d,c,t^*_{dc}) = 1$. Let $t^* = \max\{t^*_{ba}, t^*_{dc}\}$. Since the mapping $M(x,y,\cdot)$ is symmetrically nondecreasing, it follows that$M(a,b,t^*) = 1$ and $M(c,d,t^*) = 1$. Therefore we can similarly obtain
$$\sup_{\lambda\in(0,1)} \Psi\left(\lambda,a,b,c,d\right) \leq t^* < +\infty.$$

5. Common Coupled Coincidence Points

Now we are in a position to present the theorems of common coupled coincidence points. Let X be a nonempty universal set. We consider the mappings $T : X \times X \to X$ and $f : X \to X$.

- We say that the mappings T and f *commute* when
$$f(T(x,y)) = T(f(x), f(y))$$
for all $x,y \in X$.

- An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of mappings T and f when

$$T(x, y) = f(x) \text{ and } T(y, x) = f(y).$$

In particular, if $x = f(x) = T(x, y)$ and $y = f(y) = T(y, x)$, then (x, y) is called a *common coupled fixed point* of T and f.

Let X be a universal set. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings defined on $X \times X$ into X, and let f be a mapping defined on X into itself satisfying $T_n(X, X) \subseteq f(X)$ for all $n \in \mathbb{N}$. Given any two initial elements $x_0, y_0 \in X$, since $T_n(X, X) \subseteq f(X)$, there exist $x_1, y_1 \in X$ such that

$$f(x_1) = T_1(x_0, y_0) \text{ and } f(y_1) = T_1(y_0, x_0).$$

Similarly, there also exist $x_2, y_2 \in X$ such that

$$f(x_2) = T_2(x_1, y_1) \text{ and } f(y_2) = T_2(y_1, x_1).$$

Continuing this process, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that

$$f(x_n) = T_n(x_{n-1}, y_{n-1}) \text{ and } f(y_n) = T_n(y_{n-1}, x_{n-1}) \tag{5.1}$$

for $n \in \mathbb{N}$.

In the sequel, we shall study the common coupled coincidence point by separately considering the four different types of triangle inequalities. We first present some assumptions and conditions. For the mappings f and $\{T_n\}_{n=1}^{\infty}$, we consider the following assumptions:

(I) the mappings $T_n : X \times X \to X$ and $f : X \to X$ satisfy $T_n(X, X) \subseteq f(X)$ for all $n \in \mathbb{N}$;

(II) the mappings f and T_n commute, i.e., $f(T_n(x, y)) = T_n(f(x), f(y))$ for all $x, y \in X$ and all $n \in \mathbb{N}$.

Regarding the auxiliary function Ψ that is associated with the mappings f and T_1 in the sequence $\{T_n\}_{n=1}^{\infty}$, we consider the following three conditions:

($\mathfrak{a}$) there exist $x^*, y^* \in X$ satisfying

$$\sup_{\lambda \in (0,1)} \Psi\left(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)\right) < \infty; \tag{5.2}$$

($\mathfrak{b}$) there exist $x^*, y^* \in X$ satisfying

$$\sup_{\lambda \in (0,1)} \Psi\left(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)\right) < \infty.$$

($\mathfrak{c}$) there exist $x^*, y^* \in X$ satisfying

$$\sup_{\lambda \in (0,1)} \Psi\left(\lambda, f(x^*), T_1(x^*, y^*), f(y^*), T_1(y^*, x^*)\right) < \infty$$

and

$$\sup_{\lambda \in (0,1)} \Psi\left(\lambda, T_1(x^*, y^*), f(x^*), T_1(y^*, x^*), f(y^*)\right) < \infty.$$

Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$. We consider the following assumptions.

(1) the t-norm $*$ is left-continuous with respect to the first or second component;
(2) for any fixed $x, y \in X$, the mapping $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is left-continuous at each point $t \in (0, \infty)$;
(3) for any $n \in \mathbb{N}$ and $a \in [0, 1]$, the inequality $(*a)^n \geq a^n$ holds true;
(4) for any fixed $x, y \in X$, $t > 0$ and constant k, the mapping

$$\rho(\alpha) = M\left(x, y, \frac{t}{k^{\log_2 \alpha}}\right) \tag{5.3}$$

is differentiable on $(0, \infty)$ such that $\rho'(\alpha) \neq 0$ for all $\alpha \in (0, \infty)$.

Regarding the completeness and continuities, we consider the following conditions.

(a) (X, M) is $(<, \triangleright)$-complete, and f is simultaneously $(\triangleright, \triangleright)$-continuous and $(\triangleright, \triangleleft)$-continuous with respect to M;
(a°) (X, M) is $(<, \triangleright)$-complete, and f is simultaneously $(\triangleright, \triangleright)$-continuous or $(\triangleright, \triangleleft)$-continuous with respect to M;
(b) (X, M) is $(<, \triangleleft)$-complete, and f is simultaneously $(\triangleleft, \triangleright)$-continuous and $(\triangleleft, \triangleleft)$-continuous with respect to M;
(b°) (X, M) is $(<, \triangleleft)$-complete, and f is simultaneously $(\triangleleft, \triangleright)$-continuous or $(\triangleleft, \triangleleft)$-continuous with respect to M;
(c) (X, M) is $(>, \triangleright)$-complete, and f is simultaneously $(\triangleright, \triangleright)$-continuous and $(\triangleright, \triangleleft)$-continuous with respect to M;
(c°) (X, M) is $(>, \triangleright)$-complete, and f is simultaneously $(\triangleright, \triangleright)$-continuous or $(\triangleright, \triangleleft)$-continuous with respect to M;
(d) (X, M) is $(>, \triangleleft)$-complete, and f is simultaneously $(\triangleleft, \triangleright)$-continuous and $(\triangleleft, \triangleleft)$-continuous with respect to M;
(d°) (X, M) is $(>, \triangleleft)$-complete, and f is simultaneously $(\triangleleft, \triangleright)$-continuous or $(\triangleleft, \triangleleft)$-continuous with respect to M;
(e) (X, M) is $(<, \triangleright)$-complete or $(>, \triangleright)$-complete, and f is simultaneously $(\triangleright, \triangleright)$-continuous and $(\triangleright, \triangleleft)$-continuous with respect to M;
(e°) (X, M) is $(<, \triangleright)$-complete or $(>, \triangleright)$-complete, and f is $(\triangleright, \triangleright)$-continuous or $(\triangleright, \triangleleft)$-continuous with respect to M;
(f) (X, M) is $(<, \triangleleft)$-complete or $(>, \triangleleft)$-complete, and f is simultaneously $(\triangleleft, \triangleright)$-continuous and $(\triangleleft, \triangleleft)$-continuous with respect to M.
(f°) (X, M) is $(<, \triangleleft)$-complete or $(>, \triangleleft)$-complete, and f is $(\triangleleft, \triangleright)$-continuous or $(\triangleleft, \triangleleft)$-continuous with respect to M.
(g) (X, M) is $(<, \triangleright)$-complete or $(>, \triangleright)$-complete, and f is $(\triangleright, \triangleleft)$-continuous with respect to M;
(h) (X, M) is $(<, \triangleleft)$-complete or $(>, \triangleleft)$-complete, and f is $(\triangleleft, \triangleleft)$-continuous with respect to M.
(i) (X, M) is $(<, \triangleright)$-complete or $(>, \triangleright)$-complete and f is $(\triangleright, \triangleright)$-continuous with respect to M;
(j) (X, M) is $(<, \triangleleft)$-complete or $(>, \triangleleft)$-complete and f is $(\triangleleft, \triangleright)$-continuous with respect to M.

Theorem 5.1. (**Satisfying the $\bowtie$-Triangle Inequality**). *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\bowtie$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1) and (2) are satisfied;*

- *for any $x, y, u, v \in X$, the following contractive inequality is satisfied:*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(x), f(u), t) * M(f(y), f(v), t), \tag{5.4}$$

where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant k.

Then we have the following results.

(i) *Suppose that condition* $(\mathfrak{a})$ *is satisfied, and that condition* (a) *or condition* (b) *is satisfied. Then the mappings $\{T_n\}_{n=1}^{\infty}$ and f have a common coupled coincidence point (x°, y°). We further assume that the assumptions (3) and (4) are satisfied. Then we have the following properties.*
   (A) *If $(\bar{x}, \bar{y})$ is another coupled coincidence point of f and T_{n_0} for some $n_0 \in \mathbb{N}$, then $f(x^\circ) = f(\bar{x})$ and $f(y^\circ) = f(\bar{y})$.*
   (B) *There exists $(x^\circ, y^\circ) \in X \times X$ such that $(f(x^\circ), f(y^\circ)) \in X \times X$ is the common coupled fixed point of the mappings $\{T_n\}_{n=1}^{\infty}$.*

   Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.
   - *If condition* (a) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limits $f(x_n) \xrightarrow{M^{\triangleright}} x^\circ$ and $f(y_n) \xrightarrow{M^{\triangleright}} y^\circ$;*
   - *If condition* (b) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limits $f(x_n) \xrightarrow{M^{\triangleleft}} x^\circ$ and $f(y_n) \xrightarrow{M^{\triangleleft}} y^\circ$,*

   where the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (5.1).

(ii) *Suppose that condition* $(\mathfrak{b})$ *is satisfied, and that condition* (c) *or condition* (d) *is satisfied. Then we have the same result as part* (i).

Proof . We can generate two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ from the initial element $x_0 = x^*$ and $y_0 = y^*$ according to (5.1). Then we have

$$f(x^*) = f(x_0) \text{ and } f(y^*) = f(y_0)$$

and

$$T_1(x^*, y^*) = T_1(x_0, y_0) = f(x_1) \text{ and } T_1(y^*, x^*) = T_1(y_0, x_0) = f(y_1).$$

To prove part (i), from (5.1) and (5.4), we obtain

$$\begin{aligned} M(f(x_1), f(x_2), t) &= M(T_1(x_0, y_0), T_2(x_1, y_1), t) \\ &\geq M\left(f(x_0), f(x_1), \frac{t}{k_{12}}\right) * M\left(f(y_0), f(y_1), \frac{t}{k_{12}}\right) \end{aligned}$$

and

$$\begin{aligned} M(f(y_1), f(y_2), t) &= M(T_1(y_0, x_0), T_2(y_1, x_1), t) \\ &\geq M\left(f(y_0), f(y_1), \frac{t}{k_{12}}\right) * M\left(f(x_0), f(x_1), \frac{t}{k_{12}}\right). \end{aligned}$$

By induction, we can obtain

$$\begin{aligned} M(f(x_n), f(x_{n+1}), t) \geq & \left[*M\left(f(x_0), f(x_1), \frac{t}{\prod_{i=1}^{n} k_{i,i+1}}\right)\right]^{2^{n-1}} \\ & * \left[*M\left(f(y_0), f(y_1), \frac{t}{\prod_{i=1}^{n} k_{i,i+1}}\right)\right]^{2^{n-1}} \end{aligned} \tag{5.5}$$

and

$$M\left(f(y_n), f(y_{n+1}), t\right) \geq \left[*M\left(f(x_0), f(x_1), \frac{t}{\prod_{i=1}^{n} k_{i,i+1}}\right)\right]^{2^{n-1}} * \left[*M\left(f(y_0), f(y_1), \frac{t}{\prod_{i=1}^{n} k_{i,i+1}}\right)\right]^{2^{n-1}}. \tag{5.6}$$

Since the mapping $M(x, y, \cdot)$ is nondecreasing by part (i) of Proposition 2.7 and $k_{i,i+1} \leq k$ for each $i \in \mathbb{N}$, applying the increasing property of t-norm to (5.5) and (5.6), we also have

$$\begin{aligned} M\left(f(x_n), f(x_{n+1}), t\right) &\geq \left[*M\left(f(x_0), f(x_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[*M\left(f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\ &= \left[*\eta\left(f(x_0), f(x_1), f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} M\left(f(y_n), f(y_{n+1}), t\right) &\geq \left[*M\left(f(x_0), f(x_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[*M\left(f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\ &= \left[*\eta\left(f(x_0), f(x_1), f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n-1}}. \end{aligned} \tag{5.8}$$

Applying the increasing property of t-norm to (5.7) and (5.8), we have

$$\begin{aligned} \eta\left(f(x_n), f(x_{n+1}), f(y_n), f(y_{n+1}), t\right) &= M\left(f(x_n), f(x_{n+1}), t\right) * M\left(f(y_n), f(y_{n+1}), t\right) \\ &\geq \left[*\eta\left(f(x_0), f(x_1), f(y_0), f(y_1), \frac{t}{k^n}\right)\right]^{2^{n}}. \end{aligned} \tag{5.9}$$

From part (i) of Proposition 4.1, it follows that $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are $<$-Cauchy sequences. We consider the following cases

- Suppose that condition (a) is satisfied. Since (X, M) is $(<, \triangleright)$-complete, there exist $x^\circ, y^\circ \in X$ such that
$$f(x_n) \xrightarrow{M^{\triangleright}} x^\circ \text{ and } f(y_n) \xrightarrow{M^{\triangleright}} y^\circ \text{ as } n \to \infty. \tag{5.10}$$
Since f is simultaneously $(\triangleright, \triangleright)$-continuous and $(\triangleright, \triangleleft)$-continuous with respect to M, we have
$$f(f(x_n)) \xrightarrow{M^{\triangleright}} f(x^\circ) \text{ and } f(f(y_n)) \xrightarrow{M^{\triangleright}} f(y^\circ) \text{ as } n \to \infty$$
and
$$f(f(x_n)) \xrightarrow{M^{\triangleleft}} f(x^\circ) \text{ and } f(f(y_n)) \xrightarrow{M^{\triangleleft}} f(y^\circ) \text{ as } n \to \infty,$$
which say that
$$\lim_{n\to\infty} M(f(f(x_n)), f(x^\circ), t) = 1, \text{ i.e., } M(f(f(x_n)), f(x^\circ), t) \to 1- \tag{5.11}$$
$$\lim_{n\to\infty} M(f(f(y_n)), f(y^\circ), t) = 1, \text{ i.e., } M(f(f(y_n)), f(y^\circ), t) \to 1- \tag{5.12}$$
$$\lim_{n\to\infty} M(f(x^\circ), f(f(x_n)), t) = 1, \text{ i.e., } M(f(x^\circ), f(f(x_n)), t) \to 1- \tag{5.13}$$
$$\lim_{n\to\infty} M(f(y^\circ), f(f(y_n)), t) = 1, \text{ i.e., } M(f(y^\circ), f(f(y_n)), t) \to 1- \tag{5.14}$$
for all $t > 0$.

- Suppose that condition (b) is satisfied. Since (X, M) is $(<, \triangleleft)$-complete, there exist $x^\circ, y^\circ \in X$ such that
$$f(x_n) \xrightarrow{M^\triangleleft} x^\circ \text{ and } f(y_n) \xrightarrow{M^\triangleleft} y^\circ \text{ as } n \to \infty. \tag{5.15}$$
Since f is simultaneously $(\triangleleft, \triangleright)$-continuous and $(\triangleleft, \triangleleft)$-continuous with respect to M, we can similarly obtain (5.11)-(5.14).

Using (5.1) and the commutativity of T_n and f, we obtain

$$f(f(x_{n+1})) = f(T_{n+1}(x_n, y_n)) = T_{n+1}(f(x_n), f(y_n))) \tag{5.16}$$

and

$$f(f(y_{n+1})) = f(T_{n+1}(y_n, x_n)) = T_{n+1}(f(y_n), f(x_n))).$$

We shall show that $f(x^\circ) = T_n(x^\circ, y^\circ)$ and $f(y^\circ) = T_n(y^\circ, x^\circ)$ for all $n \in \mathbb{N}$. Now we have

$$\begin{aligned} &M\left(f(f(x_{n+1})), T_n(x^\circ, y^\circ), kt\right) \geq M\left(f(f(x_{n+1})), T_n(x^\circ, y^\circ), k_{n+1,n} \cdot t\right) \\ &\quad = M\left(T_{n+1}(f(x_n), f(y_n))), T_n(x^\circ, y^\circ), k_{n+1,n} \cdot t\right) \text{ (by (5.16))} \\ &\quad \geq M(f(f(x_n)), f(x^\circ), t) * M(f(f(y_n)), f(y^\circ), t) \text{ (by (5.4)).} \end{aligned} \tag{5.17}$$

Using part (ii) of Proposition 2.2 and applying (5.11) and (5.12) to (5.17), we obtain

$$\begin{aligned} &\liminf_{n\to\infty} M\left(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t\right) \\ &\quad \geq \lim_{n\to\infty} \left[M\left(f(f(x_n)), f(x^\circ), \frac{t}{k}\right) * M\left(f(f(y_n)), f(y^\circ), \frac{t}{k}\right)\right] = 1 * 1 = 1, \end{aligned}$$

which says that

$$1 \geq \limsup_{n\to\infty} M\left(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t\right) \geq \liminf_{n\to\infty} M\left(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t\right) \geq 1.$$

Therefore we obtain

$$\lim_{n\to\infty} M\left(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t\right) = 1, \text{ i.e., } M\left(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t\right) \to 1- . \tag{5.18}$$

Using the $\bowtie$-triangle inequality, we see that

$$M\left(f(x^\circ), T_n(x^\circ, y^\circ), 2t\right) \geq M\left(f(x^\circ), f(f(x_{n+1})), t\right) * M\left(f(f(x_{n+1})), T_n(x^\circ, y^\circ), t\right).$$

Applying the left-continuity of t-norm $*$ to (5.13) and (5.18), we obtain $M(f(x^\circ), T_n(x^\circ, y^\circ), 2t) = 1$ for all $t > 0$. Therefore we must have $f(x^\circ) = T_n(x^\circ, y^\circ)$ for all $n \in \mathbb{N}$. We can similarly show that $f(y^\circ) = T_n(y^\circ, x^\circ)$ for all $n \in \mathbb{N}$.

To prove property (A), let $(\bar{x}, \bar{y})$ be another coupled coincidence point of f and T_{n_0} for some $n_0 \in \mathbb{N}$, i.e., $f(\bar{x}) = T_{n_0}(\bar{x}, \bar{y})$ and $f(\bar{y}) = T_{n_0}(\bar{y}, \bar{x})$. Since the mapping $M(x, y, \cdot)$ is nondecreasing, by (5.4), we have

$$\begin{aligned} M(f(x^\circ), f(\bar{x}), t) &= M(T_{n_0}(x^\circ, y^\circ), T_{n_0}(\bar{x}, \bar{y}), t) \\ &\geq M\left(f(x^\circ), f(\bar{x}), \frac{t}{k_{n_0,n_0}}\right) * M\left(f(y^\circ), f(\bar{y}), \frac{t}{k_{n_0,n_0}}\right) \\ &\geq M\left(f(x^\circ), f(\bar{x}), \frac{t}{k}\right) * M\left(f(y^\circ), f(\bar{y}), \frac{t}{k}\right) \end{aligned} \tag{5.19}$$

and

$$
\begin{aligned}
M(f(y^\circ), f(\bar{y}), t) &= M(T_{n_0}(y^\circ, x^\circ), T_{n_0}(\bar{y}, \bar{x}), t) \\
&\geq M\left(f(y^\circ), f(\bar{y}), \frac{t}{k_{n_0,n_0}}\right) * M\left(f(x^\circ), f(\bar{x}), \frac{t}{k_{n_0,n_0}}\right) \\
&\geq M\left(f(y^\circ), f(\bar{y}), \frac{t}{k}\right) * M\left(f(x^\circ), f(\bar{x}), \frac{t}{k}\right).
\end{aligned} \tag{5.20}
$$

Therefore we obtain

$$
\begin{aligned}
1 &\geq M(f(x^\circ), f(\bar{x}), t) \\
&\geq M\left(f(x^\circ), f(\bar{x}), \frac{t}{k}\right) * M\left(f(y^\circ), f(\bar{y}), \frac{t}{k}\right) \text{ (by (5.19))} \\
&\geq \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^2}\right) * M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^2}\right)\right] * \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^2}\right) * M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^2}\right)\right] \\
&\quad \text{(by (5.19) and (5.20))} \\
&= \left[*M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^2}\right)\right]^2 * \left[*M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^2}\right)\right]^2 \\
&\geq \cdots \geq \left[*M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[*M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\
&\quad \text{(by repeating to use (5.19) and (5.20))} \\
&\geq \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\
&\quad \text{(by the assumption } (*a)^{2^{n-1}} \geq a^{2^{n-1}} \text{ and the increasing property of t-norm)} \\
&\geq \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^n}\right)\right]^{2^n} * \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^n}\right)\right]^{2^n} \\
&\quad \text{(since } M(x,y,t) \leq 1 \text{ for any } x, y \in X \text{ and } t > 0\text{),}
\end{aligned}
$$

which is equivalent to

$$
1 \geq M(f(x^\circ), f(\bar{x}), t) \geq \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^{\log_2 n}}\right)\right]^n * \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^{\log_2 n}}\right)\right]^n. \tag{5.21}
$$

Since M satisfies the canonical condition, we have

$$
\lim_{t\to\infty} M(x, y, t) = 1 \tag{5.22}
$$

for any fixed $x, y \in X$. Let

$$
\rho(\alpha) = M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^{\log_2 \alpha}}\right).
$$

Since $0 < k < 1$, by (5.22), we have $\rho(\alpha) \to 1$ as $\alpha \to \infty$. Therefore we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^{\log_2 n}} \right) \right]^n \\
&= \lim_{n\to\infty} [\rho(n)]^n = \lim_{n\to\infty} \exp\left[n \cdot \ln \rho(n) \right] = \exp\left[\lim_{n\to\infty} \frac{n}{\frac{1}{\ln \rho(n)}} \right] \\
&= \exp\left[\lim_{n\to\infty} \left(-\frac{1}{\frac{1}{(\ln \rho(n))^2} \cdot \frac{1}{\rho(n)} \cdot \rho'(n)} \right) \right] \\
&\quad \text{(using the assumption } \rho'(\alpha) \neq 0 \text{ and the l'Hospital's rule)} \\
&= 1 \qquad (5.23)
\end{aligned}$$

and

$$\lim_{n\to\infty} \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^{\log_2 n}} \right) \right]^n = 1. \tag{5.24}$$

Since

$$M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^{\log_2 n}} \right) \leq 1 \text{ and } M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^{\log_2 n}} \right) \leq 1,$$

from (5.23) and (5.24), it follows that the sequences converge to 1 from the left. Using the left-continuity of t-norm at 1 and part (ii) of Proposition 2.2, we obtain

$$\lim_{n\to\infty} \left[M\left(f(x^\circ), f(\bar{x}), \frac{t}{k^{\log_2 n}} \right) \right]^n * \left[M\left(f(y^\circ), f(\bar{y}), \frac{t}{k^{\log_2 n}} \right) \right]^n = 1 * 1 = 1.$$

From (5.21), we obtain $M(f(x^\circ), f(\bar{x}), t) = 1$ for all $t > 0$, which implies $f(x^\circ) = f(\bar{x})$. We can similarly show that $M(f(y^\circ), f(\bar{y}), t) = 1$ for all $t > 0$, which also implies $f(y^\circ) = f(\bar{y})$.

To prove property (B), using the commutativity of T_n and f, we have

$$f(T_n(x^\circ, y^\circ)) = T_n(f(x^\circ), f(y^\circ)) = T_n(T_n(x^\circ, y^\circ), T_n(y^\circ, x^\circ)) \tag{5.25}$$

and

$$f(T_n(y^\circ, x^\circ)) = T_n(f(y^\circ), f(x^\circ)) = T_n(T_n(y^\circ, x^\circ), T_n(x^\circ, y^\circ)). \tag{5.26}$$

By regarding $\bar{x}$ as $T_n(x^\circ, y^\circ)$ and $\bar{y}$ as $T_n(y^\circ, x^\circ)$, the equalities (5.25) and (5.26) say that

$$f(\bar{x}) = T_n(\bar{x}, \bar{y}) \text{ and } f(\bar{y}) = T_n(\bar{y}, \bar{x}).$$

Therefore, using property (A), we must have

$$f\left(x^\circ\right) = f(\bar{x}) = f\left(T_n(x^\circ, y^\circ)\right) = T_n\left(f(x^\circ), f(y^\circ)\right)$$

and

$$f\left(y^\circ\right) = f(\bar{y}) = f\left(T_n(y^\circ, x^\circ)\right) = T_n\left(f(y^\circ), f(x^\circ)\right),$$

which says that $(f(x^\circ), f(y^\circ)) \in X \times X$ is the common coupled fixed point of the mappings $\{T_n\}_{n=1}^\infty$.

To prove part (ii), we can similarly obtain

$$\eta\left(f(x_{n+1}), f(x_n), f(y_{n+1}), f(y_n), t\right) \geq \left[*\eta\left(f(x_1), f(x_0), f(y_1), f(y_0), \frac{t}{k^n} \right) \right]^{2^n}. \tag{5.27}$$

From part (i) of Proposition 4.1, it follows that $\{f(x_n)\}_{n=1}^\infty$ and $\{f(y_n)\}_{n=1}^\infty$ are $>$-Cauchy sequences. We consider the following cases

- Suppose that condition (c) is satisfied. Since (X, M) is $(>, \triangleright)$-complete, there exist $x^\circ, y^\circ \in X$ such that
$$f(x_n) \stackrel{M^\triangleright}{\longrightarrow} x^\circ \text{ and } f(y_n) \stackrel{M^\triangleright}{\longrightarrow} y^\circ \text{ as } n \to \infty.$$
Since f is simultaneously $(\triangleright, \triangleright)$-continuous and $(\triangleright, \triangleleft)$-continuous with respect to M, we can similarly obtain (5.11)-(5.14).

- Suppose that condition (d) is satisfied. Since (X, M) is $(>, \triangleleft)$-complete, there exist $x^\circ, y^\circ \in X$ such that
$$f(x_n) \stackrel{M^\triangleleft}{\longrightarrow} x^\circ \text{ and } f(y_n) \stackrel{M^\triangleleft}{\longrightarrow} y^\circ \text{ as } n \to \infty.$$
Since f is simultaneously $(\triangleleft, \triangleright)$-continuous and $(\triangleleft, \triangleleft)$-continuous with respect to M, we can similarly obtain (5.11)-(5.14).

The remaining proof follows from the similar argument of part (i). This completes the proof. □

Remark 5.2. The inequality (5.4) is based on the t-norm. If the t-norm in (5.4) is replaced by the arithmetic product as follows

$$M(T(x, y), T(u, v), kt) \geq M(f(x), f(u), t) \cdot M(f(y), f(v), t),$$

then we can simplify the sufficient conditions. We omit the detailed discussion.

In Theorem 5.1, since the fuzzy semi-metric M is not necessarily symmetric, if the contractive inequality (5.4) is not satisfied and alternatively the following so-called *converse-contractive inequality*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(u), f(x), t) * M(f(v), f(y), t)$$

is satisfied, then we can also obtain the desired results by assuming the different conditions.

Theorem 5.3. (**Satisfying the $\bowtie$-Triangle Inequality: Converse-Contractive Inequality**). *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\bowtie$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1) and (2) are satisfied;*

- *for any $x, y, u, v \in X$, the following converse-contractive inequality is satisfied:*
$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(u), f(x), t) * M(f(v), f(y), t),$$
where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant k.

Then we have the following results.

(i) *Suppose that condition $(\mathfrak{a})$ is satisfied, and that condition (a°) or condition (b°) is satisfied. Then the mappings $\{T_n\}_{n=1}^\infty$ and f have a common coupled coincidence point (x°, y°). We further assume that the assumptions (3) and (4) are satisfied. Then we have the following properties.*
   (A) *If $(\bar{x}, \bar{y})$ is another coupled coincidence point of f and T_{n_0} for some $n_0 \in \mathbb{N}$, then $f(x^\circ) = f(\bar{x})$ and $f(y^\circ) = f(\bar{y})$.*
   (B) *There exists $(x^\circ, y^\circ) \in X \times X$ such that $(f(x^\circ), f(y^\circ)) \in X \times X$ is the common coupled fixed point of the mappings $\{T_n\}_{n=1}^\infty$.*

Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.

- *If condition* (a°) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleright} y^\circ$;*
- *If condition* (b°) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$,*

where the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (5.1).

(ii) *Suppose that condition* ($\mathfrak{b}$) *is satisfied, and that condition* (c°) *or condition* (d°) *is satisfied. Then we have the same result as part* (i).

Remark 5.4. The assumption for the continuity of function f in Theorems 5.1 and 5.3 are different. More precisely, the assumption for the continuity of function f in Theorem 5.1 is stronger than that of Theorem 5.3.

Theorem 5.5. (**Satisfying the $\diamond$-Triangle Inequality**). *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\diamond$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1) and (2) are satisfied;*
- *for any $x, y, u, v \in X$, the following contractive inequality is satisfied:*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(x), f(u), t) * M(f(y), f(v), t),$$

where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant k.

- *condition* ($\mathfrak{c}$) *is satisfied, and that condition* (e°) *or condition* (f°) *is satisfied.*

Then the mappings $\{T_n\}_{n=1}^\infty$ and f have a common coupled coincidence point (x°, y°). We further assume that the assumptions (3) and (4) are satisfied. Then we have the following properties.

(A) *If $(\bar{x}, \bar{y})$ is another coupled coincidence point of f and T_{n_0} for some $n_0 \in \mathbb{N}$, then $f(x^\circ) = f(\bar{x})$ and $f(y^\circ) = f(\bar{y})$.*

(B) *There exists $(x^\circ, y^\circ) \in X \times X$ such that $(f(x^\circ), f(y^\circ)) \in X \times X$ is the common coupled fixed point of the mappings $\{T_n\}_{n=1}^\infty$.*

Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.

- *If condition* (e°) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleright} y^\circ$;*
- *If condition* (f°) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$,*

where the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (5.1).

Theorem 5.6. (**Satisfying the $\diamond$-Triangle Inequality: Converse-Contractive Inequality**). *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\diamond$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1) and (2) are satisfied;*
- *for any $x, y, u, v \in X$, the following converse-contractive inequality is satisfied:*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(u), f(x), t) * M(f(v), f(y), t),$$

where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant k.

- *condition* ($\mathfrak{c}$) *is satisfied, and that condition* (e) *or condition* (f) *is satisfied.*

Then the mappings $\{T_n\}_{n=1}^{\infty}$ and f have a common coupled coincidence point (x°, y°). We further assume that the assumptions (3) and (4) are satisfied. Then we have the following properties.

(A) *If $(\bar{x}, \bar{y})$ is another coupled coincidence point of f and T_{n_0} for some $n_0 \in \mathbb{N}$, then $f(x^\circ) = f(\bar{x})$ and $f(y^\circ) = f(\bar{y})$.*

(B) *There exists $(x^\circ, y^\circ) \in X \times X$ such that $(f(x^\circ), f(y^\circ)) \in X \times X$ is the common coupled fixed point of the mappings $\{T_n\}_{n=1}^{\infty}$.*

Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.

- *If condition* (e) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleright} y^\circ$;*
- *If condition* (f) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$,*

where the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (5.1).

Theorem 5.7. (**Satisfying the $\triangleright$-Triangle Inequality**). *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\triangleright$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1) and (2) are satisfied;*
- *the following contractive inequalities is satisfied*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(x), f(u), t) * M(f(y), f(v), t)$$

or the following converse-contractive inequalities is satisfied

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(u), f(x), t) * M(f(v), f(y), t),$$

where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant k.

- *condition* ($\mathfrak{c}$) *is satisfied, and that condition* (g) *or condition* (h) *is satisfied.*

Then the mappings $\{T_n\}_{n=1}^{\infty}$ and f have a common coupled coincidence point (x°, y°). We further assume that the assumptions (3) and (4) are satisfied. Then we have the following properties.

(A) *If $(\bar{x}, \bar{y})$ is another coupled coincidence point of f and T_{n_0} for some $n_0 \in \mathbb{N}$, then $f(x^\circ) = f(\bar{x})$ and $f(y^\circ) = f(\bar{y})$.*

(B) *There exists* $(x^\circ, y^\circ) \in X \times X$ *such that* $(f(x^\circ), f(y^\circ)) \in X \times X$ *is the common coupled fixed point of the mappings* $\{T_n\}_{n=1}^\infty$.

Moreover, the point $(x^\circ, y^\circ) \in X \times X$ *can be obtained as follows.*

- *If condition* (g) *is satisfied, then the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleright} y^\circ$*;*
- *If condition* (h) *is satisfied, then the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$*,*

where the sequences $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *are generated from the initial element* $(x_0, y_0) = (x^*, y^*) \in X \times X$ *according to* (5.1).

Theorem 5.8. (**Satisfying the** $\triangleleft$**-Triangle Inequality**). *Let* (X, M) *be a fuzzy semi-metric space along with a t-norm* $*$ *such that* M *satisfies the canonical condition and the* $\triangleleft$*-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1) and (2) are satisfied;*
- *the following contractive inequalities is satisfied*

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(x), f(u), t) * M(f(y), f(v), t)$$

  or the following converse-contractive inequalities is satisfied

$$M(T_i(x, y), T_j(u, v), k_{ij} \cdot t) \geq M(f(u), f(x), t) * M(f(v), f(y), t),$$

  where k_{ij} *satisfies* $0 < k_{ij} \leq k < 1$ *for all* $i, j \in \mathbb{N}$ *and for some constant* k.
- *condition* ($\mathfrak{c}$) *is satisfied, and that condition* (i) *or condition* (j) *is satisfied.*

Then the mappings $\{T_n\}_{n=1}^\infty$ *and* f *have a common coupled coincidence point* (x°, y°)*. We further assume that the assumptions (3) and (4) are satisfied. Then we have the following properties.*

(A) *If* $(\bar{x}, \bar{y})$ *is another coupled coincidence point of* f *and* T_{n_0} *for some* $n_0 \in \mathbb{N}$*, then* $f(x^\circ) = f(\bar{x})$ *and* $f(y^\circ) = f(\bar{y})$.

(B) *There exists* $(x^\circ, y^\circ) \in X \times X$ *such that* $(f(x^\circ), f(y^\circ)) \in X \times X$ *is the common coupled fixed point of the mappings* $\{T_n\}_{n=1}^\infty$.

Moreover, the point $(x^\circ, y^\circ) \in X \times X$ *can be obtained as follows.*

- *If condition* (i) *is satisfied, then the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleright} y^\circ$*;*
- *If condition* (j) *is satisfied, then the point* $(x^\circ, y^\circ) \in X \times X$ *can be obtained by taking the limit* $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ *and* $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$*,*

where the sequences $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *are generated from the initial element* $(x_0, y_0) = (x^*, y^*) \in X \times X$ *according to* (5.1).

Remark 5.9. We remark that the only difference between Theorems 5.7 and 5.8 is the continuity of function f.

6. Common Coupled Fixed Points

Let X be a nonempty universal set. Recall that an element $(x,y) \in X \times X$ is called a common coupled fixed point of mappings $T : X \times X \to X$ and $f : X \to X$ when

$$x = f(x) = T(x,y) \text{ and } y = f(y) = T(y,x).$$

It is clear that the common coupled fixed points are also the common coupled coincidence points. Since the uniqueness of common coupled coincidence points was not guaranteed, in this section, we shall provide the different arguments to prove the uniqueness of common coupled fixed points.

Theorem 6.1. (**Satisfying the $\bowtie$-Triangle Inequality**). *Let (X,M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\bowtie$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1), (2), (3) and (4) are satisfied;*
- *for any $x,y,u,v \in X$, the following contractive inequality is satisfied:*
$$M(T_i(x,y),T_j(u,v),k_{ij} \cdot t) \geq M(f(x),f(u),t) * M(f(y),f(v),t), \tag{6.1}$$
where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i,j \in \mathbb{N}$ and for some constant k.

(i) *Suppose that condition ($\mathfrak{a}$) is satisfied, and that condition* (a) *or condition* (b) *is satisfied. Then the mappings $\{T_n\}_{n=1}^{\infty}$ and f have a unique common coupled fixed point (x°,y°). Moreover, the point $(x^\circ,y^\circ) \in X \times X$ can be obtained as follows.*
  - *If condition* (a) *is satisfied, then the point $(x^\circ,y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleright} y^\circ$;*
  - *If condition* (b) *is satisfied, then the point $(x^\circ,y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$,*

  where the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0,y_0) = (x^*,y^*) \in X \times X$ according to (5.1).

(ii) *Suppose that condition ($\mathfrak{b}$) is satisfied, and that condition* (c) *or condition* (d) *is satisfied. Then we have the same result as part* (i).

Proof . According to (5.1), we can generate two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ from the initial element $x_0 = x^*$ and $y_0 = y^*$. Then we have

$$f(x^*) = f(x_0) \text{ and } f(y^*) = f(y_0)$$

and

$$T_1(x^*,y^*) = T_1(x_0,y_0) = f(x_1) \text{ and } T_1(y^*,x^*) = T_1(y_0,x_0) = f(y_1).$$

To prove part (i), from part (i) of Theorem 5.1, we have $f(x^\circ) = T_n(x^\circ,y^\circ)$ and $f(y^\circ) = T_n(y^\circ,x^\circ)$ for all $n \in \mathbb{N}$. We shall show that x° is a fixed point of f. Using (5.1), (6.1) and the nondecreasing property of $M(x,y,\cdot)$ by part (i) of Proposition 2.7, we have

$$\begin{aligned} M\left(f(x_{n+1}),f(x^\circ),t\right) &= M\left(T_{n+1}(x_n,y_n),T_n(x^\circ,y^\circ),t\right) \\ &\geq M\left(f(x_n),f(x^\circ),\frac{t}{k_{n+1,n}}\right) * M\left(f(y_n),f(y^\circ),\frac{t}{k_{n+1,n}}\right) \\ &\geq M\left(f(x_n),f(x^\circ),\frac{t}{k}\right) * M\left(f(y_n),f(y^\circ),\frac{t}{k}\right) \end{aligned} \tag{6.2}$$

and

$$\begin{aligned}
M\left(f(y_{n+1}), f(y^\circ), t\right) &= M\left(T_{n+1}(y_n, x_n), T_n(y^\circ, x^\circ), t\right) \\
&\geq M\left(f(y_n), f(y^\circ), \frac{t}{k_{n+1,n}}\right) * M\left(f(x_n), f(x^\circ), \frac{t}{k_{n+1,n}}\right) \\
&\geq M\left(f(y_n), f(y^\circ), \frac{t}{k}\right) * M\left(f(x_n), f(x^\circ), \frac{t}{k}\right). \qquad (6.3)
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
1 &\geq M(f(x_{n+1}), f(x^\circ), t) \\
&\geq M\left(f(x_n), f(x^\circ), \frac{t}{k}\right) * M\left(f(y_n), f(y^\circ), \frac{t}{k}\right) \text{ (by (6.2))} \\
&\geq \left[M\left(f(x_{n-1}), f(x^\circ), \frac{t}{k^2}\right) * M\left(f(y_{n-1}), f(y^\circ), \frac{t}{k^2}\right)\right] \\
&\quad * \left[M\left(f(x_{n-1}), f(x^\circ), \frac{t}{k^2}\right) * M\left(f(y_{n-1}), f(y^\circ), \frac{t}{k^2}\right)\right] \text{ (by (6.2) and (6.3))} \\
&= \left[*M\left(f(x_{n-1}), f(x^\circ), \frac{t}{k^2}\right)\right]^2 * \left[*M\left(f(y_{n-1}), f(y^\circ), \frac{t}{k^2}\right)\right]^2 \\
&\geq \cdots \geq \left[*M\left(f(x_1), f(x^\circ), \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[*M\left(f(y_1), f(y^\circ), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\
&\quad \text{(by repeating to use (6.2) and (6.3))} \\
&\geq \left[M\left(f(x_1), f(x^\circ), \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[M\left(f(y_1), f(y^\circ), \frac{t}{k^n}\right)\right]^{2^{n-1}} \\
&\quad \text{(by the assumption } (*a)^{2^{n-1}} \geq a^{2^{n-1}} \text{ and the increasing property of t-norm)} \\
&\geq \left[M\left(f(x_1), f(x^\circ), \frac{t}{k^n}\right)\right]^{2^n} * \left[M\left(f(y_1), f(y^\circ), \frac{t}{k^n}\right)\right]^{2^n} \qquad (6.4) \\
&\quad \text{(since } M(x, y, t) \leq 1 \text{ for any } x, y \in X \text{ and } t > 0\text{)},
\end{aligned}$$

which is equivalent to

$$1 \geq M(f(x_{n+1}), f(x^\circ), t) \geq \left[M\left(f(x_1), f(x^\circ), \frac{t}{k^{\log_2 n}}\right)\right]^n * \left[M\left(f(y_1), f(y^\circ), \frac{t}{k^{\log_2 n}}\right)\right]^n. \qquad (6.5)$$

Since M satisfies the canonical condition, we have

$$\lim_{t\to\infty} M(x, y, t) = 1 \qquad (6.6)$$

for any fixed $x, y \in X$. Let

$$\rho(\alpha) = M\left(f(x_1), f(x^\circ), \frac{t}{k^{\log_2 \alpha}}\right).$$

Since $0 < k < 1$, by (6.6), we have $\rho(\alpha) \to 1$ as $\alpha \to \infty$. Therefore we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left[M\left(f(x_1), f(x^\circ), \frac{t}{k^{\log_2 n}}\right)\right]^n \\
&= \lim_{n\to\infty}[\rho(n)]^n = \lim_{n\to\infty}\exp\left[n\cdot\ln\rho(n)\right] = \exp\left[\lim_{n\to\infty}\frac{n}{\frac{1}{\ln\rho(n)}}\right] \\
&= \exp\left[\lim_{n\to\infty}\left(-\frac{1}{\frac{1}{(\ln\rho(n))^2}\cdot\frac{1}{\rho(n)}\cdot\rho'(n)}\right)\right] \\
&\quad \text{(using the assumption } \rho'(\alpha)\neq 0 \text{ and the l'Hospital's rule)} \\
&= 1 \qquad (6.7)
\end{aligned}$$

and

$$\lim_{n\to\infty}\left[M\left(f(y_1), f(y^\circ), \frac{t}{k^{\log_2 n}}\right)\right]^n = 1. \qquad (6.8)$$

Since

$$M\left(f(x_1), f(x^\circ), \frac{t}{k^{\log_2 n}}\right) \leq 1 \text{ and } M\left(f(y_1), f(y^\circ), \frac{t}{k^{\log_2 n}}\right) \leq 1,$$

from (6.7) and (6.8), it follows that the sequences converge to 1 from the left. Using the left-continuity of t-norm at 1 and part (ii) of Proposition 2.2, we obtain

$$\lim_{n\to\infty}\left[M\left(f(x_1), f(x^\circ), \frac{t}{k^{\log_2 n}}\right)\right]^n * \left[M\left(f(y_1), f(y^\circ), \frac{t}{k^{\log_2 n}}\right)\right]^n = 1 * 1 = 1.$$

From (6.5), we obtain

$$\lim_{n\to\infty} M\left(f(x_{n+1}), f(x^\circ), t\right) = 1. \qquad (6.9)$$

We can similarly obtain

$$\lim_{n\to\infty} M\left(f(x^\circ), f(x_{n+1}), t\right) = 1. \qquad (6.10)$$

We consider the following cases.

- Suppose that condition (a) is satisfied. From (5.10), since $f(x_n) \xrightarrow{M^\triangleright} x^\circ$, i.e.,

  $$\lim_{n\to\infty} M\left(f(x_{n+1}), x^\circ, t\right) = 1, \qquad (6.11)$$

  the $\bowtie$-triangle inequality says that

  $$M\left(f(x^\circ), x^\circ, 2t\right) \geq M\left(f(x^\circ), f(x_{n+1}), t\right) * M\left(f(x_{n+1}), x^\circ, t\right).$$

  Using the left-continuity of t-norm $*$ and (6.10) and (6.11), we obtain $M\left(f(x^\circ), x^\circ, 2t\right) = 1$ for all $t > 0$ by taking $n \to \infty$.

- Suppose that condition (b) is satisfied. From (5.15), since $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$, i.e.,

  $$\lim_{n\to\infty} M\left(x^\circ, f(x_{n+1}), t\right) = 1. \qquad (6.12)$$

  The triangle inequality says that

  $$M\left(x^\circ, f(x^\circ), 2t\right) \geq M\left(x^\circ, f(x_{n+1}), t\right) * M\left(f(x_{n+1}), f(x^\circ), t\right).$$

  Using the left-continuity of t-norm $*$ and (6.9) and (6.12), we obtain $M\left(x^\circ, f(x^\circ), 2t\right) = 1$ for all $t > 0$ by taking $n \to \infty$.

The above two cases show that

$$x^\circ = f(x^\circ) = T_n(x^\circ, y^\circ)$$

for all $n \in \mathbb{N}$. We can similarly obtain

$$y^\circ = f(y^\circ) = T_n(y^\circ, x^\circ)$$

for all $n \in \mathbb{N}$.

To prove the uniqueness, let $(\bar{x}, \bar{y})$ be another common coupled fixed point of f and $\{T_n\}_{n=1}^\infty$, i.e., $\bar{x} = f(\bar{x}) = T_n(\bar{x}, \bar{y})$ and $\bar{y} = f(\bar{y}) = T_n(\bar{y}, \bar{x})$ for all $n \in \mathbb{N}$. Then, using the nondecreasing property of $M(x, y, \cdot)$, we have

$$\begin{aligned} M(x^\circ, \bar{x}, t) &= M(T_n(x^\circ, y^\circ), T_n(\bar{x}, \bar{y}), t) \\ &\geq M\left(f(x^\circ), f(\bar{x}), \frac{t}{k_{nn}}\right) * M\left(f(y^\circ), f(\bar{y}), \frac{t}{k_{nn}}\right) \text{ (by (6.1))} \\ &\geq M\left(f(x^\circ), f(\bar{x}), \frac{t}{k}\right) * M\left(f(y^\circ), f(\bar{y}), \frac{t}{k}\right) = M\left(x^\circ, \bar{x}, \frac{t}{k}\right) * M\left(y^\circ, \bar{y}, \frac{t}{k}\right) \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} M(y^\circ, \bar{y}, t) &= M(T_n(y^\circ, x^\circ), T_n(\bar{y}, \bar{x}), t) \\ &\geq M\left(f(y^\circ), f(\bar{y}), \frac{t}{k_{nn}}\right) * M\left(f(x^\circ), f(\bar{x}), \frac{t}{k_{nn}}\right) \\ &\geq M\left(f(y^\circ), f(\bar{y}), \frac{t}{k}\right) * M\left(f(x^\circ), f(\bar{x}), \frac{t}{k}\right) = M\left(y^\circ, \bar{y}, \frac{t}{k}\right) * M\left(x^\circ, \bar{x}, \frac{t}{k}\right). \end{aligned} \tag{6.14}$$

Therefore we obtain

$$\begin{aligned} 1 &\geq M(x^\circ, \bar{x}, t) \\ &\geq M\left(x^\circ, \bar{x}, \frac{t}{k}\right) * M\left(y^\circ, \bar{y}, \frac{t}{k}\right) \text{ (by (6.13))} \\ &\geq \left[M\left(x^\circ, \bar{x}, \frac{t}{k^2}\right) * M\left(y^\circ, \bar{y}, \frac{t}{k^2}\right)\right] * \left[M\left(x^\circ, \bar{x}, \frac{t}{k^2}\right) * M\left(y^\circ, \bar{y}, \frac{t}{k^2}\right)\right] \text{ (by (6.13) and (6.14))} \\ &= \left[*M\left(x^\circ, \bar{x}, \frac{t}{k^2}\right)\right]^2 * \left[*M\left(y^\circ, \bar{y}, \frac{t}{k^2}\right)\right]^2 \\ &\geq \cdots \geq \left[*M\left(x^\circ, \bar{x}, \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[*M\left(y^\circ, \bar{y}, \frac{t}{k^n}\right)\right]^{2^{n-1}} \text{ (by repeating to use (6.13) and (6.14))} \\ &\geq \left[M\left(x^\circ, \bar{x}, \frac{t}{k^n}\right)\right]^{2^{n-1}} * \left[M\left(y^\circ, \bar{y}, \frac{t}{k^n}\right)\right]^{2^{n-1}} \\ &\quad \text{(by the assumption } (*a)^{2^{n-1}} \geq a^{2^{n-1}} \text{ and the increasing property of t-norm)} \\ &\geq \left[M\left(x^\circ, \bar{x}, \frac{t}{k^n}\right)\right]^{2^n} * \left[M\left(y^\circ, \bar{y}, \frac{t}{k^n}\right)\right]^{2^n} \text{ (since } M(x,y,t) \leq 1 \text{ for any } x, y \in X \text{ and } t > 0\text{)}, \end{aligned} \tag{6.15}$$

which is equivalent to

$$1 \geq M(x^\circ, \bar{x}, t) \geq \left[M\left(x^\circ, \bar{x}, \frac{t}{k^{\log_2 n}}\right)\right]^n * \left[M\left(y^\circ, \bar{y}, \frac{t}{k^{\log_2 n}}\right)\right]^n. \tag{6.16}$$

We can similarly obtain

$$\lim_{n\to\infty}\left[M\left(x^\circ,\bar{x},\frac{t}{k^{\log_2 n}}\right)\right]^n = 1 = \left[M\left(y^\circ,\bar{y},\frac{t}{k^{\log_2 n}}\right)\right]^n.$$

Therefore, from (6.16), we have $M(x^\circ,\bar{x},t)=1$ for all $t>0$, which implies $x^\circ=\bar{x}$. We can similarly show that $M(y^\circ,\bar{y},t)=1$ for all $t>0$, which also implies $y^\circ=\bar{y}$. This proves the uniqueness. Finally, part (ii) can be obtained by applying part (ii) of Theorem 5.1 to the above argument. This completes the proof. □

Remark 6.2. The inequality (6.1) is based on the t-norm. If the t-norm in (6.1) is replaced by the arithmetic product as follows

$$M(T(x,y),T(u,v),kt)\geq M(f(x),f(u),t)\cdot M(f(y),f(v),t),$$

then we can simplify the sufficient conditions. We omit the detailed discussion.

Theorem 6.3. (**Satisfying the $\bowtie$-Triangle Inequality: Converse-Contractive Inequality**). *Let (X,M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\bowtie$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1), (2), (3) and (4) are satisfied;*
- *for any $x,y,u,v\in X$, the following converse-contractive inequality is satisfied:*

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t)\geq M(f(u),f(x),t)*M(f(v),f(y),t),$$

where k_{ij} satisfies $0<k_{ij}\leq k<1$ for all $i,j\in\mathbb{N}$ and for some constant k.

(i) *Suppose that condition $(\mathfrak{a})$ is satisfied, and that condition* (a°) *or condition* (b°) *is satisfied. Then the mappings $\{T_n\}_{n=1}^\infty$ and f have a unique common coupled fixed point (x°,y°). Moreover, the point $(x^\circ,y^\circ)\in X\times X$ can be obtained as follows.*
   - *If condition* (a°) *is satisfied, then the point $(x^\circ,y^\circ)\in X\times X$ can be obtained by taking the limit $f(x_n)\xrightarrow{M^\triangleright}x^\circ$ and $f(y_n)\xrightarrow{M^\triangleright}y^\circ$;*
   - *If condition* (b°) *is satisfied, then the point $(x^\circ,y^\circ)\in X\times X$ can be obtained by taking the limit $f(x_n)\xrightarrow{M^\triangleleft}x^\circ$ and $f(y_n)\xrightarrow{M^\triangleleft}y^\circ$,*

   where the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are generated from the initial element $(x_0,y_0)=(x^*,y^*)\in X\times X$ according to (5.1).

(ii) *Suppose that condition $(\mathfrak{b})$ is satisfied, and that condition* (c°) *or condition* (d°) *is satisfied. Then we have the same result as part* (i).

Theorem 6.4. (**Satisfying the $\triangleright$-Triangle Inequality**). *Let (X,M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\triangleright$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1), (2), (3) and (4) are satisfied;*

- *the following contractive inequalities is satisfied*

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t) \geq M(f(x),f(u),t) * M(f(y),f(v),t)$$

or the following converse-contractive inequalities is satisfied

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t) \geq M(f(u),f(x),t) * M(f(v),f(y),t),$$

where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i,j \in \mathbb{N}$ and for some constant k.

- *Suppose that condition* ($\mathfrak{c}$) *is satisfied, and that condition* (g) *or condition* (h) *is satisfied.*

Then the mappings $\{T_n\}_{n=1}^{\infty}$ and f have a unique common coupled fixed point (x°, y°). Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.

- *If condition* (g) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^{\triangleright}} x^\circ$ and $f(y_n) \xrightarrow{M^{\triangleright}} y^\circ$;*
- *If condition* (h) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^{\triangleleft}} x^\circ$ and $f(y_n) \xrightarrow{M^{\triangleleft}} y^\circ$,*

where the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (5.1).

Theorem 6.5. (**Satisfying the $\triangleleft$-Triangle Inequality**). *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\triangleleft$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1), (2), (3) and (4) are satisfied;*
- *the following contractive inequalities is satisfied*

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t) \geq M(f(x),f(u),t) * M(f(y),f(v),t)$$

or the following converse-contractive inequalities is satisfied

$$M(T_i(x,y),T_j(u,v),k_{ij}\cdot t) \geq M(f(u),f(x),t) * M(f(v),f(y),t),$$

where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i,j \in \mathbb{N}$ and for some constant k.

- *Suppose that condition* ($\mathfrak{c}$) *is satisfied, and that condition* (i) *or condition* (j) *is satisfied.*

Then the mappings $\{T_n\}_{n=1}^{\infty}$ and f have a unique common coupled fixed point (x°, y°). Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.

- *If condition* (i) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^{\triangleright}} x^\circ$ and $f(y_n) \xrightarrow{M^{\triangleright}} y^\circ$;*
- *If condition* (j) *is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^{\triangleleft}} x^\circ$ and $f(y_n) \xrightarrow{M^{\triangleleft}} y^\circ$,*

where the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (5.1).

Theorem 6.6. (**Satisfying the $\diamond$-Triangle Inequality**). *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\diamond$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1), (2), (3) and (4) are satisfied;*
- *for any $x, y, u, v \in X$, the following contractive inequality is satisfied:*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(x), f(u), t) * M(f(y), f(v), t),$$

  where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant k.
- *Suppose that condition ($\mathfrak{c}$) is satisfied, and that condition (e$^\circ$) or condition (f$^\circ$) is satisfied.*

Then the mappings T and f have a unique common coupled fixed point (x°, y°). Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.

- *If condition (e$^\circ$) is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleright} y^\circ$;*
- *If condition (f$^\circ$) is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$,*

where the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (5.1).

Theorem 6.7. (**Satisfying the $\diamond$-Triangle Inequality: Converse-Contractive Inequality**). *Let (X, M) be a fuzzy semi-metric space along with a t-norm $*$ such that M satisfies the canonical condition and the $\diamond$-triangle inequality. Suppose that the following conditions are satisfied:*

- *the assumptions (I),(II),(1), (2), (3) and (4) are satisfied;*
- *the following converse-contractive inequalities is satisfied*

$$M(T_i(x,y), T_j(u,v), k_{ij} \cdot t) \geq M(f(u), f(x), t) * M(f(v), f(y), t),$$

  where k_{ij} satisfies $0 < k_{ij} \leq k < 1$ for all $i, j \in \mathbb{N}$ and for some constant k;
- *Suppose that condition ($\mathfrak{c}$) is satisfied, and that condition (e) or condition (f) is satisfied.*

Then the mappings T and f have a unique common coupled fixed point (x°, y°). Moreover, the point $(x^\circ, y^\circ) \in X \times X$ can be obtained as follows.

- *If condition (e) is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleright} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleright} y^\circ$;*
- *If condition (f) is satisfied, then the point $(x^\circ, y^\circ) \in X \times X$ can be obtained by taking the limit $f(x_n) \xrightarrow{M^\triangleleft} x^\circ$ and $f(y_n) \xrightarrow{M^\triangleleft} y^\circ$,*

where the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are generated from the initial element $(x_0, y_0) = (x^*, y^*) \in X \times X$ according to (5.1).

References

[1] A. Aghajani and K. Nourouzi, *Convex sets in probabilistic normed spaces*, Chaos, Sol. Frac. 36 (2008) 322–328.

[2] S.S. Chang, Y.J. Cho and S.M. Kang, *Nonlinear Operator Theory in Probabilistic Metric Space*, Nova Science Publishers, New York, 2001.

[3] Y.J. Cho, S. Sedghi and N. Shobe, *Generalized fixed point theorems for compatible mappings with some types in fuzzy metric spaces*, Chaos, Sol. Frac. (2009) 2233–2244

[4] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst. 64 (1994) 395–399.

[5] A. George and P. Veeramani, *On some results ofanalysis for fuzzy metric spaces*, Fuzzy Sets Syst. 90 (1997) 365–368.

[6] V. Gregori and A. Sapena, *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets Syst. 125 (2002) 245–252.

[7] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets Syst. 27 (1988) 385–389.

[8] O. Hadžić and E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Klumer Academic Publishers, 2001.

[9] X.-Q. Hu, M.-X. Zheng, B. Damjanovi'c and X.-F. Shao, *Common coupled fixed point theorems for weakly compatible mappings in fuzzy metric spaces*, Fixed Point Theory Appl. 2013, 2013:220.

[10] O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets Syst. 12 (1984) 215–229.

[11] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika, 11 (1975) 336–344.

[12] S.A. Mohiuddine and A. Alotaibi, *Coupled coincidence point theorems for cCompatible mappings in partially ordered intuitionistic generalized fuzzy metric spaces*, Fixed Point Theory Appl. 2013, 2013:220.

[13] Z. Qiu and S. Hong, *Coupled fixed points for multivalued mappings in fuzzy metric spaces*, Fixed Point Theory and Applications 2013, 2013:265.

[14] A.P. Robertson and W. Robertson, *Topological Vector Spaces*, Cambrisge University Press, 1964.

[15] B. Schweizer and A. Sklar, *Statistical Metric Spaces*, Pacific J. Math. 10 (1960) 313–334.

[16] B. Schweizer, A. Sklar and E. Thorp, *The metrization of statistical metric spaces*, Pacific J. Math. 10 (1960) 673–675.

[17] B. Schweizer and A. Sklar, *Triangle inequalities in a class of statistical metric spaces*, J. London Math. Soc. 38 (1963) 401–406.

[18] V.M. Sehgal and A.T. Bharucha-Reid, *Fixed points of contraction mappings on probabilistic metric spaces*, Math. Syst. Theory 6 (1972) 97–102.

[19] H.-C. Wu, *Hausdorff topology induced by the fuzzy metric and the fixed point theorems in fuzzy metric spaces*, J. Korean Math. Soc. 52 (2015) 1287–1303.